Regular Matroids

Alex Nedev
Janis Bakker

Matroids have structure, presentations also

1 General discussion

Classes of matroids

Matroid
isomorphism

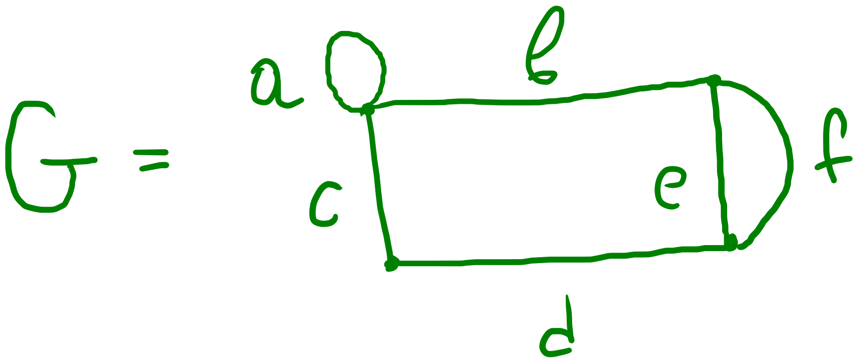
2 Theory: Representing matroids over a given field F

3 What mainly motivates the study of regular matroids?

Part 1

matroid
The ~~Marvel~~ universe

1 Graphic matroids M(G)



Ground set E = set of all edges of the graph G

The circuit set C = set of cycles in G

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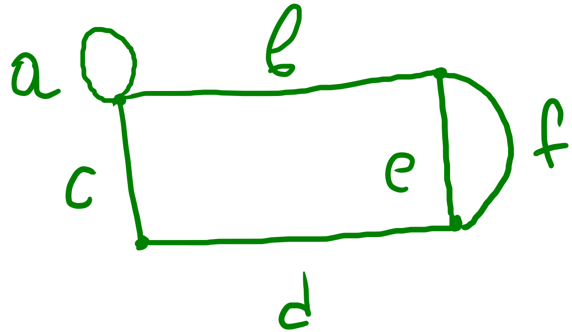

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2 Vector matroids M[Q]

$$Q = \begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{bmatrix} 0 & 3 & 0 & 0 & e^{\pi} & \sin(3) \\ 0 & 0 & 4 & 0 & e^{\pi} & \sin(3) \\ 0 & 0 & 0 & 5 & e^{\pi} & \sin(3) \end{bmatrix} \end{array}$$

E = set of all column labels

I = { subsets of the label set, s.t. the corresponding columns are linearly independent }

3 Uniform matroids $U_{n,m}$

Ground set E = arbitrary finite set of n elements

The set of bases B = the set of all m-element subsets of E

Fundamental Notion

Matroid
Equivalence

$M_1 = (E_1, \mathcal{I}_1)$ $\qquad$ $M_2 = (E_2, \mathcal{I}_2)$

M_1 isomorphic to M_2

iff

$\exists$ bijection $f : E_1 \to E_2$ such that

$A \in \mathcal{I}_1 \iff f(A) \in \mathcal{I}_2 \quad \forall A \subset E_1$

4 Matroids representable over a given field F

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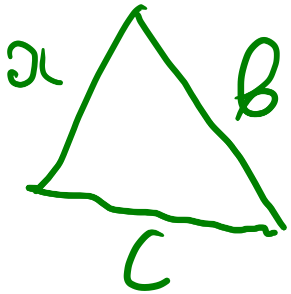
6 Unimodular matroids

Those matroids that have a vector matroid representation over the rational numbers by
a totally unimodular matrix P, i.e. any submatrix of P has determinant either 1, -1 or 0

Natural Question

How are the different matroid classes related?

