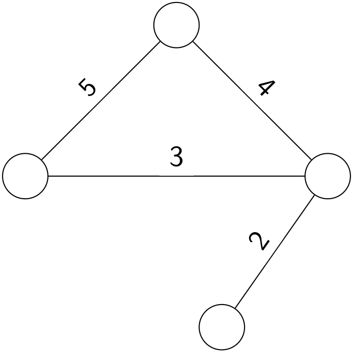
Greedy Algorithm and Matroid Intersections

by Yan Alves Radtke

July 2020

Greedy

Task Find a maximum weight spanning tree!



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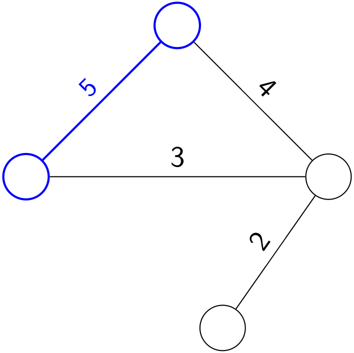

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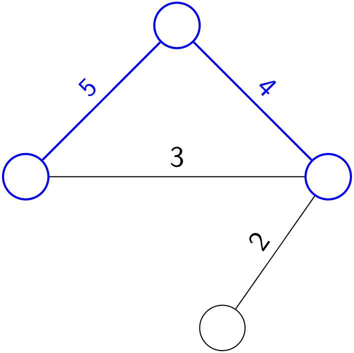

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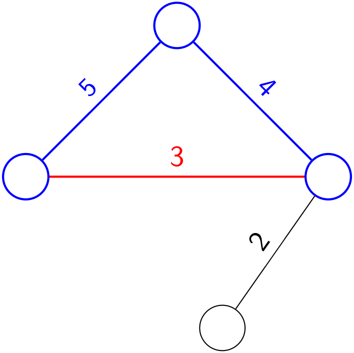

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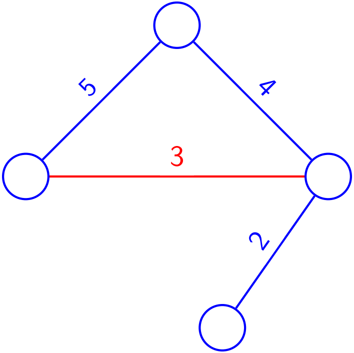

Formalization

Input: Connected graph with edge weights
Output: Maximum weight spanning tree
Initalization: $I := \emptyset$;
while *Exist edge* E *s. t.* $I \cup \{E\}$ *is a forest* **do**
  Choose such E with maximal weight;
  put $I := I \cup \{E\}$;
end
return I

Independence System

Def. $(\mathcal{S}, \mathcal{I})$ is an independence system iff

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- $\emptyset \in \mathcal{I}$
- $I \in \mathcal{I} \implies \mathcal{P}(I) \subseteq \mathcal{I}$

Greedy Algorithm

Input: $(\mathcal{S}, \mathcal{I})$ with weight function $w : \mathcal{S} \to \mathbb{R}_{\geq 0}$
Output: $I \in \mathcal{I}$ with $w(I) := \sum_{i \in I} w(i)$ maximal

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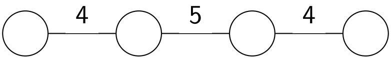
Counterexample for Greedy Algorithm

Task Find maximum weight independent edge set



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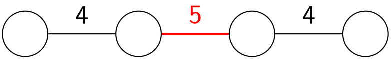

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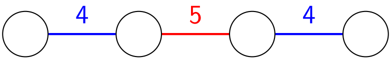

Greedy Theorem

For all Independence Systems $(\mathcal{S}, \mathcal{I})$ it holds:

$(\mathcal{S}, \mathcal{I})$ is a matroid iff the greedy algorithm returns a maximum weight independent set for all non-negative weight functions

Proof "$\Rightarrow$"

Loop invariant: I is contained in a maximum weight base B

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- Assume I is contained in a maximum weight base B
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- Case 2: $I + y \nsubseteq B$

