1

Matroids and Rigid Structures

WALTER WHITELEY

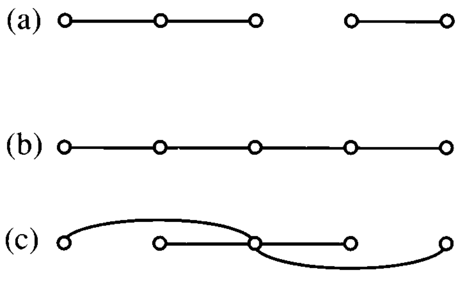
Many engineering problems lead to a system of linear equations – a represented matroid – whose rank controls critical qualitative features of the example (Sugihara, 1984; 1985; White & Whiteley, 1983). We will outline a selection of such matroids, drawn from recent work on the rigidity of spatial structures, reconstruction of polyhedral pictures, and related geometric problems.

For these situations, the combinatorial pattern of the example determines a sparse matrix pattern that has both a generic rank, for general 'independent' values of the non-zero entries, and a geometric rank, for special values for the coordinates of the points, lines, and planes of the corresponding geometric model. Increasingly, the generic rank of these examples has been studied by matroid theoretic techniques. These geometric models provide nice illustrations and applications of techniques such as matroid union, truncation, and semimodular functions. The basic unsolved problems in these examples highlight certain unsolved problems in matroid theory. Their study should also lead to new results in matroid theory.

1.1. Bar Frameworks on the Line – the Graphic Matroid

We begin with the simplest example, which will introduce the vocabulary and the basic pattern. We place a series of distinct points on a line, and specify certain *bars* – pairs of joints which are to maintain their distance – defining a *bar framework on the line.* We ask whether the entire framework is 'rigid' – i.e. does any motion of the joints along the line, preserving these distances, give all joints the same velocity, acceleration, etc.? Clearly a framework has an *underlying graph* $G=(V, E)$, with a vertex v_i for each joint p_i and an undirected edge $\{i, j\}$ for each bar $\{p_i, p_j\}$. In fact, we describe the framework as $G(\mathbf{p})$, where G is a graph without multiple edges or loops, and $\mathbf{p}$ is an assignment of points p_i to the vertices v_i. If this graph is not connected,

Figure 1.1.



then each component can move separately in the framework, and the framework is not rigid (Figure 1a). Conversely, a connected graph always leads to a rigid framework (Figure 1.1b), since each bar ensures that its two joints have the same motion on the line. This gives an informal proof of the following result.

1.1.1. Proposition. *A bar framework $G(\mathbf{p})$ on the line is rigid if and only if the underlying graph G is connected.*

To extract a matrix, we make this argument a little more formal. Assume the joints p_i move along smooth paths $p_i(t)$. The length of a bar $\|p_i(t)-p_j(t)\|$, and its square, remain constant. If we differentiate, this condition becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}[p_i(t)-p_j(t)]^2=[p_i(t)-p_j(t)][p_i'(t)-p_j'(t)]=0.$$

At $t=0$, this is written $(p_i-p_j)(p_i'-p_j')=0$. If we have distinct joints on the line, so that $(p_i-p_j)\neq 0$, this simplifies to $(p_i'-p_j')=0$.

With this in mind, we define an *infinitesimal motion* of a bar framework on the line $G(\mathbf{p})$ as an assignment of a velocity u_i along the line to each joint p_i such that $u_i-u_j=0$ for each bar $\{v_i, v_j\}$. For example, consider the framework in Figure 1.1c. The four bars lead to four equations in the unknowns $\mathbf{u}=(u_1, u_2, u_3, u_4, u_5)$:

$$\begin{bmatrix}1 & 0 & -1 & 0 & 0\\ 0 & 1 & -1 & 0 & 0\\ 0 & 0 & 1 & -1 & 0\\ 0 & 0 & 1 & 0 & -1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\\ u_3\\ u_4\\ u_5\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\\ 0\end{bmatrix}.$$

In general, this system of linear equations is written $R(G, \mathbf{p})\times\mathbf{u}^t=0$, where the *rigidity matrix* $R(G, \mathbf{p})$ has a row for each edge of the graph and a column for each vertex, and $\mathbf{u}^t$ is the transpose of the vector of velocities. We note that $R(G, \mathbf{p})$ is the transpose of the usual matrix representation for the graph over the reals: the rows are independent in $R(G, \mathbf{p})$ if and only if the corresponding edges are a forest (an independent set of edges in the cycle matroid of the graph).

A *trivial infinitesimal motion* is the derivative of a rigid motion of the line – i.e. a translation with all velocities equal. These form a one-dimensional subspace of the solutions. *An infinitesimally rigid framework on the line* has only these trivial infinitesimal motions, so the rigidity matrix has rank $|V| - 1$. This rank corresponds to a spanning tree on the vertices, or a basis for the cycle matroid of the complete graph on $|V|$ vertices. This proves the following infinitesimal version of Proposition 1.1.1.

1.1.2. Proposition. *A bar framework $G(\mathbf{p})$ on the line is infinitesmially rigid if and only if the underlying graph G is connected.*

1.2. Bar Frameworks in the Plane

A *bar framework in the plane* is a graph $G = (V, E)$ and an assignment $\mathbf{p}$ of points $\mathbf{p}_i \in \mathbb{R}^2$ to the vertices v_i such that $\mathbf{p}_i \neq \mathbf{p}_j$ if $\{i, j\} \in E$. If we differentiate the condition that bars have constant length in any smooth motion, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{p}_i(t) - \mathbf{p}_j(t)]^2 = [\mathbf{p}_i(t) - \mathbf{p}_j(t)]\cdot[\mathbf{p}_i'(t) - \mathbf{p}_j'(t)] = 0.$$

Accordingly, an *infinitesimal motion* of plane bar framework is an assignment $\mathbf{u}$ of velocities $\mathbf{u}_i \in \mathbb{R}^2$ to the joint such that

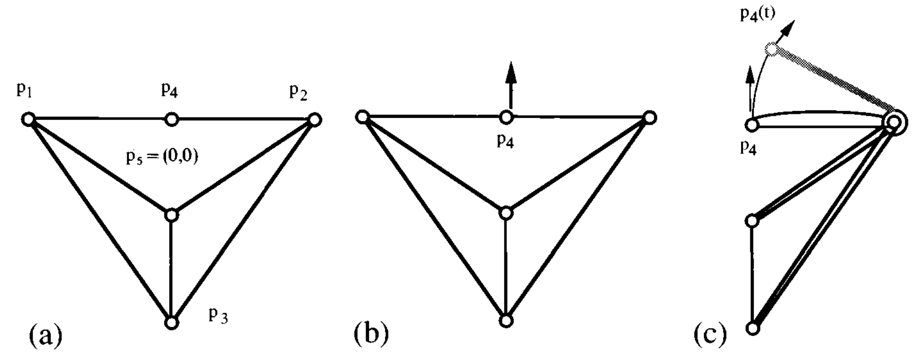
$$(\mathbf{p}_i - \mathbf{p}_j)\cdot(\mathbf{u}_i - \mathbf{u}_j) = 0 \quad \text{for each } \{i, j\} \in E.$$

A plane bar framework is *infinitesimally rigid* if all infinitesimal motions are *trivial*: $\mathbf{u}_i = \mathbf{s} + \beta(\mathbf{p}_i)^{\perp}$, where $\mathbf{s}$ is a fixed translation vector, $(\mathbf{x}, \mathbf{y})^{\perp} = (\mathbf{y}, -\mathbf{x})$ rotates the vector 90° counterclockwise, and $\beta(\mathbf{p}_i)^{\perp}$ represents a rotation about the origin. (These infinitesimal rotations and translations are the derivatives of smooth rigid motions of the plane.)

The system of equations for an infinitesimal motion has the form $R(G, \mathbf{p}) \times \mathbf{u}^t = 0$, where the *rigidity matrix* $R(G, \mathbf{p})$ now has a row for each edge of the graph and two columns for each vertex. The row for edge $\{i, j\}$ has the form

$$[0 \;\; 0 \;\; \ldots \;\; 0 \;\; 0 \;\; \mathbf{p}_i - \mathbf{p}_j \;\; 0 \;\; 0 \;\; \ldots \;\; 0 \;\; 0 \;\; \mathbf{p}_j - \mathbf{p}_i \;\; 0 \;\; 0 \;\; \ldots \;\; 0 \;\; 0]$$

Figure 1.2.



1.2.1. Example. Consider the frameworks in Figure 1.2. The framework of Figure 1.2a gives the rigidity matrix

$$
\begin{array}{c}
\{1,3\} \\ \{1,4\} \\ \{1,5\} \\ \{2,3\} \\ \{2,4\} \\ \{2,5\} \\ \{3,5\}
\end{array}
\begin{bmatrix}
x_1-x_3 & y_1-y_3 & 0 & 0 & x_3-x_1 & y_3-y_1 & 0 & 0 & 0 & 0 \\
\frac{1}{2}(x_1-x_2) & \frac{1}{2}(y_1-y_2) & 0 & 0 & 0 & 0 & \frac{1}{2}(x_2-x_1) & \frac{1}{2}(y_2-y_1) & 0 & 0 \\
x_1 & y_1 & 0 & 0 & 0 & 0 & 0 & 0 & -x_1 & -y_1 \\
0 & 0 & x_2-x_3 & y_2-y_3 & x_3-x_2 & x_3-y_2 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2}(x_2-x_1) & \frac{1}{2}(y_2-y_1) & 0 & 0 & \frac{1}{2}(x_1-x^2) & \frac{1}{2}(y_1-y^2) & 0 & 0 \\
0 & 0 & x_2 & y_2 & 0 & 0 & 0 & 0 & -x_2 & -y_2 \\
0 & 0 & 0 & 0 & x_3 & y_3 & 0 & 0 & -x_3 & -y_3
\end{bmatrix}
$$

The rows of this matrix are dependent and have rank 6. This leaves a $(10-6=4)$-dimensional space of infinitesimal motions, including the non-trivial motion shown in Figure 1.2b, which assigns zero velocity to all joints but $\mathbf{p}_4$, and gives $\mathbf{p}_4$ a velocity perpendicular to the bars at $\mathbf{p}_4$. Thus the framework is not infinitesimally rigid.

The infinitesimal motion is not the derivative of some smooth path for the vertices. The framework is *rigid* – all smooth paths, or even continuous paths, give frameworks congruent to the original framework. Figure 1.2c gives a similar framework which has the same infinitesimal motions, but is not rigid.

These examples show that there is a difference in the plane between rigid frameworks and infinitesimally rigid frameworks. A *non-rigid* plane framework will have an analytic path of positions $\mathbf{p}(t) = (\ldots, \mathbf{p}_i(t), \ldots)$, with all bar lengths of $\mathbf{p}(t)$ the same as bars in $\mathbf{p}(0)$, but $\mathbf{p}(t)$ not congruent to $\mathbf{p}(0)$, for all $0 < t < 1$ (Figure 1.2c). The first non-zero derivative of this path will be a non-trivial infinitesemal motion. However, the converse is false: many infinitesimal motions are not the derivative of an analytic path (recall Figure 1.2b). For any framework, the independence of the rows of the rigidity matrix induces a matroid on the edges of the graph. If 'rigidity' in a particular plane framework were used to define an independence structure on the edges of a graph, this need not be a matroid (see Exercise 1.6). Therefore, we will restrict ourselves, throughout this chapter, to the simpler concepts of infinitesimal motions and infinitesimal rigidity.

The space of trivial plane infinitesimal motions has dimension 3, for frameworks with at least two distinct joints. This space can be generated by two translations in distinct directions and a rotation about any fixed point. Thus an infinitesimally rigid framework with more than two joints will have an $|E|$ by $2|V|$ rigidity matrix of rank $2|V|-3$. Our basic problem is to determine which graphs G allow this matrix to have rank $2|V|-3$ for at least some plane frameworks $G(\mathbf{p})$.