Example 1: $U_{2,3}$ $\quad E = \{a, b, c\}$



Graphic

$$A = \begin{bmatrix} 1 & \cancel{0} & 1 \\ 0 & \cancel{1} & 1 \end{bmatrix} \quad \mathbb{F}$$

Example 2: $U_{2,4}$ $\quad$ $E = \{a, b, c, d\}$

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

$\mathbb{F}_3$

Part 2

Representing matroids over a given field F

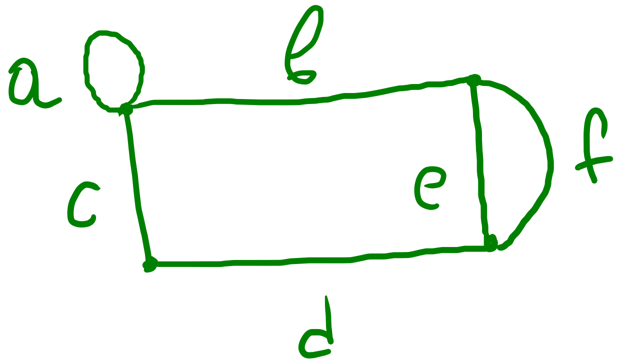
Matroid M with representation P over F

Wish

Make P nicer

M(G) represented by the matrix Q over the reals

$G =$



$$Q = \begin{array}{c} \begin{matrix} A & B & C & D & E & F \end{matrix} \\ \begin{bmatrix} 0 & 3 & 0 & 0 & e^{\pi} & \sin(3) \\ 0 & 0 & 4 & 0 & e^{\pi} & \sin(3) \\ 0 & 0 & 0 & 5 & e^{\pi} & \sin(3) \end{bmatrix} \end{array}$$

Theorem: Let P be a matrix with coefficients in F, M its associated vector matroid. Applying the following operations to P produces a matrix P' whose vector matroid M' is equivalent to M

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3. **Permute the columns (with associated labels)**

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4. **Delete trivial rows consisting of zeros only**

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1. Multiply each row/column by a nonzero element f of F

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3. Permute the columns (with associated labels)

4. Delete trivial rows consisting of zeros only

5. Replace row i by row i + (nonzero constant) x row j (j =/= i)

Important consequence

Let $M = (E, \mathcal{I})$ be a matroid of rank $r > 0$, representable over F, then $\exists$ a representation of M of the form

$$\left[I_r \mid D \right]$$

$\Rightarrow U_{2,4}$ not representable over $\mathbb{F}_2$

Lemma: Basic properties of determinants

1) Multiply some row of P by a constant c to obtain D $\Rightarrow$ $\det(D) = c \cdot \det(P)$

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3) Replace row i of P with row i + constant × row J, $J \neq i$ $\Rightarrow \det(D) = \det(P)$

Another Lemma: Pivoting on a nonzero entry x_{st} of a totally unimodular matrix X preserves total unimodularity

Proof via concrete example:

$$
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \end{array}
\xrightarrow{\text{Pivot on } x_{3,b}}
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X'

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\end{array}
$$

Important consequence

A Matroid M of rank $r > 0$ is unimodular iff M can be represented by a totally unimodular Matrix P of the form $P = [I_r \mid D]$

Main Theorem

If M has a unimodular representation X over $\mathbb{Q}$, then X also represents M over an arbitrary field F

<u>Proof:</u> $r(M) > 0$ $\quad X = [I_r \mid D]$

Let B be a set r columns of X

$\det(B) \neq 0 \quad \sum -1 + \sum 1 + \cancel{\sum Q}$

Part 3

Graphic matroids

Two Main results

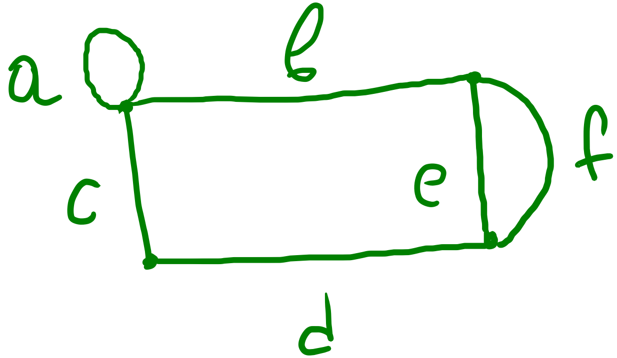
1 Graphic matroids are regular

2 Graphic matroids are unimodular

Both results follow easily from an "algorithmic" method of generating representations of a given graphic matroid

