Greedoids

by Joris Wenzel

Technische Universität Berlin

Prim's algorithm

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Goal:

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Goal: find a minimum spanning tree

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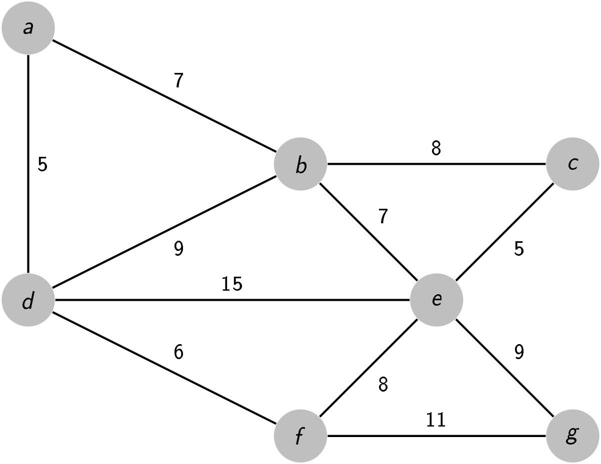

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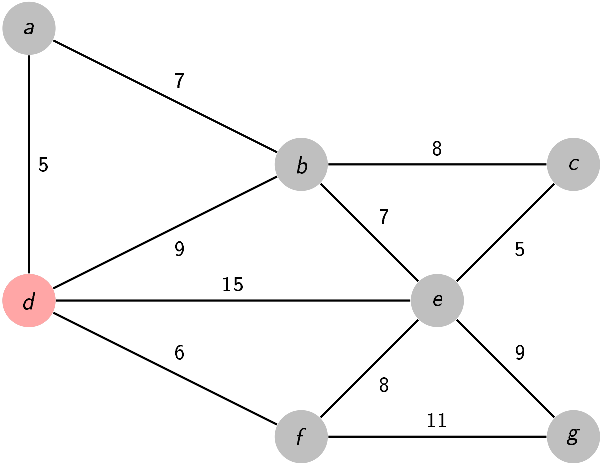

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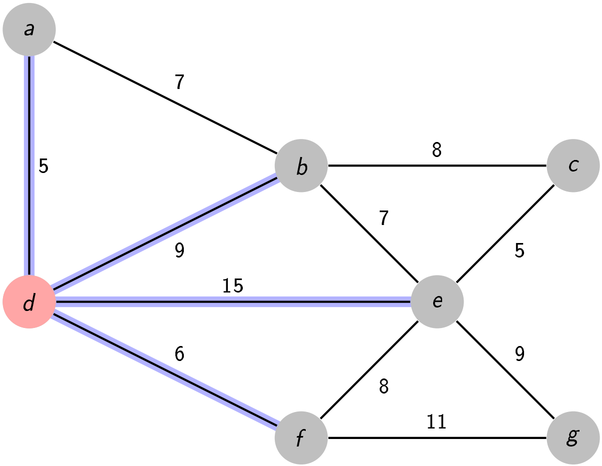

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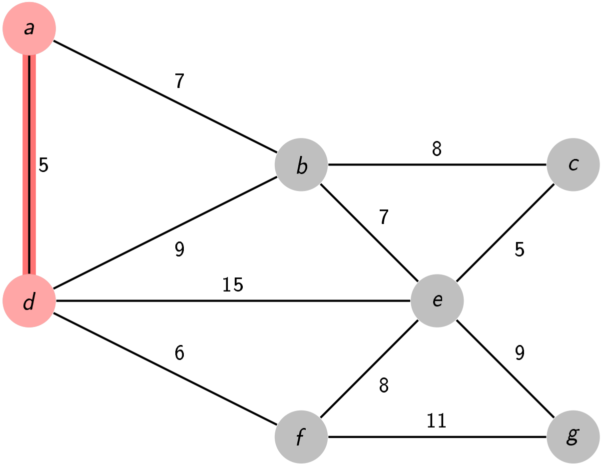

