Independent set polytopes

Fritz Geis

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Seminar Matroide und verwandte Strukturen

Linear programming

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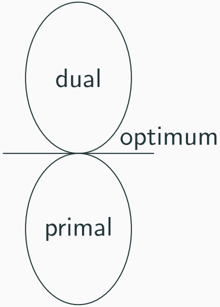
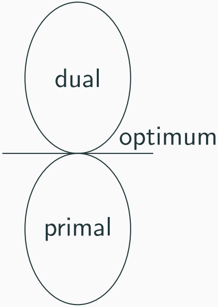


Linear programming

maximize $w^T x$

subject to $Ax \leq r$

$x \geq 0$



minimize $r^T y$

subject to $A^T y \geq w$

$y \geq 0$

Linear programming

Strong duality

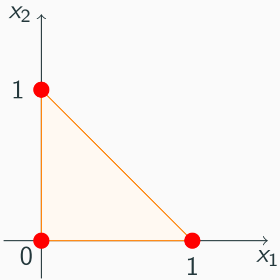
If an LP has an optimal solution, so does its dual and the optimal values are equal.

Integral Polytope

A polytope P is called **integral**, if every vertex of P is integral.

Example

$$x_1 + x_2 \leq 1$$
$$x_1, x_2 \geq 0$$



Linear programming

Totally unimodular

A matrix A is called **totally unimodular** if each square submatrix of A has determinant equal to $+1$, -1 or 0.

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Integrality condition I

Let A be a totally unimodular *mxn* matrix and let $r \in \mathbb{Z}^m$. Then the polyhedron defined by $Ax \leq r$ is integral.

Linear programming

Totally dual integral

A system $Ax \leq r$ is called **totally dual integral (TDI)**, if A and r are rational and for each $w \in \mathbb{Z}^n$, the dual of

$$\text{maximize } w^T x$$
$$\text{subject to } Ax \leq r$$

has an integer optimum solution.

Linear programming

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Integrality condition II
If $Ax \leq r$ is TDI and r integral, then $Ax \leq r$ defines an integral polyhedron.

The independent set polytope

The independent set polytope

The **independent set polytope** $P_{independent\ set}(M)$ of a matroid $M = (S, \mathcal{I})$ is defined as the convex hull of the incidence vectors of the independent sets of M.

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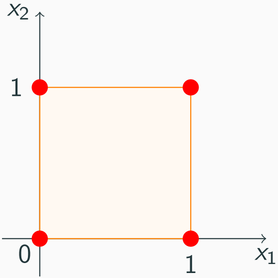
Example:

S={a,b}; $\mathcal{I} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$



The independent set polytope

Claim: $P_{independent\ set}(M)$ is fully determined by:

$$
\begin{aligned}
(1) \qquad & x_s \geq 0 && \text{for } s \in S \\
& x(U) \leq r_M(U) && \text{for } U \subseteq S
\end{aligned}
$$

where $x(U) := \sum_{u \in U} x_u$

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Remarks:

- Incidence vectors of independent sets satisfy (1)
  $\Rightarrow$ Each vector in $P_{independent\ set}(M)$ satisfies these

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<u>Remarks:</u>

- Incidence vectors of independent sets satisfy (1)
  $\Rightarrow$ Each vector in $P_{independent\ set}(M)$ satisfies these
- x integer vector satisfying (1) $\Leftrightarrow$ x incidence vector of a independent set

The independent set polytope

Theorem 40.2 Let $M = (S, \mathcal{I})$ be a matroid, with rank function r. Then for any weight function $w : S \to \mathbb{R}_+$:

$$max\{w(I) \mid I \in \mathcal{I}\} = \sum_{i=1}^{n} \lambda_i r(U_i),$$

where $U_1 \subset \cdots \subset U_n \subseteq S$ and where $\lambda_i \geq 0$ satisfy

$$w = \sum_{i=1}^{n} \lambda_i \chi^{U_i}$$

The independent set polytope

For $w : S \to \mathbb{R}$ consider following linear programming problem

$$\begin{aligned} &\text{maximize } w^T x \\ &\text{subject to } x_s \geq 0 \quad && (s \in S), \\ &\qquad x(U) \leq r_M(U) \quad && (U \subseteq S) \end{aligned}$$

and its dual:

$$\begin{aligned} &\text{minimize } \sum_{U \subseteq S} y_U r_M(U) \\ &\text{subject to } y_U \geq 0 \quad && (U \subseteq S) \\ &\qquad \sum_{U \subseteq S} y_U \chi^U \geq w \end{aligned}$$

The independent set polytope

Corollary 40.2a If $w : S \to \mathbb{Z}$, then the primal and dual have integer optimal solutions.