The independence structure of the rows of the rigidity matrix defines a matroid on the edges of the complete graph on the vertices. This matroid depends on the positions of the joints. If we vary the positions there are 'generic' positions that give a maximal collection of independent sets (for example, positions where the coordinates are algebraically independent real numbers). At these positions we have the *generic rigidity matroid for* $|V|$ *vertices in the plane.*

1.2.2. Example. Consider the framework in Figure 1.3a. With vertices as indicated we have the rigidity matrix

$$\begin{matrix} (a_1, b_1) \\ (a_1, b_2) \\ (a_1, b_3) \\ (a_2, b_1) \\ (a_2, b_2) \\ (a_2, b_3) \\ (a_2, b_1) \\ (a_3, b_2) \\ (a_3, b_3) \end{matrix}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 & -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

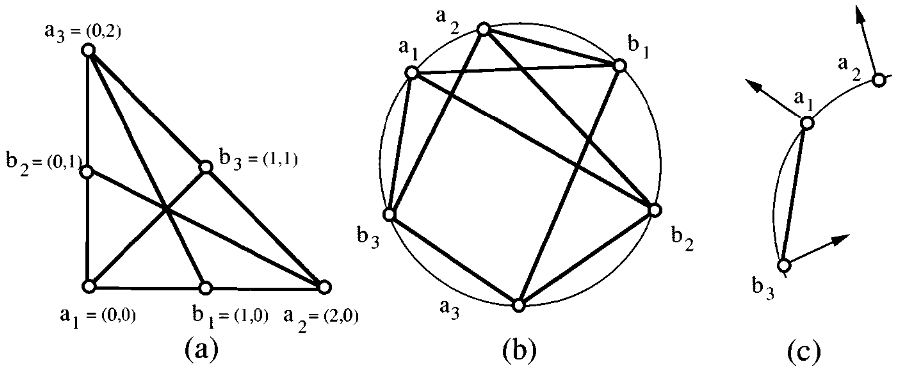
The graph of the framework has $|E|=2|V|-3$, so the framework is infinitesimally rigid if and only if the rows are independent. This independence can be checked by deleting the final three columns and seeing that the determinant of the 9×9 submatrix is non-zero. This framework is infinitesimally rigid and the graph is generically rigid, and generically independent.

Consider any realization with distinct joints $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ on a unit circle centred at the origin (Figure 1.3b). This has a non-trivial 'in–out' infinitesimal motion (Figure 1.3c):

for joints $\mathbf{a}_i$ take the velocity $\mathbf{a}_i' = \mathbf{a}_i$;

for joints $\mathbf{b}_j$ take the velocity $\mathbf{b}_j' = -\mathbf{b}_j$.

Figure 1.3.



These velocities preserve the length of all bars $(\mathbf{a}_i, \mathbf{b}_j)$, since

$$(\mathbf{a}_i - \mathbf{b}_j)\cdot(\mathbf{a}_i' - \mathbf{b}_j') = (\mathbf{a}_i - \mathbf{b}_j)\cdot(\mathbf{a}_i + \mathbf{b}_j) = (\mathbf{a}_i)\cdot(\mathbf{a}_i) - (\mathbf{b}_j)\cdot(\mathbf{b}_j) = 1 - 1 = 0.$$

This infinitesimal motion is non-trivial. Letting $\theta \neq 0$ be the angle between the unit vectors $\mathbf{a}_1$ and $\mathbf{a}_2$, we show that the distance $(\mathbf{a}_1 - \mathbf{a}_2)$ is changing instantaneously:

$$(\mathbf{a}_1 - \mathbf{a}_2)\cdot(\mathbf{a}_1' - \mathbf{a}_2') = (\mathbf{a}_1)\cdot(\mathbf{a}_1) - 2(\mathbf{a}_1)\cdot(\mathbf{a}_2) + (\mathbf{a}_2)\cdot(\mathbf{a}_2) = 1 + 1 - 2\cos\theta > 0.$$

Thus this special position is not generic (see Exercise 1.9).

We want to characterize the graphs of *isostatic plane frameworks* – minimal infinitesimally rigid frameworks in the sense that removing any one bar introduces a non-trivial infinitesimal motion. These graphs, of size $|E| = 2|V| - 3$, are the bases of the generic rigidity matroid 'of the complete graph' on the set of vertices.

Thus an isostatic framework corresponds to a row basis for the rigidity matrix of any infinitesimally rigid framework extending the framework. The independence of such a set of edges is determined by maximal minors of the rigidity matrix. This independence is *generic* in the sense that these minors are non-zero polynomials in the positions $\mathbf{p}_i$. If such a polynomial is non-zero for some position $G(\mathbf{p})$, then almost all $\mathbf{q} \in \mathbb{R}^{2|V|}$ give isostatic frameworks $G(\mathbf{q})$ (see Section 1.7).

More surprisingly, for points where this matrix and all its minors have the maximal rank achieved for $\mathbf{q} \in \mathbb{R}^{2|V|}$, infinitesimal rigidity and any reasonable form of local rigidity actually coincide (see, for example, Exercise 1.7).

We note that throughout this chapter the generic matroids defined on the complete graph of $|V|$ vertices are symmetric on the vertices – any permutation of the vertices does not change the independence of a set of edges. As a convention, we write the attached vertices for a subset of edges E' as V'.

1.2.3. Theorem. *For a graph G, with at least two vertices, the following are equivalent conditions:*

(i) *G has some positions $G(\mathbf{p})$ as an isostatic plane framework;*

(ii) *$|E| = 2|V| - 3$ and for all proper subsets of edges E' incident with vertices V', $|E'| \leqslant 2|V'| - 3$;*

(iii) *adding any edge to E (including doubling an edge) gives an edge set covered by two edge-disjoint spanning trees.*

Proof. (i) $\Rightarrow$ (ii): For an isostatic plane framework $G(\mathbf{p})$ on at least two vertices, the rows of the rigidity matrix have rank $|E| = 2|V| - 3$. If any proper subset of edges has $|E'| > 2|V'| - 3$, the corresponding rows are dependent. Since $G(\mathbf{p})$ is independent, we conclude that $|E'| \leqslant 2|V'| - 3$ for all proper subsets.

(ii) $\Leftrightarrow$ (iii): The count $f(E') = 2|V'| - 3$ defines a non-decreasing semimodular function on sets of edges, which is non-negative on non-empty sets (see Exercise 1.1). This semimodular function defines a matroid by the standard property:

E is independent if and only if $|E'| \leqslant f(E')$ for all proper subsets E'. (1.1)

This count has the form $f(E') = (2|V'| - 2) - 1$ which shows that the matroid for f is a Dilworth truncation of the matroid defined by the semimodular function $g(E') = 2(|V'| - 1)$. In turn, the semimodular function g represents a matroid union of two copies of the matroid given by the semimodular function $h(E') = |V| - 1$ (the cycle matroid of the graph). Thus a graph is independent in the matroid of f if and only if adding any edge (including doubling an edge) gives a graph covered by two edge-disjoint forests.

Before we prove (iii) $\Rightarrow$ (i), we need a lemma about a simpler matrix that has rank $2|V| - 2$ (matching the function g). For a graph $G = (V, E)$, including possible multiple edges, a *2-frame* $G(\mathbf{d})$ is an assignment of directions $\mathbf{d}_e \in \mathbb{R}^2$ to the edges. An *infinitesimal motion of the 2-frame* $G(\mathbf{d})$ is an assignment of velocities $\mathbf{u}_i \in \mathbb{R}^2$ to the vertices such that

$$\mathbf{d}_e \cdot (\mathbf{u}_i - \mathbf{u}_j) = 0 \quad \text{for every edge } e \text{ joining } v_i \text{ and } v_j \ (i < j).$$

This system of equations defines the *rigidity matrix* $R(G, \mathbf{d})$ for the 2-frame.

1.2.4. Lemma. *The rows of the rigidity matrix of a generic 2-frame $G(\mathbf{d})$ are independent if and only if G is the union of two edge-disjoint forests.*

Proof. Take the two forests F_1 and F_2. For all edges in the first forest, we assign the direction (1, 0). For all edges in the second forest, we assign the direction (0, 1). If we reorder the rows and columns of this rigidity matrix, placing all second columns of vertices to the right, and all rows for the second forest at the bottom, we have a pattern:

$$\begin{bmatrix} [F_1] & [0] \\ [0] & [F_2] \end{bmatrix}$$

where $[F_1]$ and $[F_2]$ are the standard matrices representing the two forests over the reals. Thus the blocks $[F_1]$ and $[F_2]$ have non-zero minors on all their rows, and the entire matrix also has a non-zero minor on all the rows. We conclude that the rows are independent.

Conversely, if the rows are independent, we can reorder the columns as above. The independence guarantees a non-zero minor on all the rows. Using a Laplace expansion on the two blocks, there are non-zero minors on complementary sets of rows. Each of these non-zero minors on the matrix representing the graphic matroid must correspond to a forest, as required. □

Proof of Theorem 1.2.3 (continued). (iii)⇒(i) Assume that adding any edge to E (including doubling an edge) gives an edge set covered by two edge-disjoint forests. We show that this gives independent rows in the rigidity matrix for some (almost all) choices of the points.

Take a 2-frame $G(\mathbf{d})$ with algebraically independent directions for the edges. By our assumption, adding any edge (or doubling any edge) between any pair of vertices gives an independent 2-frame E^*. Therefore, the 2-frame on E has an infinitesimal motion $\mathbf{u}_{ij}$ that has different velocities on the two vertices of the added edge. Taking linear combinations of these $\mathbf{u}_{ij}$ there is an infinitisemal motion $\mathbf{u}$ that assigns distinct velocities $\mathbf{u}_i = (s_i, t_i)$ to each of the vertices of G.

To create the independent framework $G(\mathbf{p})$, we set $\mathbf{p}_i = (-t_i, s_i)$. Since $\mathbf{u}_i \neq \mathbf{u}_j$, we have $\mathbf{p}_i \neq \mathbf{p}_j$ for each edge $\{i, j\}$, as required in a framework. We claim that the rigidity matrix of $G(\mathbf{p})$ has rows parallel to the rows of the original 2-frame $G(\mathbf{d})$. Clearly

$$(\mathbf{p}_i - \mathbf{p}_j)\cdot(\mathbf{u}_i - \mathbf{u}_j) = (-t_i + t_j, s_i - s_j)\cdot(s_i - s_j, t_i - t_j) = 0 = \mathbf{d}_e\cdot(\mathbf{u}_i - \mathbf{u}_j).$$

Since $\mathbf{u}_i - \mathbf{u}_j \neq \mathbf{0}$, this means $(\mathbf{p}_i - \mathbf{p}_j) = \beta_e \mathbf{d}_e$ for some non-zero scalar β_e. The rigidity matrix of the framework is equivalent to the rigidity matrix of the independent 2-frame. We conclude that a set of edges satisfying condition (1.1) has been realized as an independent (therefore isostatic) bar framework. (The infinitesimal motion $\mathbf{u}$ of the 2-frame is a rotation of this framework around the origin.)

This completes the proof. □

Figure 1.4 shows some other examples of the graphs of isostatic frameworks in the plane (Figure 1.4a) and graphs of circuits in the plane generic rigidity matroid (Figure 1.4b) with $|E| = 2|V| - 2$, and $|E'| \leqslant 2|V'| - 3$ for proper subsets.

The semimodular count of Theorem 1.2.3(ii) converts to a criterion for graphs of infinitesimally rigid frameworks. We state the theorem without proof.

1.2.5. Corollary. (*Lovász & Yemini, 1982*) *A graph has realizations as an infinitesimally rigid plane framework if and only if for every partition of the* *edges into non-empty subsets* $(\ldots E^j \ldots)$, *with vertices* V^j *incident with the edges* E^j,

$$\sum_j (2|V^j| - 3) \geqslant 2|V| - 3.$$

Figure 1.5 gives a simple example of a 5-connected graph, in a vertex sense, which is never infinitesimally rigid by this criterion: take the eight K_5 graphs as sets of the partition, and all other edges as singletons and apply Corollary 1.2.5. However, every graph which is 6-connected in a vertex sense is generically rigid (Exercise 1.16).

The dependence of rows in the rigidity matrix, or *dependence of bars in the framework*, also has a physical interpretation. A *self-stress* on a framework is an assignment of scalars ω_{ij} to the bars $\{\mathbf{p}_i, \mathbf{p}_j\}$ such that for each joint $\mathbf{p}_i$ there is an equilibrium (Figures 1.6a, b):

Figure 1.4. (a) $|E| = 2|V| - 3$, $|E'| \leqslant 2|V'| - 3$; (b) $|E| = 2|V| - 2$, $|E'| \leqslant 2|V'| - 3$.

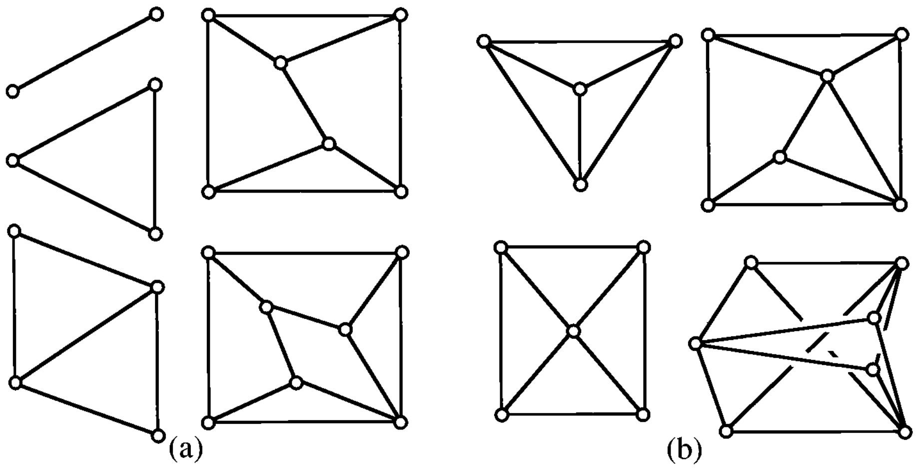


Figure 1.5.