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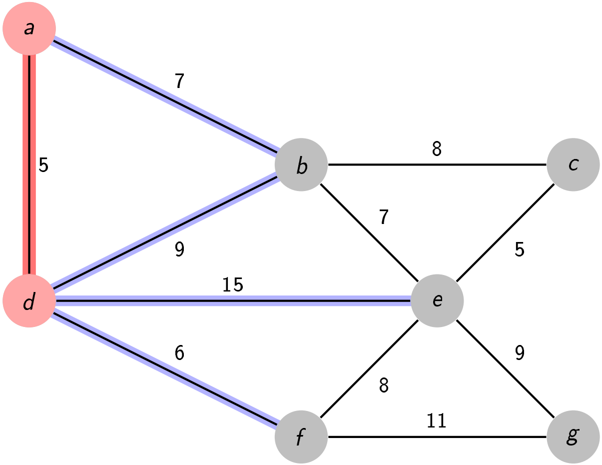

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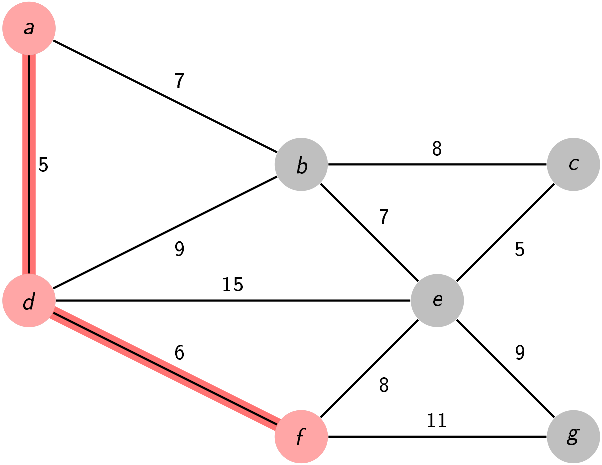

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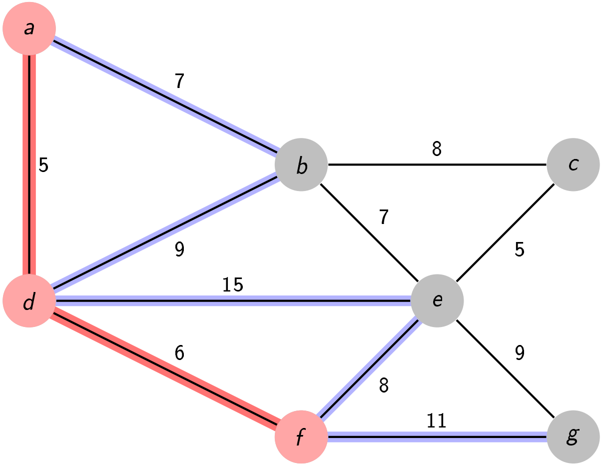

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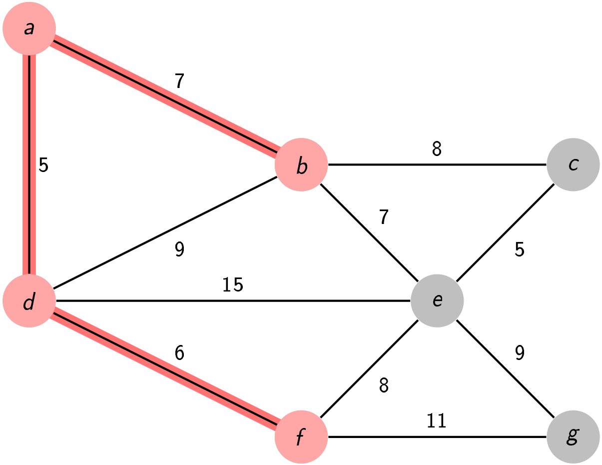

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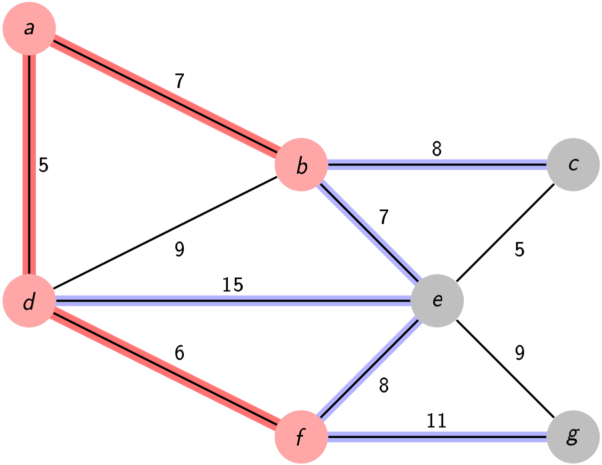