The independent set polytope

Primal:

$$\begin{aligned} &\text{maximize } w^T x \\ &\text{subject to } x_s \geq 0 \quad && (s \in S), \\ &\qquad x(U) \leq r_M(U) \quad && (U \subseteq S) \end{aligned}$$

Dual:

$$\begin{aligned} &\text{minimize } \sum_{U \subseteq S} y_U r_M(U) \\ &\text{subject to } y_U \geq 0 \quad && (U \subseteq S) \\ &\qquad \sum_{U \subseteq S} y_U \chi^U \geq w \end{aligned}$$

Thm. 40.2

$max\{w(I) | I \in \mathcal{I}\} = \sum_{i=1}^{n} \lambda_i r(U_i)$

$w = \sum_{i=1}^{n} \lambda_i \chi^{U_i}$

The independent set polytope

Corollary 40.2a If $w : S \to \mathbb{Z}$, then the primal and dual have integer optimal solutions.

Corollary 40.2b The independent set polytope is determined by

$$x_s \geq 0 \qquad \text{for } s \in S$$
$$x(U) \leq r_M(U) \qquad \text{for } U \subseteq S$$

Intersection of the independent set polytopes

Intersection of the independent set polytopes

Common independent set polytope

Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be two Matroids. Then we define $P_{common\ independent\ set}(M_1, M_2)$ as the convex hull of the incidence vectors of common independent sets of M_1 and M_2.

Intersection of the independent set polytopes

Claim: Let $M_1 = (S, \mathcal{I})$ and $M_2 = (S, \mathcal{I})$ be two matroids, with rank function r_1 and r_2. Then the common independent set polytope is fully determined by following system,

$$
\begin{aligned}
(2) \quad & x_s \geq 0 && \text{for } s \in S \\
& x(U) \leq r_i(U) && \text{for } i = 1, 2 \textit{ and } U \subseteq S
\end{aligned}
$$

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Remarks

- (2) contains the convex hull of incidence vectors of common independent sets of M_1 and M_2

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Remarks

- (2) contains the convex hull of incidence vectors of common independent sets of M_1 and M_2
- Every integer vector satisfying (2) is an incidence vector of a common independent set
- (2) determines $P_{independent\ set}(M_1) \cap P_{independent\ set}(M_2)$

Intersection of the independent set polytopes

Chain

A family of sets $\mathcal{F}$ is called a **chain**, if for any pair of subsets $U, T \in \mathcal{F}$ holds, that either $T \subseteq U$ or $U \subseteq T$

Intersection of the independent set polytopes

Chain

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Laminar family

A family of sets $\mathcal{C}$ is called **laminar**, if
$Y \subseteq Z$ or $Z \subseteq Y$ or $Y \cap Z = \emptyset$ for all $Y, Z \in \mathcal{C}$.

Intersection of the independent set polytopes

Totally unimodular

A matrix A is called **totally unimodular** if each square submatrix of A has determinant equal to +1, -1 or 0.

Intersection of the independent set polytopes

Theorem 41.11 Let $\mathcal{C}$ be the union of two laminar families of subsets of a set X. Let A be the $\mathcal{C} \times X$ incidence matrix of $\mathcal{C}$. Then A is totally unimodular.

Intersection of the independent set polytopes

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has an integer optimum solution.

Box-totally dual integral

The linear system $Ax \leq b$ is **box-totally dual integral (box TDI)**, if for every choice of rational vectors u and l the system $Ax \leq b;\ u \leq x \leq l$ is TDI.

Intersection of the independent set polytopes

Thm. 41.12 The system

$$(2) \qquad \begin{aligned} & x_s \geq 0 && \text{for } s \in S \\ & x(U) \leq r_i(U) && \text{for } i = 1,2 \textit{ and } U \subseteq S \end{aligned}$$

is box-totally dual integral.

Intersection of the independent set polytopes

Choose $w \in \mathbb{Z}^S$

Primal

$$\text{maximize} \;\; w^T x$$
$$\text{subject to } x(U) \leq r_i(U)$$

Dual

$$\text{minimize} \;\; \sum_{U \subseteq S} (y_1(U) r_1(U) + y_2(U) r_2(U))$$
$$\text{subject to } y_1, y_2 \geq 0$$
$$\sum_{U \subseteq S} (y_1(U) + y_2(U)) \chi^U = w$$

Intersection of the independent set polytopes

Corollary 41.12a $P_{common\ independent\ set}(M_1, M_2)$ is determined by

$$
\begin{aligned}
(2) \quad & x_s \geq 0 && \text{for } s \in S \\
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Intersection of the independent set polytopes

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Corollary 41.12b

$P_{common\ independent\ set}(M_1, M_2) = P_{independent\ set}(M_1) \cap P_{independent\ set}(M_2)$

Cunningham

Theorem 40.3 Let $M = (S, \mathcal{I})$ be a matroid and let $x \in \mathbb{Q}_+^S$. Then

$$
\begin{aligned}
&max\{z(S) | z \in P_{independent\ set}(M), z \leq x\} \\
&= min\{r_M(U) + x(S \setminus U) | U \subseteq S\}
\end{aligned}
$$

The independent set polytope

given: a matroid $M = (S, \mathcal{I})$, by an independence testing oracle, and a $x \in \mathbb{Q}_+^S$;

find: a $z \in P_{independent\ set}(M)$ with $z \leq x$ maximizing $z(S)$, with a decomposition of z as a convex combination of incidence vectors of independent sets, and a subset $U \subseteq S$ satisfying $z(S) = r_M(U) + x(S \setminus U)$.

Thank you for your attention!