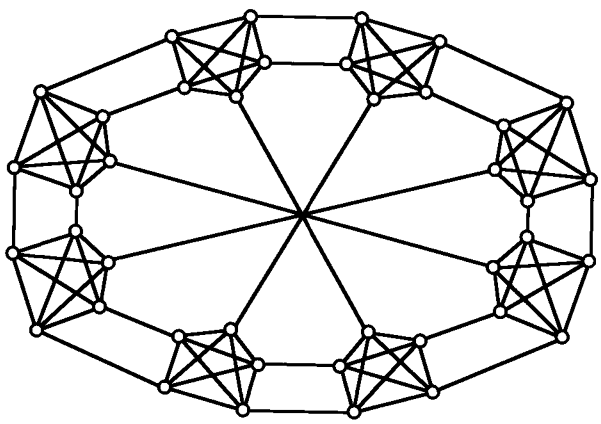
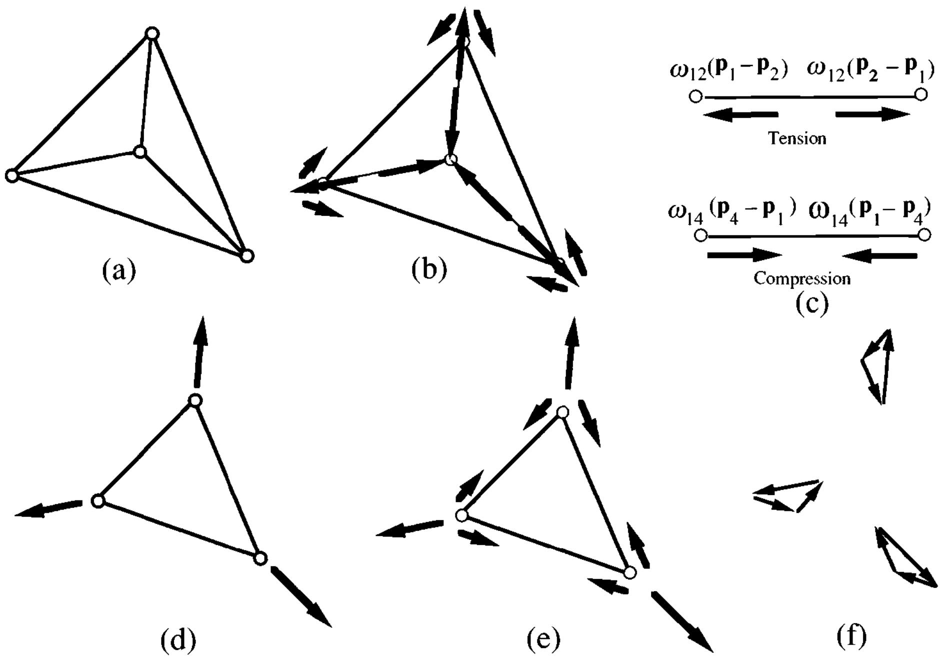


Figure 1.6.



$$\sum_j \omega_{ij}(\mathbf{p}_j - \mathbf{p}_i) = \mathbf{0} \quad \text{(sum over all edges } \{i, j\} \text{ incident with } \mathbf{p}_i\text{).}$$

Thus a self-stress is equivalent to a row dependence. If $\omega_{ij} < 0$, we interpret this as a *compression* in the bar – a force $\omega_{ij}(\mathbf{p}_j - \mathbf{p}_i)$ at $\mathbf{p}_i$, and a force $\omega_{ij}(\mathbf{p}_i - \mathbf{p}_j)$ at $\mathbf{p}_j$. If $\omega_{ij} > 0$, this is a *tension* in the bar (Figure 1.6c).

In the same spirit, the row space of the rigidity matrix is interpreted as the space of loads $\mathbf{L}_i$ *resolved* by forces in the bars of the framework (Figure 1.6d, e, f):

$$\mathbf{L}_i + \sum_j \omega_{ij}(\mathbf{p}_j - \mathbf{p}_i) = \mathbf{0} \quad \text{(sum over all edges } \{i, j\} \text{ incident with } \mathbf{p}_i\text{).}$$

These resolved loads satisfy an additional property of global static equilibrium, defined below (see Exercise 1.17). Thus an *equilibrium load* is an assignment $\mathbf{L}_i$ of vectors to the vertices that satisfies the three equilibrium equations

$$\sum_i \mathbf{L}_i = \mathbf{0} \quad \text{and} \quad \sum_i \mathbf{L}_i \times \mathbf{p}_i = \mathbf{0}$$

where $\times$ represents a cross product in 3-space. A framework is *statically rigid* if all equilibrium loads on its joints are resolved.

We note that a single point is trivially both infinitesimally rigid and statically rigid in the plane. A single bar has only a one-dimensional space of equilibrium loads: $\alpha(\mathbf{p}_1 - \mathbf{p}_2)$ at $\mathbf{p}_1$ and $\alpha(\mathbf{p}_2 - \mathbf{p}_1)$ at $\mathbf{p}_2$. Since these are resolved, the bar is statically rigid and infinitesimally rigid. Moreover, the equilibrium loads on any framework with at least two distinct joints form a space of dimension $2|V|-3$, and all resolved loads are equilibrium loads (Exercise 1.17). These observations prove the following.

1.2.6. Theorem. *For a bar framework $G(\mathbf{p})$ in the plane, on at least two joints, the following are equivalent conditions*:

(i) *the framework is statically rigid*;
(ii) *the rigidity matrix has rank* $2|V|-3$;
(iii) *the framework is infinitesimally rigid.*

Such dual static concepts arise for the infinitesimal mechanics of most types of structure, giving dual static and infinitesimal mechanical theories for the row and column ranks of the corresponding matrix. Of course the non-zero self-stresses correspond to the dependences of the underlying matroid on the edges.

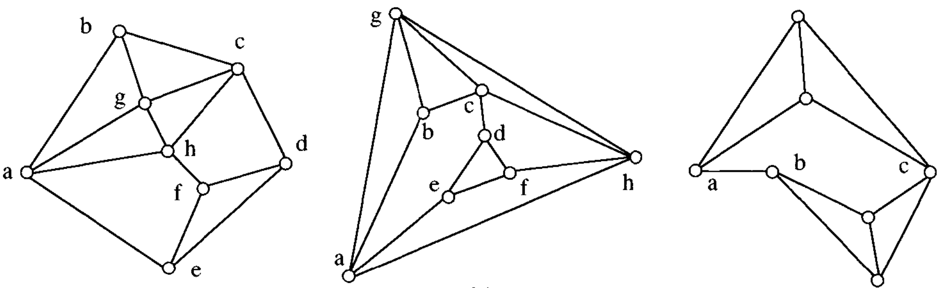
1.3. Plane Stresses and Projected Polyhedra

There is a classical theorem that interprets self-stresses in a framework with a planar graph as instructions for building a 'spatial polyhedron' with this projection. The ideas of the proof are simple, but we need a few topological definitions for our 'spherical polyhedra'.

We begin with a simple class of graphs, for which we can associate the face, edge, and vertex structures of abstract polyhedra. A *planar graph* is a graph G that can be drawn in the plane with the edges as straight lines, disjoint except at shared vertices. A graph is *vertex 3-connected* (or 3-connected for short) if deleting any two vertices leaves the graph connected. A planar drawing of a 3-connected planar graph $G=(V;E)$ divides up the plane into a set of polygonal discs, and a single infinite region with a polygonal hole. These polygonal regions oriented counterclockwise, and the infinite region, with its polygonal hole oriented counterclockwise, form the *oriented faces*, $(\ldots, f^h, \ldots)$, of the associated polyhedron. (Note that a theorem of graph theory says that, up to reversing all these orientations, the face structure for a 3-connected planar graph is unique.) We call this collection of vertices, edges, and faces a *3-connected abstract spherical polyhedron* $S=(V, E, F)$.

1.3.1. Example. Figures 1.7a and 1.7b show two planar drawings of the same 3-connected graph. The reader can check that they define the same oriented faces. Figure 1.7c shows a planar graph that is not 3-connected: removing vertices a and c, or b and c, separates the graph. The results of this section actually extend to these 2-connected planar graphs – see Exercise 1.18.

Figure 1.7.



(a) (b) (c)

We are interested in projections of spatial polehydra. We assume that we are projecting vertically down into the xy-plane, and that no face is a vertical plane. Thus the face planes can be written $Ax + By + C = z$, or (A, B, C) for short.

A *spatial polyhedron* $S(\mathbf{Q}, \mathbf{q})$ is an assignment of a plane $\mathbf{Q}^h = (A^h, B^h, C^h)$ to each face f^h, and points $\mathbf{q}_j = (x_j, y_j, z_j)$ to each vertex v_j, such that for each vertex v_k on a face f^i

$$A^i x_k + B^i y_k + C^i = z_k.$$

For convenience, we also assume that for any edge $\{v_j, v_k\}$ of the polyhedron, $\mathbf{q}_j \neq \mathbf{q}_k$. We do not assume that the two faces at an edge have to have distinct planes, nor that the faces have to be topological discs, convex, etc. in space.

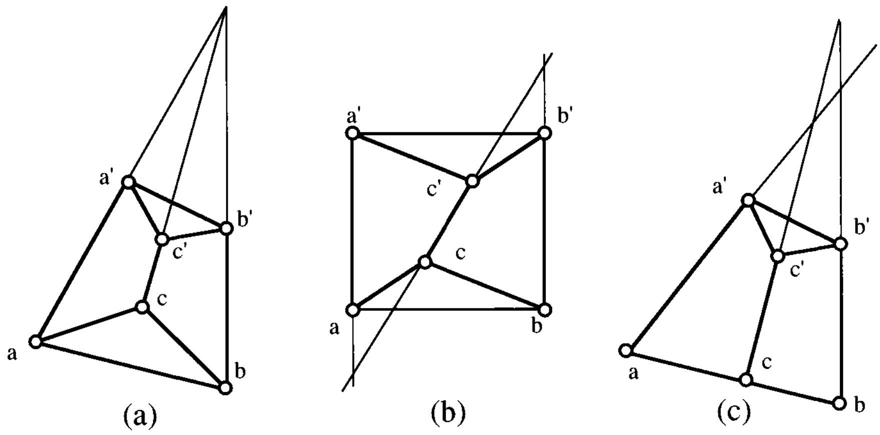
A spatial polyhedron $S(\mathbf{Q}, \mathbf{q})$ *projects* to a plane framework $G(\mathbf{p})$ if G is the graph of edges and vertices of S, and if for each vertex with $\mathbf{q}_j = (x_j, y_j, z_j)$, $\mathbf{p}_j = (x_j, y_j)$.

1.3.2. Example. What geometric condition(s) on the drawings in Figure 1.8 correspond to a projected polyhedron? If a spatial polyhedron has three distinct planes for the faces $aa'b'b$, $bb'c'c$, and $cc'a'a$, these planes intersect in a single point, even in the projection. Therefore the three edges aa', bb', and cc' must have a common point. The reader can mentally reconstruct the triangular prism over Figure 1.8a. Figure 1.8b is a generic drawing, and the only spatial polyhedra are trivial, with all faces in the same plane.

Figure 1.8c corresponds to a 'degenerate' spatial polyhedron, which assigns one plane to face abc, and another plane to all other faces. This object satisfies our definition and is non-trivial.

Consider a plane framework on the graph of the polyhedron $G(\mathbf{p})$, with distinct vertices for each edge. A polyhedron in space that projects to this framework assigns a plane $z = A^h x + B^h y + C^h$ to each face f^h. If two faces f^h, f^i share an edge over the line through $\{\mathbf{p}_j, \mathbf{p}_k\}$, we have the conditions

Figure 1.8.



$$A^h x_j + B^h y_j + C^h = A^i x_j + B^i y_j + C^i$$

and

$$A^h x_k + B^h y_k + C^h = A^j x_k + B^i y_k + C^i$$

or equivalently, by Cramer's rule,

$$(A^h, B^h, C^h) - (A^i, B^i, C^i) = \omega_{jk}((y_j - y_k), (x_k - x_j), (x_j y_k - x_k y_j))$$

for some scalar ω_{jk}. We use the orientation of the polyhedron to associate an order of the faces (h, i) with the order of the vertices (j, k), following the order of vertices in the cycle of face h. This orientation also associates (i, h) with (k, j). This generates a consistent scalar $\omega_{jk} = \omega_{kj}$ for each edge. The two planes are distinct if and only if $\omega_{jk} \neq 0$.