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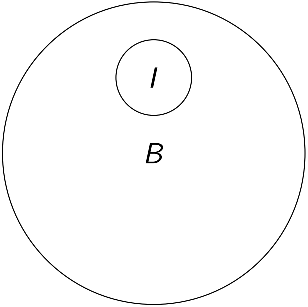

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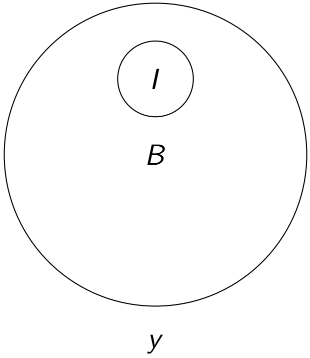

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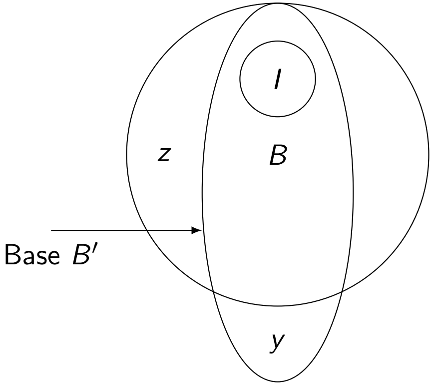

Proof "$\Rightarrow$"

$$w(B') - w(B) = w(y) - w(z) \geq 0$$

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$w(B') - w(B) = w(y) - w(z) \geq 0$, since the greedy algorithm chose y over z and $I + z \subseteq B$

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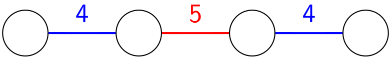

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- $|J| > |I| =: k$
- $I + z \notin \mathcal{I}$ for all $z \in J \setminus I$

Proof "$\Leftarrow$"

$$w(s) := \begin{cases} k+2 & \text{if } s \in I \\ k+1 & \text{if } s \in J \setminus I \\ 0 & \text{else} \end{cases}$$

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Matroid Intersections

- Two matroids $\mathcal{M}_1 = (\mathcal{S}, \mathcal{I}_1)$ and $\mathcal{M}_2 = (\mathcal{S}, \mathcal{I}_2)$

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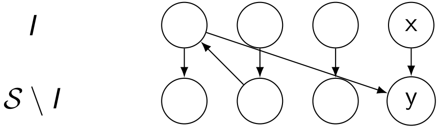
- $(\mathcal{S}, \mathcal{I}_1 \cap \mathcal{I}_2)$ is generally **not** a matroid
- But it is a independence system

$D_{\mathcal{M}_1,\mathcal{M}_2}(I)$

For given $\mathcal{M}_1, \mathcal{M}_2$ and common independent set I, $D_{\mathcal{M}_1,\mathcal{M}_2}(I)$ is



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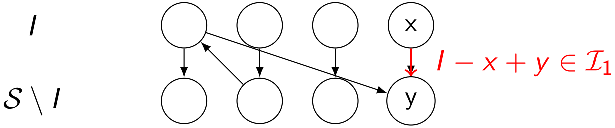

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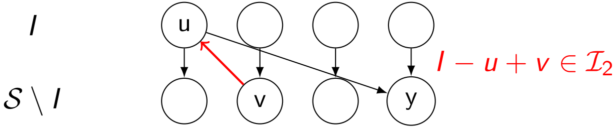
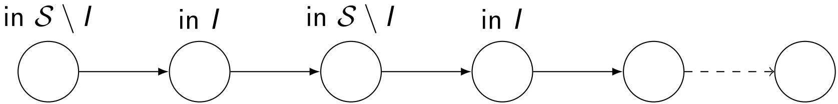


Paths in $D_{\mathcal{M}_1,\mathcal{M}_2}(I)$



Performing swaps along Path P on I gives us $I \triangle VP$

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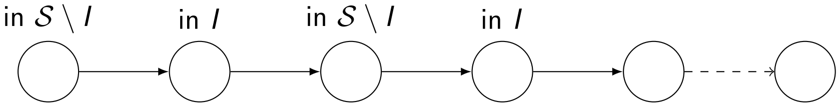

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This isn't necessary a common independent set!