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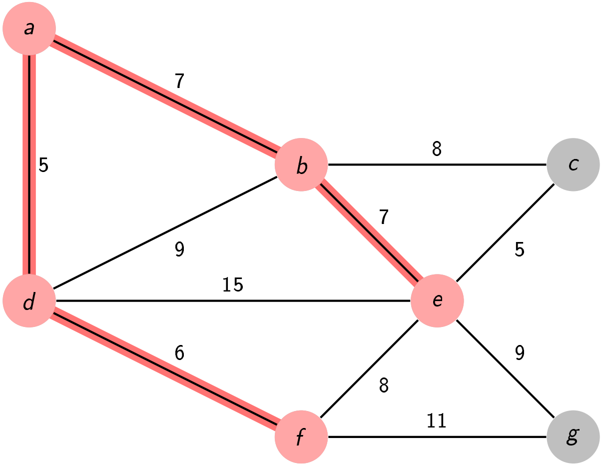

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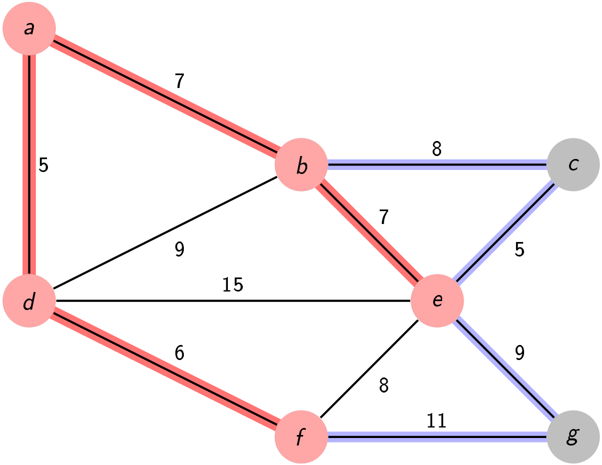

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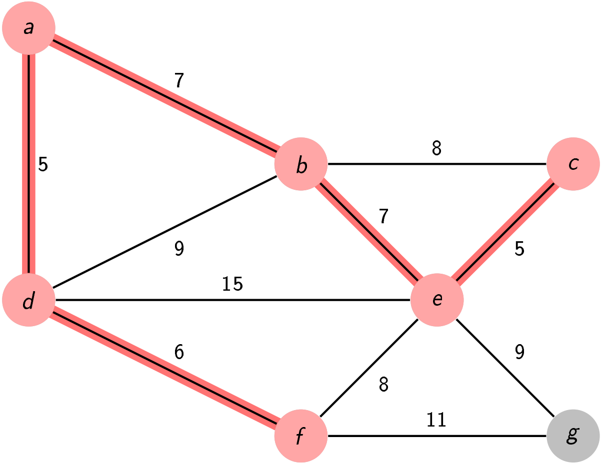

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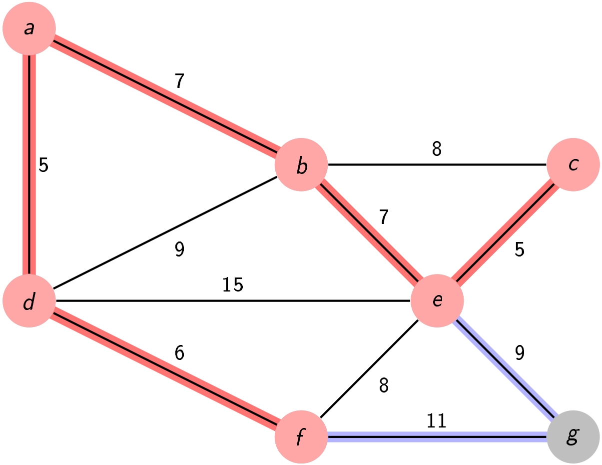