If we follow a cycle of faces and edges around a vertex v_j of the spatial polyhedron, the first face equals the last face. This gives the equation

$$\sum \omega_{jk}((y_j - y_k), (x_k - x_j), (x_j y_k - x_k y_j)) = (0, 0, 0) \quad \text{(sum over the cycle of edges)}.$$

These scalars look suspiciously like a self-stress on the framework $G(\mathbf{p})$.

1.3.3. Maxwell's theorem. *Given a spatial spherical polyhedron $S(\mathbf{Q}, \mathbf{q})$ projecting to the plane framework $G(\mathbf{p})$, there is a self-stress of the framework $G(\mathbf{p})$ such that an edge separates faces in distinct planes in the polyhedron if and only if the self-stress has a non-zero scalar on the edge.*

Proof. We have seen that a spatial polyhedron leads to scalars ω_{jk} satisfying

$$\sum \omega_{jk}(y_j - y_k, x_k - x_j, x_j y_k - x_k y_j) = (0, 0, 0) \quad \text{(sum over the edges at } v_j).$$

Clearly this implies that

$$\sum \omega_{jk}(x_k - x_j, y_k - y_j) = (0, 0) \quad \text{(sum over the edges at } v_j).$$

Therefore the projection of a polyhedron gives a self-stress that is non-zero on exactly the edges separating distinct planes. □

1.3.2. Example (continued). By counting vertices and edges, it is found that the graph of Figure 1.8 is generically independent – hence a generic framework (Figure 1.8b) is not the projection of a spatial polyhedron. If the three edges *aa'*, *bb'*, and *cc'* have a common point, Figure 1.8a, the spatial triangular prism proves that the framework has a self-stress that is non-zero on all edges.

Figure 1.8c has a self-stress, non-zero only on edges of the collinear triangle *abc*. This corresponds to the degenerate spatial polyhedron that assigns one plane to face *abc* and another plane to all other faces.

We have a converse to Maxwell's theorem. The next lemma presents a crucial property of all self-stresses, which we need in the proof of this converse.

1.3.4. Cut lemma. *Given a framework* $G(\mathbf{p})$ *with a self-stress* ω, *and a set of bars* $(\mathbf{a}_i, \mathbf{b}_i)$ *such that they separate the graph, with all* $\mathbf{a}_i$ *in one component, and all* $\mathbf{b}_i$ *in other components, then*

$$\sum \omega_i(\mathbf{b}_i - \mathbf{a}_i) = \mathbf{0} \quad \textit{and} \quad \sum \omega_i(\mathbf{b}_i \times \mathbf{a}_i) = \mathbf{0}.$$

Proof. We have used the shorthand $(\mathbf{b}_i \times \mathbf{a}_i)$ for the third component of $(\mathbf{b}_i, 0) \times (\mathbf{a}_i, 0)$. For a cut set that isolates a single vertex v_j, we have

$$\begin{aligned}(0, 0, 0) \times (x_j, y_j, 0) &= \sum_k \omega_{jk}(x_k - x_j, y_k - y_j, 0) \times (x_j, y_j, 0) \\ &= \sum_k \omega_{jk}(0, 0, x_j y_k - x_k y_j).\end{aligned}$$

as required. Therefore:

$$\sum_k \omega_{jk}(y_j - y_k, x_k - x_j, x_j y_k - x_k y_j) = (0, 0, 0) \quad \text{(sum over the edges at } v_j).$$

Take the component created by the cut set. If we add these equations over all vertices of the component, the terms cancel on all edges joining two ends in the component containing all the $\mathbf{a}_i$, and leave the required sums on the cut set $(\mathbf{a}_i, \mathbf{b}_i)$. □

1.3.5. Converse of Maxwell's theorem. *Given a self-stress on the plane framework* $G(\mathbf{p})$ *built on the graph of a spherical polyhedron, there is a spatial polyhedron projecting to* $G(\mathbf{p})$ *such that an edge separates faces in distinct planes in the polyhedron if and only if the corresponding self-stress is non-zero on the edge.*

Proof. Assume that we have a self-stress $(\ldots, \omega_{jk}, \ldots)$ on the framework $G(\mathbf{p})$. We choose the arbitrary plane $z = A^0x + B^0y + C^0$ for an initial face f^0. For every other face f^n we take a face–edge path Π from f^0 to f^n and define the plane for f^n by

$$(A^n, B^n, C^n) = (A^0, B^0, C^0) + \sum_{\Pi} \omega_{jk}(y_j - y_k,\ x_k - x_j,\ x_j y_k - x_k y_j)$$

We must prove that this answer is well defined, that is, independent of the path used.

Consider two different paths Π and Π' from f^0 to f^n. Together, the path Π, $-\Pi'$ forms a face–edge cycle on the polyhedron (i.e. a cycle on the dual graph). Such a cycle is the disjoint union of simple cycles, and each simple cycle on a spherical polyhedron is a cut set satisfying Lemma 1.3.4. Summing over all these oriented cycles simultaneously:

$$\sum_{\Pi} \omega_{jk}(x_j - x_k,\ y_j - y_k,\ x_j y_k - x_k y_j)$$
$$- \sum_{\Pi'} \omega_{jk}(x_j - x_k.\ y_j - y_k,\ x_j y_k - x_k y_i) = (0, 0, 0).$$

It is a simple exercise to rewrite the order and the signs of coordinates in this sum, giving:

$$\sum_{\Pi} \omega_{jk}(y_j - y_k,\ x_k - x_j,\ x_j y_k - x_k y_j) = \sum_{\Pi'} \omega_{jk}(y_j - y_k,\ x_k - x_j,\ x_j y_k - x_k y_j).$$

This proves that the plane for face f^n is well defined and that these spatial planes have the required incidences in space over the vertices to form a spatial polyhedron projecting to $G(\mathbf{p})$.

For any edge $-jk$ of the polyhedron, we still have

$$(A^h, B^h, C^h) - (A^i, B^i, C^i) = \omega_{jk}(y_j - y_k,\ x_k - x_j,\ x_j y_k - x_k y_j).$$

Therefore the edge separates two distinct planes if and only if there is a non-zero scalar in the self-stress. □

1.3.2. Example (completed). Consider the frameworks in Figure 1.8. We saw that Figure 1.8a was the projection of a polyhedron with distinct faces – hence it has a self-stress. Figure 1.8b is not the projection of a polyhedron – so it must be independent, by Theorem 1.3.5.

The self-stresses on a framework $G(\mathbf{p})$ actually form a vector space. The polyhedra that project to this framework, with non-vertical faces and with a face f^0 held in a fixed plane $z = 0$, also form a vector space. (Note that we include polyhedra with all faces coplanar, corresponding to the zero self-stress.) Theorems 1.3.3 and 1.3.5 actually give an isomorphism of these spaces.

1.3.6. Corollary. *The space of spatial polyhedra over a plane drawing of the graph $G(\mathbf{p})$, with the plane $z = 0$ assigned to a fixed face f^0, is isomorphic to the vector space of self-stresses of $G(\mathbf{p})$.*

1.4. Bar Frameworks in 3-space

The definitions of a framework, the rigidity matrix, and infinitesimal rigidity in the plane easily generalize to bar frameworks in $\mathbb{R}^3$. Trivial infinitesimal motions now have the form $\mathbf{u}_i = \mathbf{s} + \mathbf{r} \times \mathbf{p}_i$. These infinitesimal motions form a space of dimension 6 for any framework whose joints are not all on a line, a space generated by three non-coplanar translations and by three non-coplanar rotations about any point. Thus a spatial framework with more than two joints is infinitesimally rigid if and only if the rigidity matrix has rank $3|V| - 6$.

For statics in 3-space we have the similar modification that equilibrium loads satisfy the six equations $\sum_i \mathbf{L}_i = \mathbf{0}$ and $\sum_i \mathbf{L}_i \times \mathbf{p}_i = \mathbf{0}$. These equations are independent provided that the joints are not all collinear. The row space of the rigidity matrix still corresponds to the resolved equilibrium loads.

We also observe that a framework on the line with more than two joints cannot be statically or infinitesimally rigid even in the plane. These observations lead directly to

1.4.1. Theorem. *For bar framework $G(\mathbf{p})$ in space, with at least three joints, the following are equivalent conditions:*

(i) *$G(\mathbf{p})$ is infinitesimally rigid;*
(ii) *the rigidity matrix $RG(\mathbf{p})$ has rank $3|V| = 6$;*
(iii) *$G(\mathbf{p})$ is statically rigid.*

1.4.2. Corollary. *If a framework in space with at least three joints is isostatic then $|E| = 3|V| - 6$ and $|E'| \leqslant 3|V'| - 6$ for all subsets of at least two edges.*

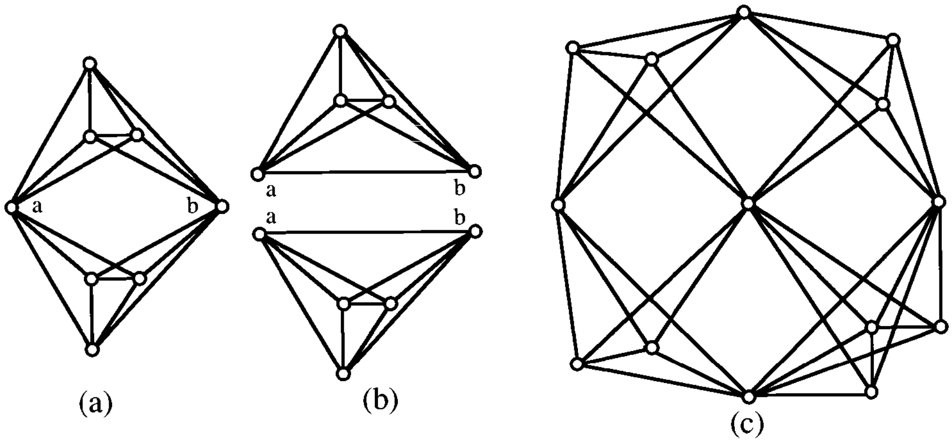
When we seek the converse to this corollary, we run into a fundamental problem: some graphs that satisfy the count are never realized as isostatic frameworks. Figure 1.9a gives a simple example, which clearly has a non-trivial motion based on rotations around the 'hinge' *ab*. If we take two copies of K_5, which are generic circuits in space (Figure 1.9b), and do a circuit exchange on the common bar *ab* we obtain this generic circuit. Tay (1986) surveys other generic circuits in 3-space that are not infinitesimally rigid (Figure 1.9c, Exercise 1.22).

An alternative vision of our difficulty comes from the semimodular function $f(E) = 3|V| - 6$ on sets of edges. Since $f(E)$ is negative on single edges, this function does not directly define a matroid. Instead we ask the more subtle question:

> Is there a maximal matroid $M(G)$ on graphs that has rank $3|V| - 6$ on all complete graphs of at least three vertices?

Maximal means that if E'' is independent in some matroid on graphs satisfying this count, then E'' is independent in M.

Figure 1.9.



1.4.3. Graver's conjecture. *The rigidity matroid in 3-space is maximal: any independent set in another matroid on finite graphs with rank* $3|V|-6$ *on each complete graph of* $|V|>2$ *vertices will be independent in the rigidity matroid.*

This question can be approached through a set of inductive constructions building up the number of vertices in an isostatic set. The graph of an isostatic framework in 3-space that is larger than a triangle has a vertex of valence 3, 4, or 5. The edge set satisfies $|E|=3|V|-6$ and $|E'|\leqslant 3|V'|-6$ for all subsets of at least three vertices. Since each edge is incident with two vertices, we have $2|E|=\sum_i \mathrm{val}(v_i)<6|V|$. There must be vertices of valence less than 6. On the other hand, if we remove vertex v_i attached to edges E_i, leaving at least three vertices V', we have

$$|E'|\leqslant 3|V'|-6 \quad \text{and} \quad |E'\cup E_i|=|E'|+|E_i|=3|V|-6=3|V'|-6+3.$$

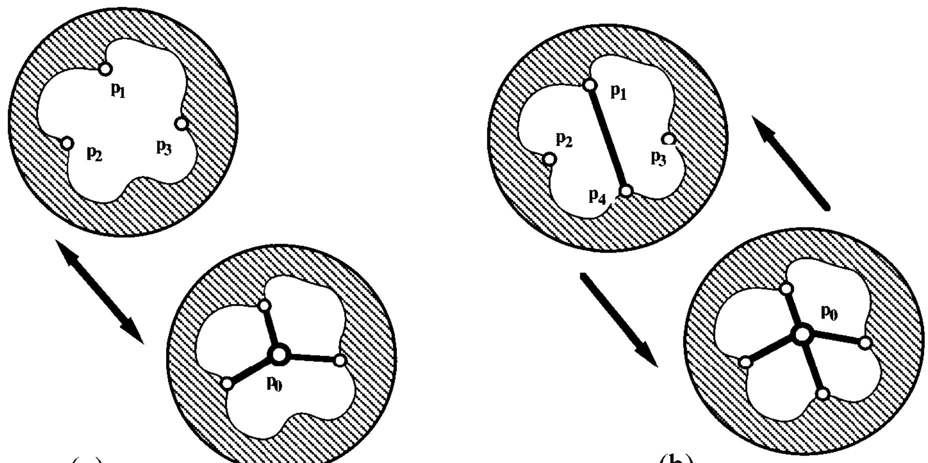
Thus, if the graph is bigger than a triangle, each vertex must have valence of at least 3.