Maximum weight algorithm

For a given weight function w we can define:

$$\ell(x) := \begin{cases} w(x) \text{ if } x \in I \\ -w(x) \text{ if } x \in \mathcal{S} \setminus I \end{cases}$$

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$$\implies w(J) = w(I \triangle VP) = w(I) - \ell(P)$$

Maximum weight algorithm for matroid intersections

Input: $\mathcal{M}_1$ and $\mathcal{M}_2$, an extreme common independent set I and a weight function w

Output: An extreme common independent set J with $|J| = |I| + 1$ if any exists, else I

Construct $D_{\mathcal{M}_1,\mathcal{M}_2}(I)$;

Define $X_i := \{x \in \mathcal{S} \setminus I | I \cup \{x\} \in \mathcal{I}_i\}$;

if X_1-X_2 *Path exists in* $D_{\mathcal{M}_1,\mathcal{M}_2}(I)$ **then**

  P:=minimal weight X_1-X_2 Path with minimal number of arcs;

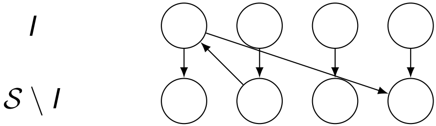
  return $I \triangle VP$

else

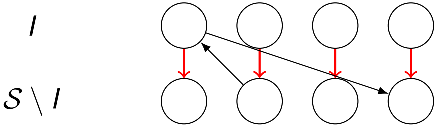
  return I

end

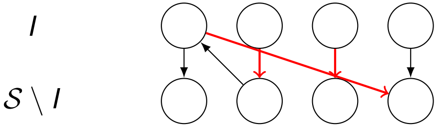
Matchings



Perfect Matchings



Not Perfect Matchings



Properties of perfect matchings

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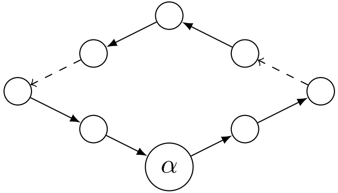
$\implies J \in \mathcal{I}_1$

Lemma

Let $C \ni \alpha$ be a circuit s.t. $I \triangle VC$ is not a common independent set



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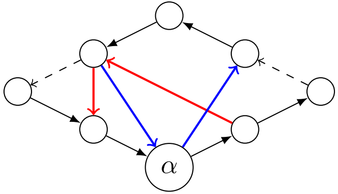
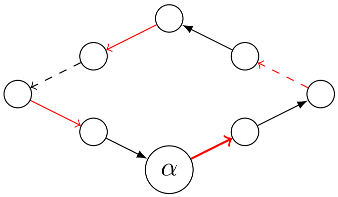


Then there exists:

Negative length circuit C' with $VC' \subsetneq VC$

or $C' \ni \alpha$ s.t. $\ell(C') \leq \ell(C)$ with $VC' \subsetneq VC$

Proof of Lemma



Matching

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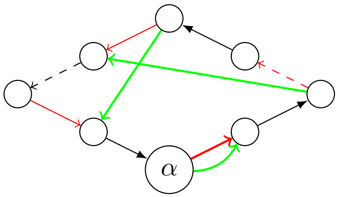

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Since not independent: another matching

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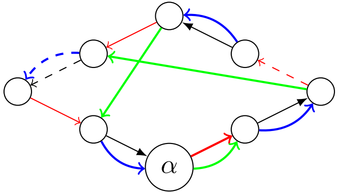

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$$\ell(C_1) + \ell(C_2) \leq \sum_{i=1,\dots,j} \ell(C_i) = 2\ell(C)$$

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$$\ell(C_1) + \ell(C_2) \leq \sum_{i=1,\dots,j} \ell(C_i) = 2\ell(C)$$

$$\implies \ell(C_1) \leq \ell(C) \text{ or } \ell(C_2) \leq \ell(C)$$

Extreme set - negative circuit theorem

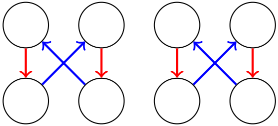
Statement: $D_{\mathcal{M}_1,\mathcal{M}_2}(I)$ has no negative length circuit $\Leftrightarrow$ I is an extreme set

