Rigidity Matroids

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Motivation

Motivation

Many engineering problems deal with rigidity of frameworks. The fundamental problem is how to predict the rigidity of a structure by theoretical analysis, without having to build it.

Figure: Truss Bridge

Rigidity Matroid

What is a rigid Framework?

Definition (d-Dimensional Frameworks)

1. A $d-$*dimensional framework* is a pair (G, p), where $G = (V, E)$ is a graph and p is a map from V to $\mathbb{R}^d$.
2. We consider a framework to be a straight line realization of G in $\mathbb{R}^d$.
3. A framework (G, p) is said to be *generic*, if all the coordinates of the points are algebraically independent over the rationals.

In the following we will consider straight line generic frameworks.

What is a rigid Framework?

Definition (Congruent and equivalent frameworks)

1. Two frameworks (G, p) and (G, q) are *equivalent* if $\|p(u) - p(v)\| = \|q(u) - q(v)\|$ holds for all pairs $u, v \in V$ with $uv \in E$.

2. (G, p) and (G, q) are *congruent* if $\|p(u) - p(v)\| = \|q(u) - q(v)\|$ holds for all pairs $u, v \in V$.

What is a rigid Framework?

Definition (rigid frameworks)

The framework (G, p) is rigid if there exists an $\epsilon > 0$ such that if (G, p) is equivalent to (G, q) and $\|q(v) - p(v)\| < \epsilon$ for all $v \in V$ then (G, q) is congruent to (G, p).

The rigidity of (G, p) only depends on the Graph G if (G, p) is generic.
A graph G is rigid in $\mathbb{R}^d$ if every generic realization of G in $\mathbb{R}^d$ is rigid.

Minimally rigid

Definition (Minimally rigid)

The graph G is said to be *minimally rigid* if G is rigid and $G - e$ is not rigid for all $e \in E$.

Theorem of Laman

Theorem

A graph $G = (V, E)$ *is minimally rigid in* $\mathbb{R}^2$ *if and only if* $|E| = 2|V| - 3$ *and*
$|E_{[X]}| \leq 2|X| - 3$ *for all* $X \subset V$ *with* $|X| \geq 2$

Note that every rigid graph has a minimally rigid spanning subgraph.

Redundantly rigid

Definition (Redundantly rigid)

A Graph G is *redundantly rigid* in $\mathbb{R}^d$ if deleting any edge of G results in a Graph wich is rigid in $\mathbb{R}^d$.
Graphs, which are redundantly rigid in $\mathbb{R}^2$ and have the minimum number of edges $2|V| - 2$, we call *M-ciruits*.

Graph extensions

Definition

1. The operation $0 - \textit{extension}$ adds a new vertex v and two edges vu and vw with $u \neq w$.
2. The operation $1 - \textit{extension}$ subdivides an edge uw by a new vertex v and adds a new edge vz for some $z \neq v, w$.
3. An *extension* is either a 0-extension or a 1-extension.

Characterization of minimally rigid graphs

Theorem

Each of the following conditions on a Graph $G = (V, E)$ is a characterization of minimally rigid graphs:

1. *G can be produced from a single edge by a sequence of extensions*
2. *for any two vertices $v \neq w$, with $vw \in E$ the (multi)-graph with edges $E \cup (v, w)$ is the union of two disjoint spanning trees.*
3. $|E| = 2|V| - 3$ *and*
   $|E_{[X]}| \leq 2|X| - 3$ *for all* $X \subset V$ *with* $|X| \geq 2$

Proof.