Accordingly, the inductive techniques describe the addition and deletion of 3-, 4-, or 5-valent vertices. Unfortunately the techniques we outline remain incomplete for 5-valent vertices in 3-space.

1.4.4. Proposition. *If an isostatic bar framework in 3-space has three non-collinear joints* $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$, *then inserting a 3-valent joint* $\mathbf{p}_0$ *out of this plane with bars* $(\mathbf{p}_0, \mathbf{p}_1), (\mathbf{p}_0, \mathbf{p}_2), (\mathbf{p}_0, \mathbf{p}_3)$ *creates an isostatic bar framework.*

Conversely, given any isostatic framework in 3-space with a 3-valent joint, then deleting this joint with its bars leaves an isostatic bar framework (Figure 1.10a).

Figure 1.10.



(a) (b)

Proof. (1) For convenience, we assume that the added vertex v_0 is at the left side of the rigidity matrix, and the added bars are on the top:

$$\begin{bmatrix} \mathbf{p}_1-\mathbf{p}_0 & \mathbf{p}_0-\mathbf{p}_1 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{p}_2-\mathbf{p}_0 & \mathbf{0} & \mathbf{p}_0-\mathbf{p}_2 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{p}_3-\mathbf{p}_0 & \mathbf{0} & \mathbf{0} & \mathbf{p}_0-\mathbf{p}_3 & \cdots & \mathbf{0} \\ \mathbf{0} & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}.$$

It is clear that this vertex adds rank 3 to the lower rigidity matrix if and only if the three vectors $\mathbf{p}_1-\mathbf{p}_0$, $\mathbf{p}_2-\mathbf{p}_0$, and $\mathbf{p}_3-\mathbf{p}_0$ are independent, that is, the four points are not coplanar.

(2) Conversely, removing a 3-valent joint keeps the independence of the rows of the rigidity matrix and keeps $|E'|=|E|-3=3|V|-9=3|V'|-6$. □

1.4.5. Proposition. *If an isostatic bar framework in 3-space has four non-coplanar joints $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, $\mathbf{p}_4$ and a bar $(\mathbf{p}_1, \mathbf{p}_4)$ then removing this bar and inserting a joint $\mathbf{p}_0$ with bars $(\mathbf{p}_0, \mathbf{p}_1)$, $(\mathbf{p}_0, \mathbf{p}_2)$, $(\mathbf{p}_0, \mathbf{p}_3)$, $(\mathbf{p}_0, \mathbf{p}_4)$ creates an isostatic bar framework for almost all positions of $\mathbf{p}_0$.*

Conversely, given any isostatic bar framework in 3-space with a 4-valent joint $\mathbf{p}_0$ attached to four non-coplanar joints, then there is a bar $(\mathbf{p}_i, \mathbf{p}_j)$ among the vertices adjacent to $\mathbf{p}_0$ such that deleting this joint and inserting $(\mathbf{p}_i, \mathbf{p}_j)$ leaves an isostatic bar framework (Figure 1.10b).

Proof. (1) Again, we add the vertex v_0 at the left side of the rigidity matrix, with three bars to $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, so that $\mathbf{p}_1-\mathbf{p}_0$ is parallel to $\mathbf{p}_1-\mathbf{p}_4$:

$$\begin{bmatrix} \mathbf{p}_1-\mathbf{p}_0 & \mathbf{p}_0-\mathbf{p}_1 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{p}_2-\mathbf{p}_0 & \mathbf{0} & \mathbf{p}_0-\mathbf{p}_2 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{p}_3-\mathbf{p}_0 & \mathbf{0} & \mathbf{0} & \mathbf{p}_0-\mathbf{p}_3 & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{p}_4-\mathbf{p}_1 & \mathbf{0} & \mathbf{0} & \mathbf{p}_1-\mathbf{p}_4 & \mathbf{0} \\ \mathbf{0} & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}.$$

It is clear that this vertex adds rank 3 to the lower rigidity matrix if and only if the three vectors $\mathbf{p}_1-\mathbf{p}_0$, $\mathbf{p}_2-\mathbf{p}_0$, and $\mathbf{p}_3-\mathbf{p}_0$ are independent. However, we can now row reduce with row I and row IV to get the desired independent rigidity matrix:

$$\begin{bmatrix} \mathbf{p}_1-\mathbf{p}_0 & \mathbf{p}_0-\mathbf{p}_1 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{p}_2-\mathbf{p}_0 & \mathbf{0} & \mathbf{p}_0-\mathbf{p}_2 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{p}_3-\mathbf{p}_0 & \mathbf{0} & \mathbf{0} & \mathbf{p}_0-\mathbf{p}_3 & \cdots & \mathbf{0} \\ \mathbf{p}_4-\mathbf{p}_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{p}_0-\mathbf{p}_4 & \mathbf{0} \\ \mathbf{0} & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}.$$

Since we have $|E'| = |E| + 4 - 1 = 3|V| - 3 = 3|V'| - 6$, the framework is isostatic.

(2) Conversely, deleting a 4-valent joint $\mathbf{p}_0$ leaves

$$|E'| = = |E| - 4 = 3|V| - 10 = 3|V'| - 7$$

independent bars. If some bar $(\mathbf{p}_i, \mathbf{p}_j)$, $1 \leqslant i, j \leqslant 4$, is independent, the proof is complete. If not, all six such bars are already implicit – as linear combinations of the remaining bars. These six implicit rows, plus the original four to $\mathbf{p}_0$ give a set of $|E'| = 10$ rows on five joints. This set with $|E'| > 3|V'| - 6$ rows is dependent – contradicting the assumption that the original framework was isostatic. □

A 5-valent joint can be replaced by two bars in several different patterns (Figure 1.11a, Exercise 1.24). We have *conjectured* that the replacement procedures of Figure 1.11b preserve isostatic frameworks for generic positions of the joints. This would imply the basic Conjecture 1.4.3.

In the absence of a general characterization, we present another inductive technique that covers some higher valent vertices in triangulated polyhedra (Figure 1.12). Given a graph G with a vertex v_1 incident to the edges (1, 2), (1, 3), (1, 4), ..., (1, k), (1, $k+1$), ..., (1, $k+m$), then a *vertex split* of v_1 on the edges (1, 2), (1, 3) is the modified graph with edges (1, 4), ..., (1, k) removed and an added vertex p_0 incident with new edges (0, 1), (0, 2), (0, 3), (0, 4), ..., (0, k).

1.4.6. Vertex splitting theorem. *Given an independent framework* $G(\mathbf{p})$ *in 3-space with a joint* $\mathbf{p}_1$, *and all incident bars* (1, 2), (1, 3), (1, 4), ..., (1, k), *(1,* $k+1$*), ..., (1,* $k+m$*) with* $\mathbf{p}_1-\mathbf{p}_2$ *not parallel to* $\mathbf{p}_1-\mathbf{p}_3$*, then for any* $k+m \geqslant 2$*, the new framework on the vertex split on (1, 2), (1, 3) is independent for almost all positions for the new joint* $\mathbf{p}_0$*.*

Proof. We choose, as a limiting initial case, to add $\mathbf{p}_0$ at $\mathbf{p}_1$, with the 'bar' $(\mathbf{p}_0, \mathbf{p}_1)$ assigned a direction $\mathbf{d}_{01}$ not in the plane of

$$\mathbf{d}_{02} = \mathbf{p}_2 - \mathbf{p}_0 = \mathbf{p}_2 - \mathbf{p}_1 = \mathbf{d}_{12} \quad \text{and} \quad \mathbf{d}_{03} = \mathbf{p}_3 - \mathbf{p}_0 = \mathbf{p}_3 - \mathbf{p}_1 = \mathbf{d}_{13}.$$

This is not a bar framework, but a '3-frame' with a rigidity matrix – and it is the limit of frameworks with variable $\mathbf{p}_0$ (see Exercise 1.43). This creates the following matrix:

$$\begin{bmatrix} \mathbf{d}_{01} & -\mathbf{d}_{01} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{d}_{02} & \mathbf{0} & -\mathbf{d}_{02} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{d}_{03} & \mathbf{0} & \mathbf{0} & -\mathbf{d}_{03} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{d}_{12} & -\mathbf{d}_{12} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{d}_{13} & \mathbf{0} & -\mathbf{d}_{13} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{p}_4-\mathbf{p}_1 & \mathbf{0} & \mathbf{0} & \mathbf{p}_1-\mathbf{p}_4 & \cdots \\ \mathbf{0} & \cdots & & & & \end{bmatrix}.$$

Figure 1.11.

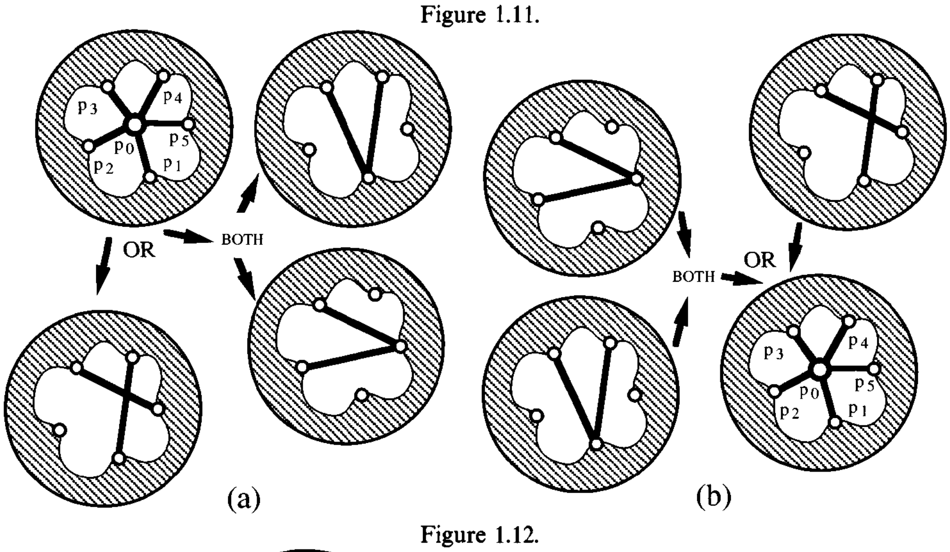
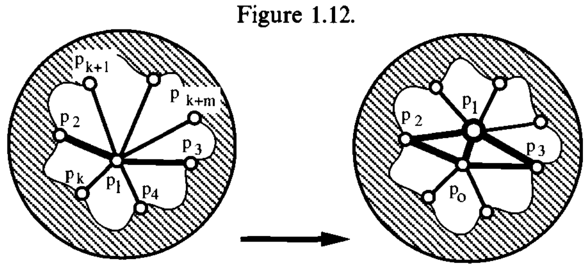


Figure 1.12.



It is clear that this vertex adds rank 3 to the lower rigidity matrix if and only if the three vectors $\mathbf{d}_{01}$, $\mathbf{d}_{02} = \mathbf{p}_2 - \mathbf{p}_0$, and $\mathbf{d}_{03} = \mathbf{p}_3 - \mathbf{p}_0$ are independent. If we add row IV to row II, as well as row V to row III, we get the row equivalent matrix

$$\begin{bmatrix} \mathbf{d}_{01} & -\mathbf{d}_{01} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{d}_{02} & -\mathbf{d}_{02} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{d}_{03} & -\mathbf{d}_{03} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{d}_{12} & -\mathbf{d}_{12} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{d}_{13} & \mathbf{0} & -\mathbf{d}_{13} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{p}_4 - \mathbf{p}_1 & \mathbf{0} & \mathbf{0} & \mathbf{p}_1 - \mathbf{p}_4 & \cdots \\ \mathbf{0} & \cdots & & & & \end{bmatrix}.$$

With these first three independent rows, and with $\mathbf{p}_0 - \mathbf{p}_4 = \mathbf{p}_1 - \mathbf{p}_4$, we can row reduce on rows such as row VI, to change the non-zero entries to:

$$[\mathbf{p}_4 - \mathbf{p}_0 \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{p}_0 - \mathbf{p}_4 \quad \mathbf{0}]$$

We can then return the top five rows to their original pattern, giving a matrix for the graph of the vertex split. Finally since this matrix is independent, it remains independent if we move $\mathbf{p}_0$ to any nearby position with $\mathbf{p}_1 - \mathbf{p}_0 = \mathbf{d}_{01}$. This completes the proof. □

The converse is not true: if we split a vertex on two edges, we can turn a generically non-rigid graph (Figure 1.13a) into a generically rigid graph (Figure 1.13b). Figure 1.13c proves the generic rigidity of Figure 1.13b by a sequence of vertex splits from a tetrahedron which include one application of Proposition 1.4.5.