Proof $\Rightarrow$

Let J be a common independet set with $|J| = |I|$

$I \setminus J$

$J \setminus I$



Proof $\Rightarrow$

$$w(J) = w(I) - \ell(I \triangle J) = w(I) - \sum \ell(C_i) \leq w(I)$$

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Proof $\Leftarrow$

Let C negative length circuit minimal nodes and I extreme set

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Reminder of Lemma

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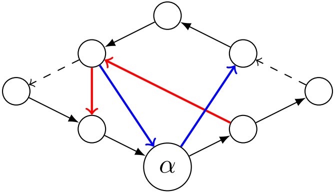

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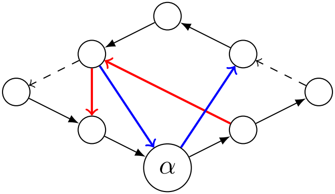

Since $\ell(C) < 0$ this implies:

Negative length circuit C' with $VC' \subsetneq VC$

Proof $\Leftarrow$

Let C negative length circuit minimal nodes and I extreme set

Then $w(I \triangle VC) = w(I) - \ell(VC) > w(I)$

Then $I \triangle VC$ is not a commmon independent set

- By Lemma C' is a negative length circuit with less nodes ↯

Maximum weight algorithm

Input: $\mathcal{M}_1$ and $\mathcal{M}_2$, an extreme common independent set I and a weight function w

Output: An extreme common independent set J with $|J| = |I| + 1$ if any exists, else I

Construct $D_{\mathcal{M}_1,\mathcal{M}_2}(I)$;

Define $X_i := \{x \in \mathcal{S} \setminus I | I \cup \{x\} \in \mathcal{I}_i\}$;

if X_1-X_2 *Path exists in* $D_{\mathcal{M}_1,\mathcal{M}_2}(I)$ **then**

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Proof $I + t - x + y \in \mathcal{I}'_i \Leftrightarrow I - x + y \in \mathcal{I}_i$

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Proof $I + t - x + y \in \mathcal{I}'_i \Leftrightarrow I - x + y \in \mathcal{I}_i$

Claim $N(t) = X_1 \cup X_2$

Auxiliary Matroid

Def. $\mathcal{M}'_i := (\mathcal{S} + \mathrm{t}, \{U \subseteq \mathcal{S} + \mathrm{t} | U - t \in \mathcal{I}_i\})$

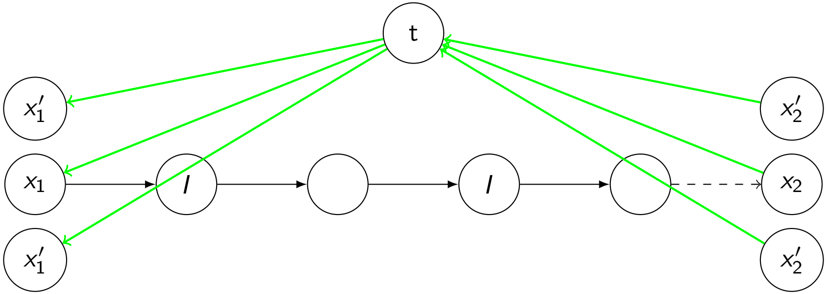
Claim $D_{\mathcal{M}'_1,\mathcal{M}'_2}(I + t)[\mathcal{S}] = D_{\mathcal{M}_1,\mathcal{M}_2}(I)$

Proof $I + t - x + y \in \mathcal{I}'_i \Leftrightarrow I - x + y \in \mathcal{I}_i$

Claim $N(t) = X_1 \cup X_2$

Proof $I + t - t + x \in \mathcal{I}'_i \Leftrightarrow I + x \in \mathcal{I}'_i \Leftrightarrow I + x \in \mathcal{I}_i \Leftrightarrow x \in X_i$