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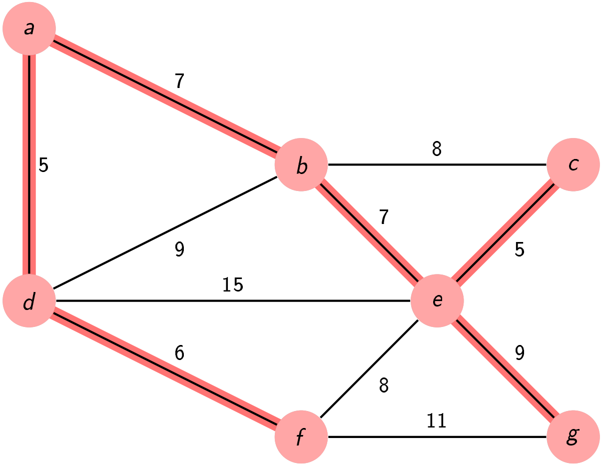

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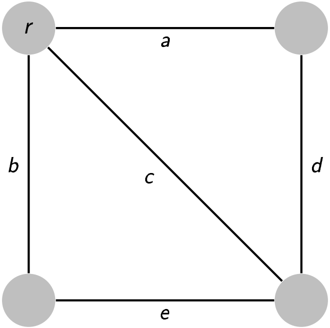

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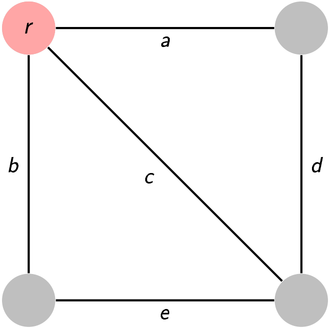

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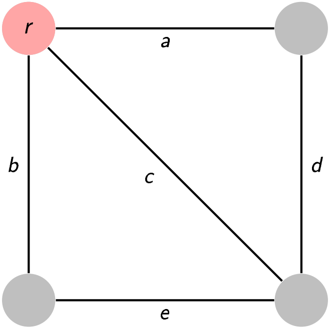

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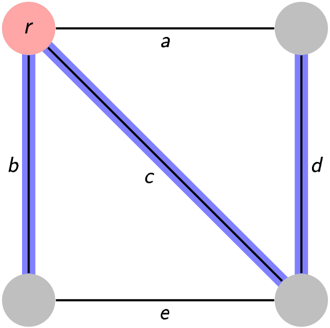

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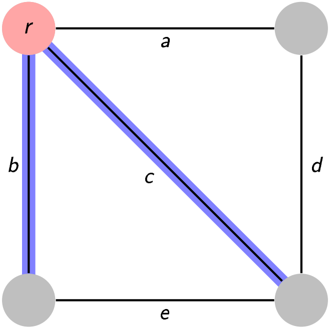

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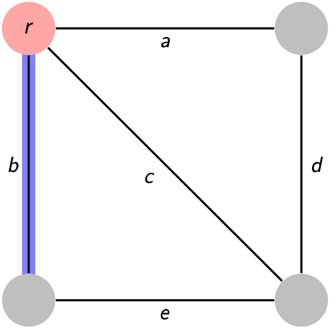

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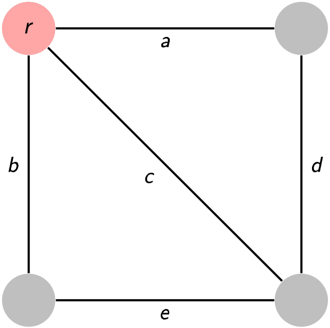

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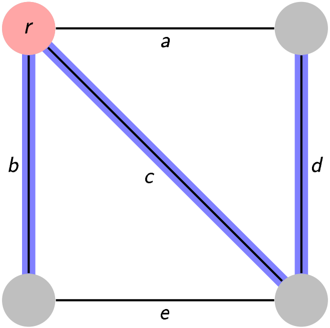

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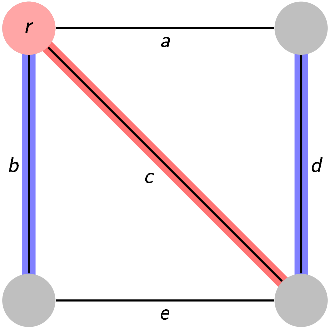