A selected vertex split on a triangulated surface will create a larger triangulated surface of the same topological type. Let v be a vertex of a triangulation of a 2-manifold (Figure 1.14a). The *star* of v is the union of the closed triangles meeting v, and the *link* of v is the simple polygon of edges in the star that do not contain v. The link can be split into two simple paths M and N, meeting at vertices x and y (Figure 1.14b). A *vertex split* of the triangulation at the vertex v consists of replacing the triangles of the star of v by two new vertices v' and v'' and the triangles determined by v' and the edges of M, v'' and the edges of N, and the two triangles $v'v''x$, $v'v''y$ (Figure 1.14c).

This vertex split in a polyhedron can be reversed by shrinking an edge $\mathbf{p}_0\mathbf{p}_1$ and identifying the vertices – provided that this edge is not part

Figure 1.13.

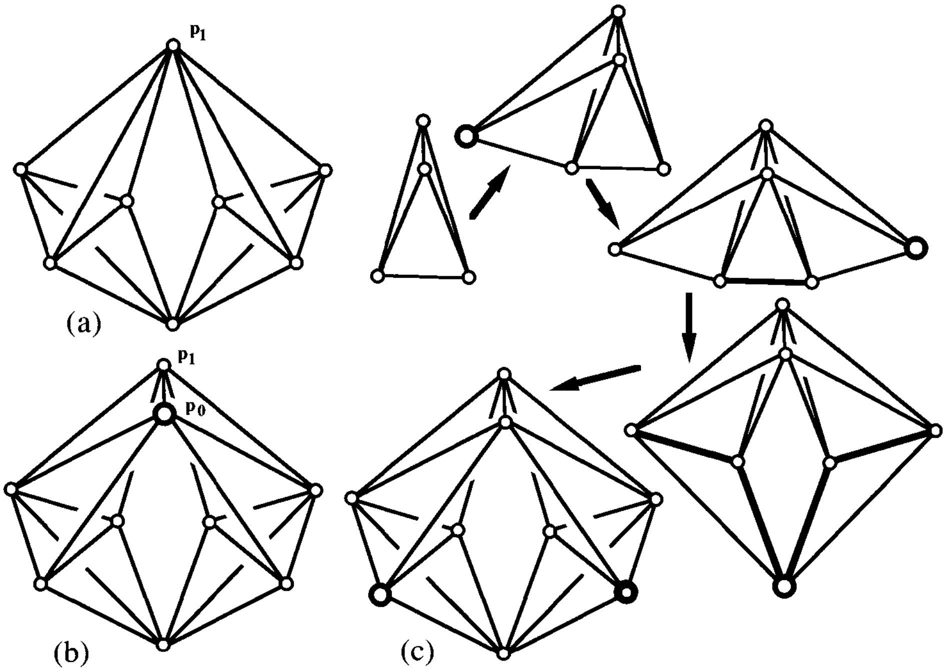
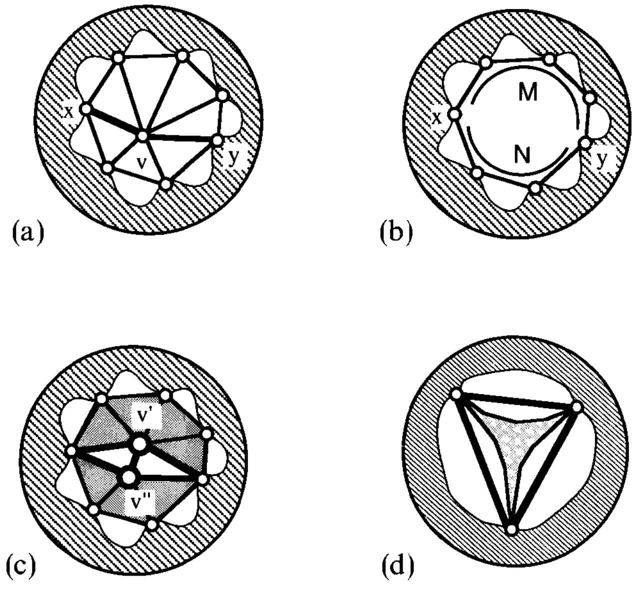


Figure 1.14.



of a non-facial triangle of the surface (Figure 1.14d) and the 2-manifold is not the tetrahedron. We illustrate this process of vertex splitting within the class of triangulated surfaces with a quick proof that all triangulated spherical polyhedra are generically rigid.

1.4.7. Generic Cauchy's theorem. *Every triangulated spherical polyhedron is oenerically rigid in 3-space.*

Proof. We prove this by induction on $|V|$, the number of vertices in the triangulated sphere. If $|V| = 4$, we have a tetrahedron, which is generically rigid in 3-space.

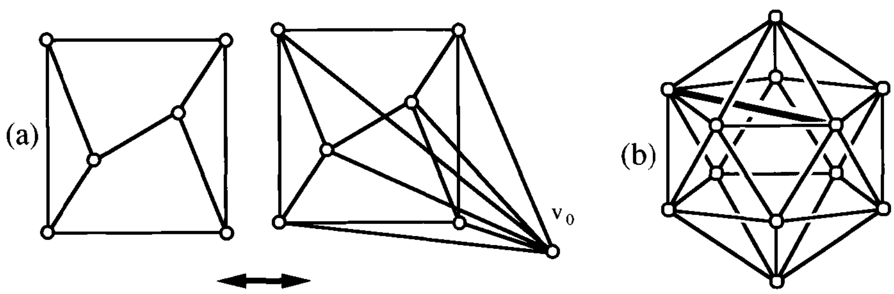
Assume that generic rigidity holds for all triangulated spheres with $<|V|$ vertices and take any triangulate sphere with $|V| > 4$ vertices. We first show that there is a 'shrinkable edge'. Any *non-shrinkable edge e* is part of a non-facial triangle separating a disc of N interior vertices from the remaining triangulated sphere. We claim that there is a shrinkable edge inside each such disc. If $N = 1$, this vertex is 3-valent, and every interior edge is shrinkable.

Assume a shrinkable edge occurs for $N = k$. If a disc has $k + 1$ vertices, then there is an interior edge e. If e is shrinkable, the proof is complete. If not, e is part of a non-facial triangle – which separates the disc, with the previous boundary triangle on one side. The other component is a triangulated disc with fewer than k interior vertices, which must contain a shrinkable edge.

Take any shrinkable edge in the triangulated sphere with $|V|$ vertices. Shrinking this edge creates a smaller generically rigid triangulated sphere. Splitting back along this edge, Theorem 1.4.6 shows that the initial triangulated sphere is also generically rigid in 3-space. □

In the exercises we offer a few other examples that can be proven by similar inductions (Figure 1.15, Exercises 1.23, 1.27). However, the central problem of characterizing the generic rigidity matroid in 3-space remains unsolved.

Figure 1.15.



1.5. A Matroid from Splines

Consider a planar drawing of a 3-connected planar graph, with a triangular boundary. In Section 1.3 we saw that this drawing is the projection of a spatial polyhedron if and only if the drawing has a self-stress as a bar framework. If we ignore the exterior triangle (which is always a plane), such a spatial polyhedron is a globally continuous (C^0) function that is piecewise linear over the plane regions (the 'faces'). In approximation theory such a function is called a *bivariate* C^0_1-spline. Maxwell's theorem and its converse give a correspondence between C^0_1-splines and self-stresses for planar dissections of a boundary triangle.

Consider the analogous problem of finding globally C^1 functions over the planar drawing, which are piecewise second degree polynomials – *bivariate* C^1_2-*splines*. We shall extract a central matroid for these splines that has a deep analogy to the rigidity matroid for spatial frameworks. We outline how the techniques of sections 1.3 and 1.4 extend to this new matroid to characterize the dimension of this space of C^1_2-splines, for generic planar drawings with a triangular boundary.

A *plane geometric graph* $G(\mathbf{p})$ is a graph $G=(V; E)$, with no loops or multiple edges, and an assignment $\mathbf{p}$: $V\rightarrow\mathbb{R}^2$, $\mathbf{p}(v_i)=\mathbf{p}_i$, giving distinct points to the two ends of each edge. For our final application to splines these graphs will be chosen to be planar embeddings of planar graphs. However, in the general theory, including the construction of some special splines, the graph need not even be planar. A *generic* geometric graph uses algebraically independent real numbers for the coordinates of the vertices.

For each edge we have a line with equation

$$p_{ij}x+q_{ij}y+r_{ij}=(y_j-y_i)x+(x_i-x_j)y+(x_jy_i-x_iy_j)=0.$$

The *double-line vector* for a directed edge (i, j) is the vector

$$\mathbf{D}^2_{ij}=(p^2_{ij}, 2p_{ij}q_{ij}, q^2_{ij})=((y_j-y_i)^2, 2(y_j-y_i)(x_i-x_j), (x_i-x_j)^2) \quad \text{for } i<j,$$

and

$$\mathbf{D}^2_{ji}=-\mathbf{D}^2_{ij} \quad \text{for} \quad i<j.$$

A *conic-dependence* of a geometric graph is an assignment of scalars ω_{ij} to the directed edges, $\omega_{ji}=\omega_{ij}$, such that at each vertex $\mathbf{p}_i$ there is a balance:

$$\sum_j \omega_{ij}\mathbf{D}^2_{ij}=\mathbf{0} \quad \text{(sum over edges incident with } \mathbf{p}_i).$$

A geometric graph is *conic-independent* if the only conic-dependence is trivial (all $\omega_{ij}=0$).

These dependences correspond to a matrix equation $\omega RC(G, \mathbf{p})=\mathbf{0}$, where $RC(G, \mathbf{p})$ is the $|E|\times 3|V|$ *conic-rigidity matrix* for a geometric graph. For edges (1, 2) and (i, j) the rows of $RC(G, \mathbf{p})$ have the form:

$$[\mathbf{D}_{12}^2 \quad -\mathbf{D}_{12}^2 \quad 0\,0\,0 \quad \ldots \quad 0\,0\,0 \quad \ldots \quad 0\,0\,0 \quad \ldots \quad 0\,0\,0]$$
$$[0\,0\,0 \quad 0\,0\,0 \quad 0\,0\,0 \quad \ldots \quad \mathbf{D}_{ij}^2 \quad \ldots \quad -\mathbf{D}_{ij}^2 \quad \ldots \quad 0\,0\,0]$$

1.5.1. Example. Consider a plane triangle with vertices $\mathbf{p}_1 = (0, 0)$, $\mathbf{p}_2 = (0, 1)$, $\mathbf{p}_3 = (1, 0)$. The edges of this triangle have the double line vectors

$$\mathbf{D}_{12}^2 = ((y_2 - y_1)^2, 2(y_2 - y_1)(x_1 - x_2), (x_1 - x_2) = (1, 0, 0)$$
$$\mathbf{D}_{13}^2 = (0, 0, 1) \qquad \mathbf{D}_{23}^2 = (1, 2, 1)$$

This gives the conic-rigidity matrix

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 & 1 & -1 & -2 & -1 \end{bmatrix}.$$

Clearly this matrix is independent with rank 3, so there is only the trivial conic-dependence.

Accordingly, the solution set to the homogeneous system $RC(G, \mathbf{p}) \times \mathbf{u}^t = \mathbf{0}$ has dimension 6. The reader can check that this solution space is spanned by the six independent vectors:

$$\mathbf{T}_1 = (1,0,0,\ 1,0,0,\ 1,0,0)$$
$$\mathbf{T}_2 = (0,1,0,\ 0,1,0,\ 0,1,0)$$
$$\mathbf{T}_3 = (0,0,1,\ 0,0,1,\ 0,0,1)$$
$$\mathbf{T}_4 = (0,0,0,\ 0,1,0,\ 2,0,0)$$
$$\mathbf{T}_5 = (0,0,0,\ 0,0,2,\ 0,1,0)$$
$$\mathbf{T}_6 = (0,0,0,\ 0,0,1,\ 1,0,0)$$

The reader can also check that a collinear triangle gives a conic-rigidity matrix of rank 2: all the $\mathbf{D}_{ij}^2$ are parallel vectors, and there is a conic-dependence.