Proof of extremity of $I \triangle VP$



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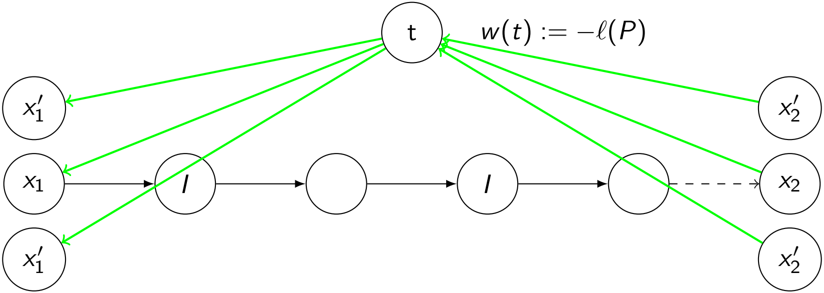

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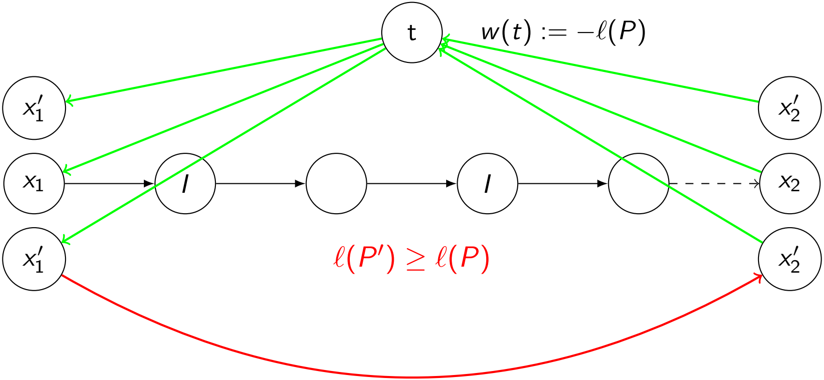

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$$w(I + t) = w(I) + w(t) = w(I) - \ell(P) = w(I \triangle P) = w(J)$$

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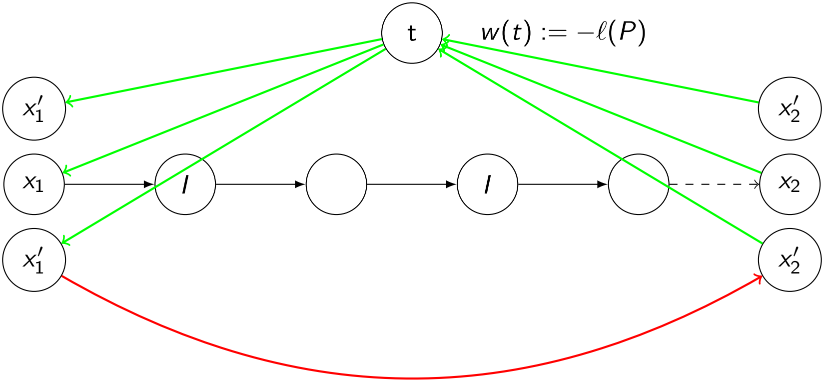
A relaxation of our original problem has a maximum weight of $w(J)$

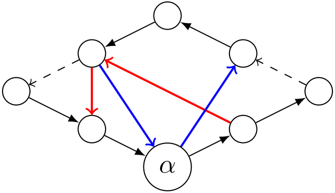
J common independent $\implies$ J is extreme common independent

Proof of independency of $I \triangle VP$



Reminder of Lemma

Let $C \ni t$ be a circuit s.t. $I \triangle VC$ is not a common independent set

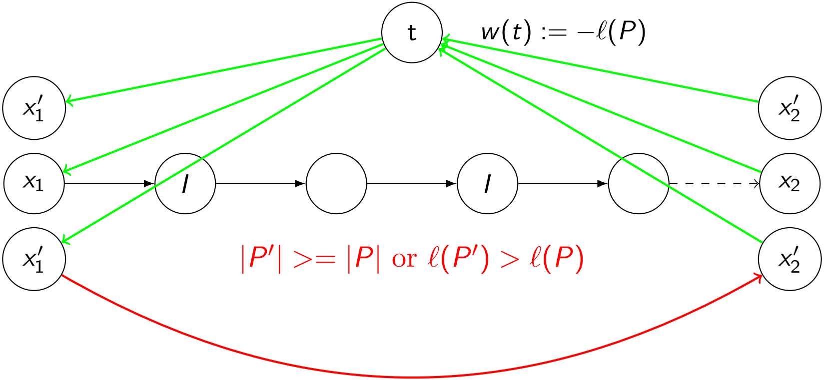


Then there exists:

~~Negative length circuit C' with $VC' \subsetneq VC$~~

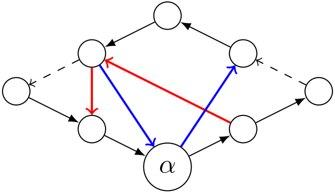
or $C' \ni t$ s.t. $\ell(C') \leq \ell(C)$ with $VC' \subsetneq VC$

Proof of independence of $I \triangle VP$



Reminder of Lemma

Let $C \ni \alpha$ be a circuit s.t. $I \triangle VC$ is not a common independent set



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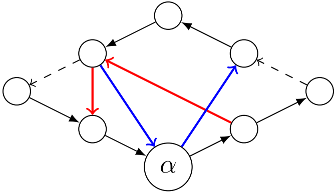

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Proof of independence

$(I + t \triangle VP + t) = I \triangle VP$ is a common independent set

Proof of independence

$(I + t \triangle VP + t) = I \triangle VP$ is a common independent set

$\implies J = I \triangle VP$ is an extreme common independent set

Maximum weight algorithm

Input: $\mathcal{M}_1$ and $\mathcal{M}_2$, an extreme common independent set I and a weight function w

Output: An extreme common independent set J with $|J| = |I| + 1$ if any exists, else I

Construct $D_{\mathcal{M}_1,\mathcal{M}_2}(I)$;

Define $X_i := \{x \in \mathcal{S} \setminus I | I \cup \{x\} \in \mathcal{I}_i\}$;

if X_1-X_2 *Path exists in* $D_{\mathcal{M}_1,\mathcal{M}_2}(I)$ **then**

  P:=minimal weight X_1-X_2 Path with minimal number of arcs;

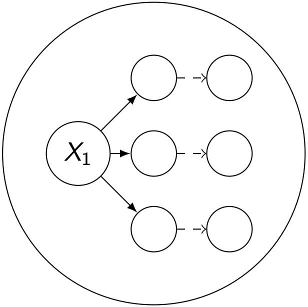
  return $I \triangle VP$

else

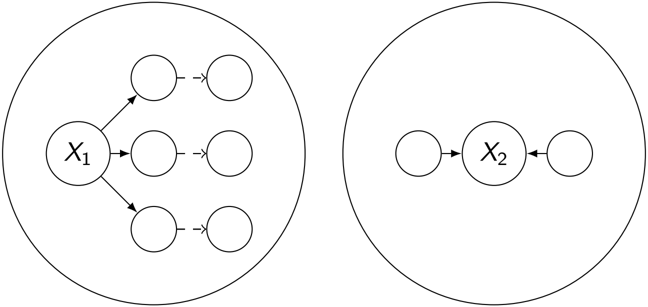
  return I

end

Case 2



Case 2



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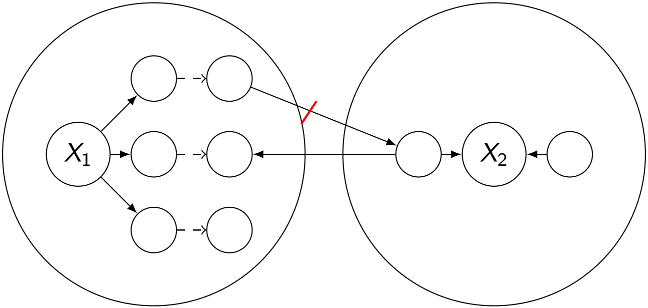