$(1 \Rightarrow 2)$

1. Idea: Build trees T_1 and T_2 along extensions.
2. We start the extensions with the edge (v, w).
3. Let G_0 be the initial Graph, duplicate (v, w). $T_1 = \{(v, w)\}$, $T_2 = \{(v, w)\}$ are the spanning trees.
4. Let G^+ be the 0-extension of G, adding a new vertex v_0 and two new edges (v_0, v_i) and (v_0, v_j).
5. $T_1^+ = T_1 \cup \{(v_0, v_i)\}$ and $T_2^+ = T_1 \cup \{(v_0, v_j)\}$ This is a partition of $E^+ \cup \{(v, w)\}$ into two spanning trees.
6. Let $G^+ = (V \cup \{v_0\}, E \backslash \{(v_i, v_j)\} \cup \{(v_0, v_i), (v_0, v_j), (v_0, v_k)\}$ be a 1-extension of G.
7. Assume $E \cup \{(v, w)\}$ is the union of two spanning trees and $(v_i, v_j) \in T_1$.
8. Let $T_1^+ = T_1 \backslash \{(v_i, v_j)\} \cup \{(v_0, v_i), (v_0, v_j)\}$ and $T_2^+ = T_2 \cup \{(v_0, v_k)\}$.
9. This is a partition of $E^+ \cup \{(v, w)\}$ in two spanning trees.

Proof.

$(3 \Rightarrow 1)$

1. Define $b(X) = 2|X| - 3 - |E_{[X]}|$ which allows to state the Laman Property as $b(V) = 0$ and $b(X) \geq 0$ for all $X \subset V$.
2. Let G be a Graph with the Laman property. By induction it is enauph to show there is a G' with one vertex less, s.t. G' has the Laman property and G can be optained from G' by extensions.
3. A Graph with Laman property must have a vertex of deg 2 or 3.
4. if $deg(z) = 2$, then removing z and the two incident edges gives G' with the Laman property. G is optained from G' by 0-extension.
5. Suppose $deg(z) = 3$ and let $N(z) = \{u, v, w\}$. Observations:
   1. $|E_{[u,v,w]}| = 2$
   2. If $\{u, v, w\} \subset X$ and $z \notin X$ then $b(X) > 0$

□

Case 1: $|E_{[u,v,w]}| = 2$

1. Let (u, v) and (u, w) be the edges. We claim the Graph G' obtained by deleting z and adding (v, w) has the Laman property.
2. Assume G' is not Laman. Then there is $X \subset V(G')$ s.t. $b_{G'}(X) < 0$.
3. $\Rightarrow b_{G'}(X) \neq b(X)$
4. hence $v, w \in X$, $z \notin X$ and $b(X) = 0$
5. With observation 2 we obtain $u \notin X$
6. It follows $b(X + u + z) = 2(|X| + 2) - 3 - |E_{[X]}| - \#(edges\ in\ E_{[X+u+z]}\ incident\ to\ u\ or\ z) \leq b(X) + 4 - 5 < 0$
7. The contradiction $b(X + u + z) < 0$ shows that G' has the Laman property.

Rigidity Matroid

Definition (Rigidity Matroid)

Let $G = (V, E)$ be a graph. Let $F \subset E$, $F \neq \emptyset$ U be the set of vertices incident with F, and $H = (U, F)$ be a subgraph of G induced by F.

1. We say that F is *independent* if $|E_{[X]}| \leq 2|X| - 3$ for all $X \subset U$ with $|X| \geq 2$.
2. The empty set is also independent.
3. The *rigidity matroid* $\mathcal{M}(G) = (E, \mathcal{I})$ is defined on the edge set of G by

$$\mathcal{I} = \{F \subset E | F \textit{ is independent in } G\} \tag{1}$$

Rank of a rigidity matroid

Lemma

Let $G = (V, E)$ be a graph. Then $\mathcal{M}(G)$ is a matroid, in which the rank of a non-empty set $E' \subset E$ of edges is given by

$$r(E') = min\left\{\sum_{i=1}^{t}(2|X_i| - 3)\right\} \tag{2}$$

where the minimum is taken over all collections of subsets $\{X_1, \cdots, X_t\}$ of V such that $\{E_G(X_1), \cdots E_G(X_t)\}$ partitions E'.

$G = (V, E)$ is rigid if $r(E) = 2|V| - 3$ in $\mathcal{M}(G)$. The graph is minimally rigid if it is rigid and $|E| = 2|V| - 3$.