1.5.2. Proposition. *The conic-rigidity matrix of a plane geometric graph* $G(\mathbf{p})$*, non-collinear, has a maximum rank* $3|V| - 6$.

Proof. We show that the homogeneous system $RC(G, \mathbf{p}) \times \mathbf{u}^t = \mathbf{0}$ has a six-dimensional space of solutions generated by:

$$\mathbf{T}_1 = (1,0,0,\ 1,0,0,\ \ldots,\ 1,0,0)$$
$$\mathbf{T}_2 = (0,1,0,\ 0,1,0,\ ..,\ 0,1,0)$$
$$\mathbf{T}_3 = (0,0,1,\ 0,0,1,\ ..,\ 0,0,1)$$
$$\mathbf{T}_4 = (2x_1, y_1, 0,\ 2x_2, y_2, 0,\ \ldots,\ 2x_{|V|}, y_{|V|}, 0)$$
$$\mathbf{T}_5 = (0, x_1, 2y_1,\ 0, x_2, 2y_2,\ \ldots,\ 0, x_{|V|}, 2y_{|V|})$$
$$\mathbf{T}_6 = (x_1^2, x_1 y_1, y_1^2,\ x_2^2, x_2 y_2, y_2^2,\ \ldots,\ x_{|V|}^2, x_{|V|} y_{|V|} y_{|V|}^2).$$

The first three vectors, which correspond to the 'translations' in 3-space, are clearly independent solutions. We check that the last three, which correspond to 'rotations', are solutions for a typical edge joining points (x_1, y_1) and (x_2, y_2) along the corresponding line

$$(y_2 - y_1)x + (x_1 - x_2)y + (x_2 y_1 - x_1 y_2) = 0.$$

For $\mathbf{T}_4$ we have

$$\begin{aligned}
2x_1(y_2 - y_1)^2 &+ (y_1)2(y_2 - y_1)(x_1 - x_2) + 0 \\
&- [2x_2(y_1 - y_2)^2 + (y_2)2(y_1 - y_2)(x_2 - x_1) + 0] \\
&= 2(y_2 - y_1) \\
&\quad \times [x_1(y_2 - y_1) + y_1(x_1 - x_2) + x_2(y_1 - y_2) + y_2(x_2 - x_1)] \\
&= 2(y_2 - y_1)(0) = 0.
\end{aligned}$$

A similar check works for $\mathbf{T}_5$.

For $\mathbf{T}_6$ we have

$$\begin{aligned}
x_1^2(y_2 - y_1)^2 &+ 2x_1 y_1(y_2 - y_1)(x_1 - x_2) + y_1^2(x_1 - x_2)^2 \\
&- [x_2^2(y_1 - y_2)^2 + 2x_2 y_2(y_1 - y_2)(x_2 - x_1) + y_2^2(x_2 - x_1)^2] \\
&= (x_1^2 - x_2^2)(y_2 - y_1)^2 + (2x_1 y_1 - 2x_2 y_2)(y_2 - y_1)(x_1 - x_2) \\
&+ (y_1^2 - y_2^2)(x_1 - x_2)^2 \\
&= (y_2 - y_1)(x_1 - x_2) \\
&\quad \times [(x_1 + x_2)(y_2 - y_1) + (2x_1 y_1 - 2x_2 y_2) - (y_1 + y_2)(x_1 - x_2)] \\
&= (y_2 - y_1)(x_1 - x_2)(0) = 0.
\end{aligned}$$

In Example 1.5.1 we checked that these vectors were independent solutions for the three vertices of the triangle. The same check holds for any three non-collinear points. □

By analogy with spatial frameworks, we call a plane geometric graph with at least three vertices *conic-rigid* if the conic-rigidity matrix has this maximal rank $3|V| - 6$. A single edge with distinct vertices is also conic-rigid, as is a single point with no edges. Example 1.5.1 shows that a non-collinear triangle is conic-rigid. All the inductive techniques developed for frameworks in 3-space extend to conic-rigidity (see below, and Exercises 1.35, 1.36) and some techniques only conjectured for frameworks apply to conic-rigidity. (Exercise 1.37).

With these similarities in counts and in inductive techniques, we can ask if the conic-rigidity matrix $RC(G, \mathbf{p})$ for a generic embedding in the plane, and the rigidity matrix $R(G, \mathbf{q})$ for a generic embedding in 3-space, create the same matroid on the edges.

1.5.3. Conjecture. *A set E of edges is independent for generic embeddings of the graph as a bar framework in 3-space if and only if the set is conic-independent for generic embeddings as a geometric graph in the plane.*

If the rigidity matroid is maximal, as conjectured in the previous section, Conjecture 1.5.3 claims that the generic conic-rigidity matroid for a graph is also maximal.

We have shown that a triangle is generically conic-rigid. We will show that the graph of any triangulated sphere is also generically conic-rigid, following the approach in section 1.4.

1.5.4. Vertex splitting theorem for conic-rigidity. *Given a conic independent geometric graph $G(\mathbf{p})$ in the plane with a joint $\mathbf{p}_1$, and all incident bars $(1, 2), (1, 3), (1, 4), .., (1, k), (1, k+1), \ldots, (1, k+m)$ with $\mathbf{p}_1 - \mathbf{p}_2$ not parallel to $\mathbf{p}_1 - \mathbf{p}_3$, then for any $k + m \geqslant 2$, the new geometric graph on the vertex split on $(1, 2), (1, 3)$ is conic-independent for almost all positions for the new vertex $\mathbf{p}_0$.*

Proof. We follow the proof of Theorem 1.4.6, choosing, as a limiting initial case, to add $\mathbf{p}_0$ at $\mathbf{p}_1$, with the 'line' $\mathbf{p}_0, \mathbf{p}_1$ assigned a direction $\mathbf{D}_{01}$ so that $\mathbf{D}_{02}^2 = \mathbf{D}_{12}^2$, $\mathbf{D}_{03}^2 = \mathbf{D}_{13}^2$, and $\mathbf{D}_{01}^2$ are independent 3-vectors. This creates the conic-rigidity matrix

$$\begin{bmatrix} \mathbf{D}_{01}^2 & \mathbf{D}_{01}^2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{D}_{02}^2 & \mathbf{0} & -\mathbf{D}_{02}^2 & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{D}_{03}^2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{D}_{03}^2 & \cdots \\ \mathbf{0} & \mathbf{D}_{12}^2 & -\mathbf{D}_{12}^2 & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{D}_{13}^2 & \mathbf{0} & -\mathbf{D}_{13}^2 & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{D}_{14}^2 & \mathbf{0} & \mathbf{0} & -\mathbf{D}_{14}^2 & \cdots \\ \mathbf{0} & \cdots & & & & \end{bmatrix}.$$

It is clear that this vertex adds rank 3 to the previous conic-rigidity matrix if and only if the three vectors $\mathbf{DE}_{01}^2$, $\mathbf{D}_{02}^2$, and $\mathbf{D}_{03}^2$ are independent. The remaining steps duplicate the steps of Theorem 1.4.6, creating the pattern of a conic-rigidity matrix for the vertex split. Again this matrix remains independent if we move $\mathbf{p}_0$ to any nearby position with $\mathbf{p}_1 - \mathbf{p}_0 = \mathbf{D}_{01}$. This completes the proof. □

Theorem 1.4.7 now extends immediately to conic-rigidity, since the proof was essentially topological.

1.5.5. Proposition. *For every graph of a triangulated sphere the geometric graph $G(\mathbf{p})$ at a generic point $\mathbf{p}$ has rank $3|V|-6$ and is conic-independent.*

What is the significance of these results? Consider a plane drawing of a 3-connected planar graph – with its associated faces. A C_2^1-spline assigns a quadratic function

$$z = A^h x^2 + B^h xy + C^h y^2 + D^h x + E^h y + F^h$$

for each face f^h (except the exterior unbounded region). If two faces f^h, f^i share an edge along the line $\mathbf{L}_{jk}: px + qy + r = 0$ then the C^1 requirement of a common tangent plane at all points on this edge becomes a simple algebraic condition on the functions:

$$\begin{aligned}(A^h x^2 + B^h xy + C^h y^2 + D^h x + E^h y + F^h)\\ -(A^j x^2 + B^i xy + C^i y^2 + D^i x + E^i y + F^i)\\ = \omega^{hi}(p^2x^2 + 2pqxy + q^2y^2 + 2prx + 2qry + r^2).\end{aligned}$$

for some scalar ω^{hi} (see Exercise 1.38). Moreover, if we follow a cycle of faces and edges around an interior vertex of the disc, the net difference is zero, giving the equation

$$\sum \omega^{hi}(p^2x^2 + 2pqxy + q^2y^2 + 2prx + 2qry + r^2) = 0 \quad \text{(sum over the vertex cycle)}.$$

Thus a C_2^1-spline corresponds to a choice for a quadratic function for an initial face, and set of scalars for the edges, satisfying these vertex conditions. (The complete proof reproduces the arguments of Lemma 1.3.4 and Theorem 1.3.5 for self-stresses on a plane framework.)

These scalars look suspiciously like our conic-dependences, although these satisfy the six conditions:

$$\sum \omega^{hi}(p^2, 2pq, q^2, 2pr, 2qr, r^2) = (0, 0, 0, 0, 0, 0) \quad \text{(sum over the vertex cycle)}.$$

We can use the orientation of the polyhedron to convert from scalars for a pair of adjacent faces to scalars for a pair of adjacent vertices.

We must show that this extension from three equations to six is automatic at any vertex. Given a point $\mathbf{p}_0 = (x_0, y_0)$, and the three coordinates $(A, B, C) = (p^2, 2pq, q^2)$ for a line $px + qy + r = 0$ through $\mathbf{p}_0$, there is a unique 6-vector defined as follows:

$$\begin{aligned}&(x_0, y_0, 1)\#(A, B, C)\\ &\quad = (A, B, C, -2Ax_0 - By_0, -Bx_0 - 2Cy_0, Ax_0^2 + Bx_0y_0 + Cy_0^2)\\ &\quad = (p^2, 2pq, q^2, 2pr, 2qr, r^2) = \mathbf{L}^2.\end{aligned}$$

Thus if $\sum_j \omega_{ij}(A_{ij}, B_{ij}, C_{ij}) = \mathbf{0}$ then $\sum_j \omega_{ij}(x_i, y_i, 1)\#(A_{ij}, B_{ij}, C_{ij}) = \mathbf{0}$. (The 6-vector

$$(A, B, C, -2Ax_0 - By_0, -Bx_0 - 2Cy_0, Ax_0^2 + Bx_0y_0 + Cy_0^2)$$

is interpreted as the coordinates of a quadratic function that is tangent to the xy-plane at the point $\mathbf{p}_0$.)

This gives a set of scalars for all edges except the exterior boundary edges. If the boundary is a non-degenerate triangle, then this triangle has rank $3|V-6|=3$. This rank gives conic-rigidity – so the interior scalars extend to scalars on the boundary edges (see Exercise 1.38). The conic-dependence on the interior extends to a conic-dependence on the whole.

1.5.6. Proposition. *For a triangulated disc in the plane, with a triangular boundary and a graph G*(**p**)*, the space of C_2^1-splines with the zero quadric over a fixed triangle is isomorphic to the vector space of conic-dependences of G*(**p**).

This gives, as a corollary, the basic theorem for the dimension of the space of C_2^1-splines.

1.5.7. Corollary. *A generic triangulated plane disc with a triangular boundary has only trivial C_2^1-splines (all quadrics equal).*

An analogous matroid exists for C_3^2-splines, giving a matroid similar, but not identical, to the generic rigidity matroid in 4-space (see Exercises 1.40–1.42).

1.6. Scene Analysis of Polyhedral Pictures

In section 1.3 we demonstrated a connection between polyhedral pictures and the plane rigidity matroid. We shall now introduce a matroid that characterizes pictures of 'general polyhedral scenes'. Basic similarities to rigidity, at the level of matroid theory, will again be evident.