M(G) represented by the matrix Q over the reals

$G =$



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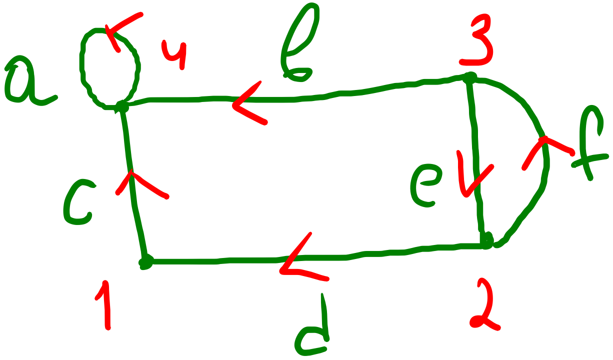

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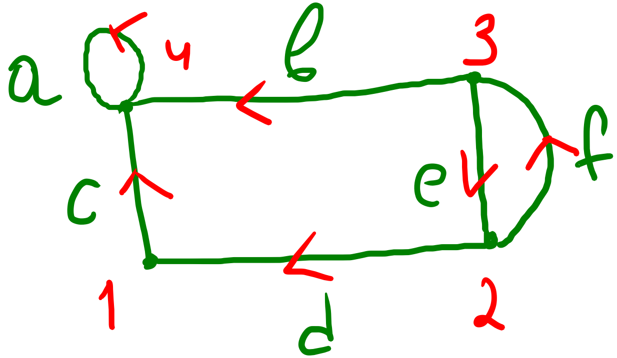

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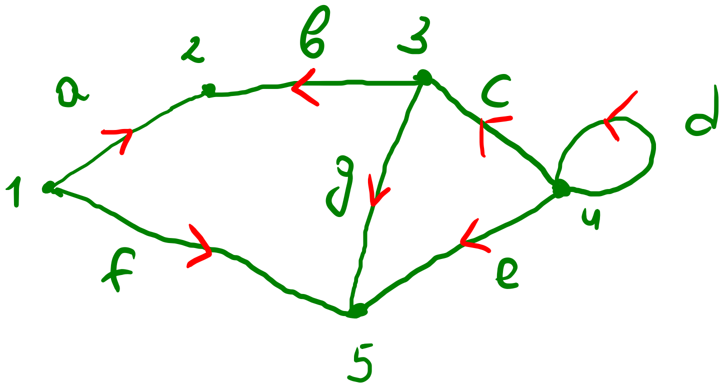
0 & 0 & 0 & 1 & -1 & 1 \\
0 & 1 & 0 & 0 & 1 & -1 \\
0 & -1 & -1 & 0 & 0 & 0
\end{bmatrix}
\end{array}
$$

Proof of regularity via concrete example

$G =$ 

$a\,b\,g\,f$

E

$x \qquad y$

$\deg(v) \geq 2 \qquad X \subset y$

$$X = \begin{array}{c|ccccccc} & a & b & c & d & e & f & g \\ \hline 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \end{array}$$

$\begin{bmatrix} 0 \\ \\ \\ \end{bmatrix} \quad y \subset X$

Proof of unimodularity via induction on |E(G)|

Proof

Assume $|E(G)| = n$

Assume Th. holds for $|E(G')| < n$

e

$m \times m$

1 0 1

$m < n$

X

$m = n$

$G \setminus e$

$m=n$

$$X = i\;\begin{bmatrix} 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots \end{bmatrix}$$

e_j $\quad$ Ce_j $\quad$ x''

✓ 1 Trivial row e_j → $\det X' = 0$

2 No trivial rows

✓ 1) Just one $\neq 0$ ~~x~~ is some row

2) Each row has at least 2 nonzero

X' has at least $2n$ nonzero entries $\Rightarrow X'$ has exactly $2n$ nonzero entries G

G'

$$\left[\quad X' \quad \right]$$

QED