Circuit

Definition (Circuits)

Given A Graph $G = (V, E)$, a subgraph $H = (W, C)$ is said to be an *M-circuit* in G if C is a minimal dependent set in $\mathcal{M}(G)$.

A graph G is redundantly rigid if and only if G is rigid and each edge of G belongs to a circuit in $\mathcal{M}(G)$. i.e. an M-circuit of G.

Infinitesimally Rigidity

Infinitesimally rigididty

Definition (infinitesimally rigid)

1. An *infinitesimal motion* of a plane framework is an assignment of velocities $v_i \in \mathbb{R}^2$ to each vertex i such that for every edge $(i, j) \in E$

$$\langle p_i - p_j, v_i - v_j \rangle = 0 \; \textit{for all} \; (i, j) \in \; E \tag{3}$$

2. A *trivial motion* is a motion which comes from a rigid transformation of the hole plane. A plane framework is *infinitesimally rigid* if every infinitesimal motion is trivial.

Rigidity Matrix

Definition (Rigidity matrix)

The *rigidity matrix* of a plane framework $G(p)$ is an $|E|x2|V|$ matrix $\mathbf{R}_{G(p)}$. Each vertex has two columns in $\mathbf{R}_{G(p)}$ representing the two coordinates.

1. This allows us to write the condition for infinitesimal motion $v : V \to \mathbb{R}^2$ as
$$\mathbf{R}_{G(p)} \cdot v = 0. \quad (4)$$
2. Every infinitesimal motion is an element of the kernel of $\mathbf{R}_{G(p)}$.
3. Since we have 3 trivial motions in the plane, the rank $of\,\mathbf{R}_{G(p)}$ from a rigid framework needs to be $2|V| - 3$

Generic rigidity

Definition (Generic rigidity)

A Graph G is *generically rigid*, if for almost all embeddings p of G the rigidity matrix has rank $2|V| - 3$.

An embedding is generic if for every point we can find an open neighbourhood in which the rank of the rigidity matrix is not changing.

Generic rigidity Matroid

Definition (Generic rigidity Matroid)

1. The independence structure of the rows of the rigidity matrix defines a matroid on the edges of the complete graph on the vertices.
2. This matroid depends on the positions of the joints.
3. There are *generic* positions that give a maximal collection of independent sets.
4. At these points we have the *generic rigidity matroid for* $|V|$ *vertices in the plane*.

Isostatic plane frameworks

Definition (Isostatic plane frameworks)

1. *Isostatic plane frameworks* are minimal infinitisemally rigid frameworks.
2. Removing any one bar introduces a non-trivial infinitesmal motion.
3. These graphs, of size $|E| = 2|V| - 3$, are the bases on the generic rigidity matroid of the complete graph on the set of vertices.

Thus an isostatic framework corresponds to a row basis for the rigidity matrix of any infinitesmally rigid framework extending the framework.

Characterizations of an isostatic framework

Theorem

For a Graph G, with at least two vertices the following are equivalent conditions:

1. *G has some positions $G(p)$ as an isostatic plane framework;*
2. $|E| = 2|V| - 3$ *and for all proper subsets of edges* $|E'|$ *incident with vertices* $|V|$, $|E'| \leq 2|V'| - 3$
3. *adding any edge to E gives an edge set covered by two edge-disjoint spanning trees.*

How are the rigidity definitions connected?

Rigidity and infinitesamally rigid

Proposition

A non- rigid framework can not be infinitesamally rigid. The opposite is not true: many infinitesamlly motions are not the deriviative of an analytic path.
If the framework is generic, a graph is rigid if and only if it is infinitesamlly rigid.

Generic rigidity Matroid

Proposition

If G is a minimal generically rigid graph and p a generic embedding, $G(p)$ is an isostatic framework.

Roundup

1. Rigidity in the plane is a property of a Graph if the embedding of the Graph is generic.
2. There are different ways to characterize rigidity, and to define independence structures and Matroids
3. For generic graph embeddings these rigidity definitions are equivalent