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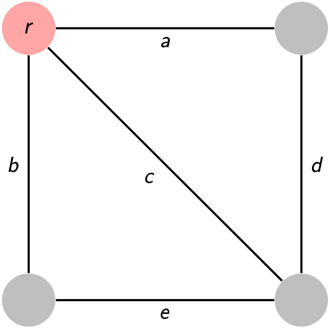

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undirected branching greedoid

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antimatroids
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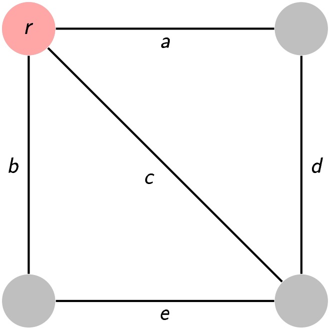

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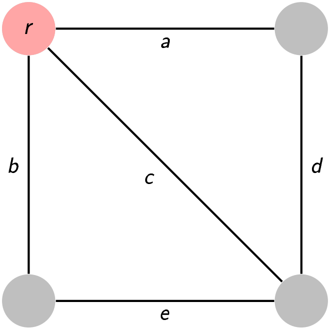

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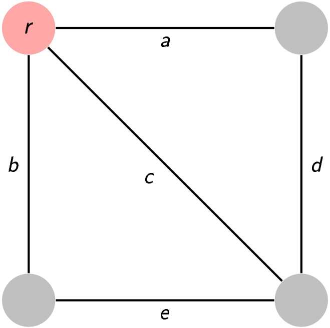

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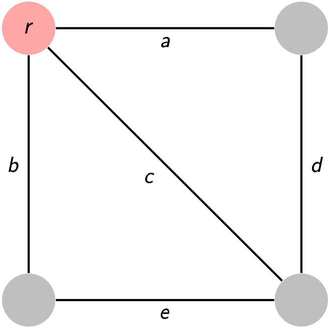

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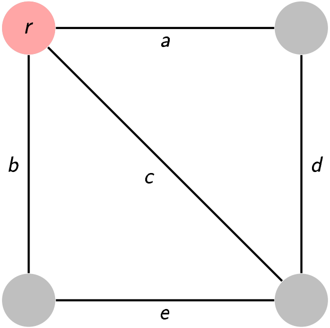

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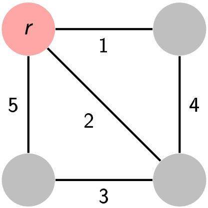

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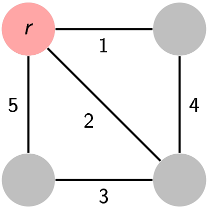

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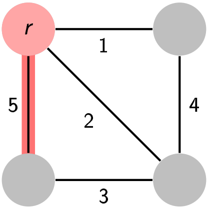

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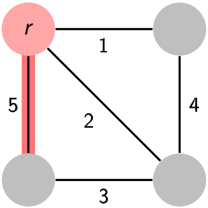

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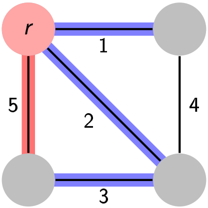

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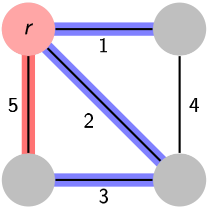

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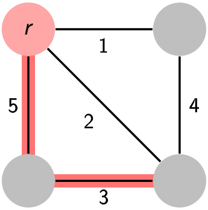

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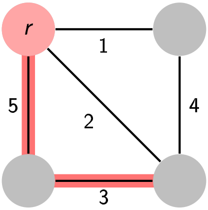

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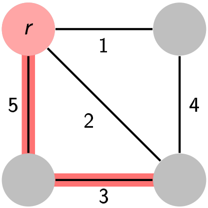

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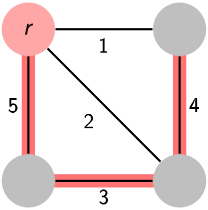

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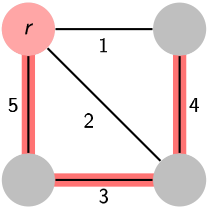

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Korte & Lovász (1984). Greedoids, a structural framework for the greedy algorithm

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$(E, \mathcal{L})$ is interval greedoid, ω compatible $\Rightarrow$ greedy works:

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claim: $\gamma = \delta = \alpha$
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Questions?