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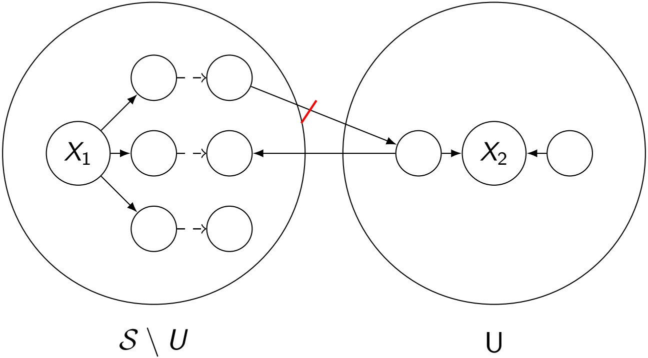

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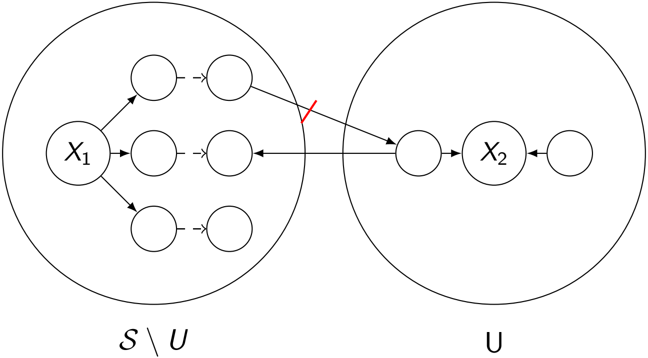

$X_1 \cap U = \emptyset$, $X_2 \subseteq U$

Matroid Intersection Theorem

The maximum size of a set in $\mathcal{I}_1 \cap \mathcal{I}_2$ is

$$\min_{U \subseteq \mathcal{S}} r_1(U) + r_2(\mathcal{S} \setminus U)$$

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$$\min_{U \subseteq \mathcal{S}} r_1(U) + r_2(\mathcal{S} \setminus U)$$

Use Case: Partition of base set can certify an upper bound

Proof "$\leq$"

For any common independent set I and any $U \subseteq \mathcal{S}$

$$|I| = |I \cap U| + |I \setminus U|$$

Proof "$\leq$"

For any common independent set I and any $U \subseteq \mathcal{S}$

$$|I| = |I \cap U| + |I \setminus U| = r_1(I \cap U) + r_2(I \setminus U)$$

Proof "$\leq$"

For any common independent set I and any $U \subseteq \mathcal{S}$

$$|I| = |I \cap U| + |I \setminus U| = r_1(I \cap U) + r_2(I \setminus U) \leq r_1(U) + r_2(\mathcal{S} \setminus U)$$

Idea of Proof "$\geq$"

Proof by induction over $|\mathcal{S}|$

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Proof by induction over $|\mathcal{S}|$

- $|\mathcal{S}| = 1$ just 3 cases
- Use restrictions and contractions to construct submatroids on U and $\mathcal{S} \setminus U$
- Get common independent sets of size $r_1(U)$ and $r_2(\mathcal{S} \setminus U)$ on U and $\mathcal{S} \setminus U$

Case 2

Claim $r_1(U) + r_2(\mathcal{S} \setminus U) \leq |I|$

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Suppose $r_1(U) > |I \cap U|$

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Claim $r_1(U) + r_2(\mathcal{S} \setminus U) \leq |I|$

- $r_1(U) \leq |I \cap U|$

Suppose $r_1(U) > |I \cap U|$

$\implies$ There is $x \in U \setminus I$ s.t. $I \cap U + x \in \mathcal{I}_1$

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$\implies$ $x \notin X_1 \implies I + x \notin \mathcal{I}_1$

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$\implies$ There is $y \in I \setminus U$ s.t. $I - y + x \in \mathcal{I}_1$

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$\implies$ There is $y \in I \setminus U$ s.t. $I - y + x \in \mathcal{I}_1$

$\implies$ (y, x) enters U $\lightning$

Similarly $r_2(\mathcal{S} \setminus U) \leq |I \setminus U|$

- So it follows: I is a maximum cardinality common independet set

Thank you for your attention