Consider an *elementary plane picture* formed by a set P of distinct points $\mathbf{p}_j$, and a set F of unordered sets of four distinct points $f^i=\{\mathbf{p}_h, \mathbf{p}_j, \mathbf{p}_k, \mathbf{p}_m\}$. The points $\mathbf{p}_j$ are lifted to heights z_j, with the constraint that the four points of each face must remain coplanar in this *spatial scene*. For four non-collinear points, each face imposes one linear equation on the heights:

$$\begin{vmatrix} x_h & x_j & x_k & x_m \\ y_h & y_j & y_k & y_m \\ z_h & z_j & z_k & z_m \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0 \quad \text{or} \quad [\bar{\mathbf{p}}_j\bar{\mathbf{p}}_k\bar{\mathbf{p}}_m]z_h - [\bar{\mathbf{p}}_h\bar{\mathbf{p}}_k\bar{\mathbf{p}}_m]z_j - [\bar{\mathbf{p}}_h\bar{\mathbf{p}}_j\bar{\mathbf{p}}_k]z_m = 0$$

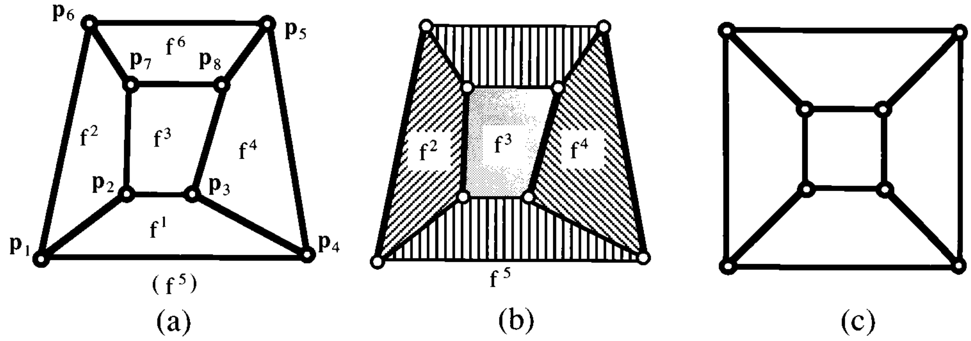
where the bracket $[\bar{\mathbf{p}}_j\bar{\mathbf{p}}_k\bar{\mathbf{p}}_m]$ is the corresponding minor of the determinant, recording the oriented area of the plane triangle $\mathbf{p}_j\mathbf{p}_k\mathbf{p}_m$. (If the four points are collinear, then a vertical plane will fit any selection of heights, so there is no condition.)

We actually record a plane picture as an abstract *hypergraph* $H = (V, F)$, with all faces as unordered quadruples of vertices, and an assignment $\mathbf{P}$ of plane points to the vertices. The constraints on the spatial scenes define an $|F| \times |V|$ homogeneous system $M(H, \mathbf{P}) \times Z^t = 0$, and a corresponding matroid on the faces. Every picture whose points span the plane has a 3-space of *trivial scenes* – scenes with all points coplanar. Thus an independent set F of faces must satisfy

$$|F'| \leqslant |V'| - 3 \quad \text{for every non-empty subset } F'.$$

Figure 1.16 shows some elementary plane pictures, with the faces shown as quadrilateral regions, using heavy lines between faces sharing two vertices.

Figure 1.16.



1.6.1. Example. Consider the elementary scene in Figure 1.16a, with six faces and eight vertices. The matrix for this picture has the form:

$$\begin{bmatrix}
[\bar{\mathbf{p}}_2\bar{\mathbf{p}}_3\bar{\mathbf{p}}_4] & -[\bar{\mathbf{p}}_1\bar{\mathbf{p}}_3\bar{\mathbf{p}}_4] & [\bar{\mathbf{p}}_1\bar{\mathbf{p}}_2\bar{\mathbf{p}}_4] & -[\bar{\mathbf{p}}_1\bar{\mathbf{p}}_2\bar{\mathbf{p}}_3] & 0 & 0 & 0 & 0 \\
[\bar{\mathbf{p}}_2\bar{\mathbf{p}}_6\bar{\mathbf{p}}_7] & -[\bar{\mathbf{p}}_1\bar{\mathbf{p}}_6\bar{\mathbf{p}}_7] & 0 & 0 & 0 & [\bar{\mathbf{p}}_1\bar{\mathbf{p}}_2\bar{\mathbf{p}}_7] & -[\bar{\mathbf{p}}_1\bar{\mathbf{p}}_2\bar{\mathbf{p}}_6] & 0 \\
0 & [\bar{\mathbf{p}}_3\bar{\mathbf{p}}_7\bar{\mathbf{p}}_8] & [-\bar{\mathbf{p}}_2\bar{\mathbf{p}}_7\bar{\mathbf{p}}_8] & 0 & 0 & 0 & [\bar{\mathbf{p}}_2\bar{\mathbf{p}}_3\bar{\mathbf{p}}_8] & -[\bar{\mathbf{p}}_2\bar{\mathbf{p}}_3\bar{\mathbf{p}}_7] \\
0 & 0 & [\bar{\mathbf{p}}_4\bar{\mathbf{p}}_5\bar{\mathbf{p}}_8] & -[\bar{\mathbf{p}}_3\bar{\mathbf{p}}_5\bar{\mathbf{p}}_8] & [\bar{\mathbf{p}}_3\bar{\mathbf{p}}_4\bar{\mathbf{p}}_8] & 0 & 0 & -[\bar{\mathbf{p}}_3\bar{\mathbf{p}}_4\bar{\mathbf{p}}_5] \\
[\bar{\mathbf{p}}_4\bar{\mathbf{p}}_5\bar{\mathbf{p}}_6] & 0 & 0 & -[\bar{\mathbf{p}}_1\bar{\mathbf{p}}_5\bar{\mathbf{p}}_6] & [\bar{\mathbf{p}}_1\bar{\mathbf{p}}_4\bar{\mathbf{p}}_6] & -[\bar{\mathbf{p}}_1\bar{\mathbf{p}}_4\bar{\mathbf{p}}_5] & 0 & 0 \\
0 & 0 & 0 & 0 & [\bar{\mathbf{p}}_6\bar{\mathbf{p}}_7\bar{\mathbf{p}}_8] & -[\bar{\mathbf{p}}_5\bar{\mathbf{p}}_7\bar{\mathbf{p}}_8] & [\bar{\mathbf{p}}_5\bar{\mathbf{p}}_6\bar{\mathbf{p}}_8] & -[\bar{\mathbf{p}}_6\bar{\mathbf{p}}_7\bar{\mathbf{p}}_8]
\end{bmatrix}$$

The matrix has rank 5 for generic pictures, giving a linear dependence and leaving only the trivial scenes with all vertices (and faces) coplanar. Even with one face removed, general pictures are independent, but have only trivial scenes. However, with two faces removed (any two), the matrix has rank 4 for a generic picture (Figure 1.16b), giving non-trivial scenes, with distinct planes for each face.

Consider a special picture, such as Figure 1.16c, with the positions:

$$\mathbf{p}_1 = (0, 0) \quad \mathbf{p}_2 = (1, 1) \quad \mathbf{p}_3 = (2, 1) \quad \mathbf{p}_4 = (3, 0)$$

$$\mathbf{p}_5 = (3, 3) \quad \mathbf{p}_6 = (0, 3) \quad \mathbf{p}_7 = (1, 2) \quad \mathbf{p}_8 = (2, 2).$$

The picture matrix for all six faces now has the form

$$\begin{bmatrix} -1 & 3 & -3 & 1 & 0 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 & 0 & 1 & -3 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 3 & -1 & 1 & 0 & 0 & -3 \\ 9 & 0 & 0 & -9 & 9 & -9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 3 & -3 \end{bmatrix}.$$

This matrix has rank 4, giving non-trivial scenes with distinct planes for each face.

We want to characterize the *generic elementary picture matroid*, which occurs for general (e.g. algebraically independent) points in the plane.

1.6.2. Theorem. *A hypergraph H of quadruples has realizations as an independent plane picture if and only if* $|F'| \leqslant |V'| - 3$ *for every non-empty subset of faces* F'.

Proof. We already have seen that an independent plane picture satisfies $|F'| \leqslant |V'| - 3$ for every non-empty subset of faces F'.

Assume that a hypergraph satisfies

$$|F'| \leqslant |V'| - 3 \quad \text{for every non-empty subset of faces } F'.$$

We begin with the simpler transversal matroid, defined by the semimodular function $g(F) = |V|$. The transversal matroid is represented over the reals by the modified incidence matrix $A(F)$:

$$a_{ij} = t_{ij} \quad \text{if} \quad v_j \in f_i \quad \text{and} \quad a_{ij} = 0 \quad \text{if} \quad v_j \notin f_i$$

for any set of algebraically independent reals t_{ij}.

By our assumption, F is also independent in this transversal matroid, giving independent rows in the transversal matrix. Since this matrix is $|F| \times |V|$, the system $A \times X = 0$ has a solution space of dimension at least 3. Moreover, if we add three copies of any edge of F, we get a set F'' that is still independent in the transversal matroid, so that we can choose a solution set of dimension 3 when restricted to the vertices of any edge. We write a basis for this chosen solution space as the $|F| \times 3$ matrix Y, with rows $Y_j = [x_j y_j w_j]$, satisfying $A \times Y = 0$.

We can choose the matrix Y so that no values of w_i are zero (take linear combinations of the columns). If we divide each row Y_j by the scalar w_j, creating Y', and multiply the corresponding column A^j of A by w_j, we obtain another representation A' of the same matroid, with $A' \times Y' = 0$ and $Y_j' = [x_j' y_j' 1]$. We check that this matrix A' is the picture matrix up to multiplication of rows by non-zero constants.

For a typical face of the form $f = \{v_1, v_2, v_3, v_4\}$, the equation $A' \times Y' = 0$ gives the equations

$$t_1x_1+t_2x_2+t_3x_3+t_4x_4=0 \quad t_1y_1+t_2y_2+t_3y_3+t_4y_4=0$$

$$t_11+t_21+t_31+t_41=0$$

Given the matrix Y', these form three independent equations in the t_i, for which the only solutions are scalar multiples of

$$t_1=\begin{vmatrix} x_2 & x_3 & vx_4 \\ y_2 & y_3 & y_4 \\ 1 & 1 & 1 \end{vmatrix}, \quad t_2=\begin{vmatrix} x_1 & x_3 & x_4 \\ y_1 & y_3 & y_4 \\ 1 & 1 & 1 \end{vmatrix},$$

$$t_3=\begin{vmatrix} x_1 & x_2 & x_4 \\ y_1 & y_2 & y_4 \\ 1 & 1 & 1 \end{vmatrix}, \quad t_4=\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}.$$

Therefore, the row of A' for edge f is:

$$\beta([\bar{\mathbf{p}}_2\bar{\mathbf{p}}_3\bar{\mathbf{p}}_4],\ -[\bar{\mathbf{p}}_1\bar{\mathbf{p}}_3\bar{\mathbf{p}}_4],\ [\bar{\mathbf{p}}_1\bar{\mathbf{p}}_2\bar{\mathbf{p}}_4],\ -[\bar{\mathbf{p}}_1\bar{\mathbf{p}}_2\bar{\mathbf{p}}_3]).$$

If we take the picture with $\mathbf{p}_i=(x_i, y_i)$, this row of A' is a multiple of the row of picture matrix $M(F, \mathbf{P})$ for the edge f. Thus an independent set of rows in A' corresponds to independent rows in $M(F, \mathbf{P})$.

We conclude that F has a realization as an independent elementary picture.

□

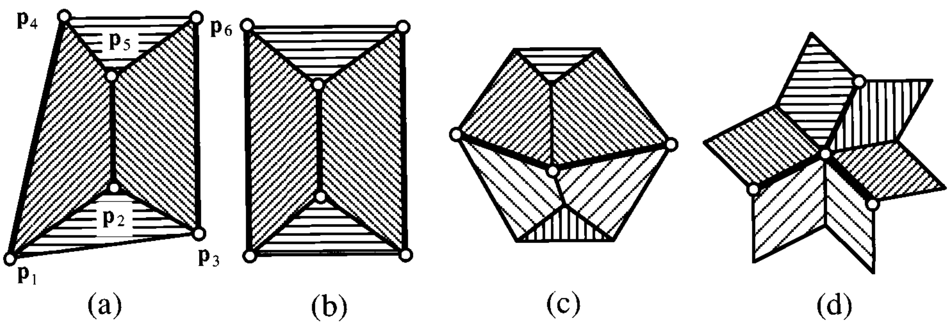
This is an elementary result for elementary pictures. A general *plane picture* involves faces with arbitrary (finite) sets of points which are constrained to be coplanar in a general *spatial scene*. There is an alternate system of equations that is traditionally used for these scenes. Each face f^i in the scene is recorded as a non-vertical plane $a^ix+b^iy+z+c^i=0$ or as a triple (a^i, b^i, c^i). If vertex $v_j \in f^i$, then we have the single linear equation in the variables z_j, a^i, b^i, c^i:

$$a^ix_j+b^iy_j+z_j+c^i=0$$

The corresponding system of equations creates a matroid on the rows of the coefficient matrix, or a matroid on the incidences $I \subset V \times F$. A *plane picture* $S(\mathbf{P})$ is an *incidence structure* (or bipartite graph) $S=(V, F; I)$ with an assignment $\mathbf{P}$ of plane points to the vertices V. The 3-space of trivial scenes continues to exist, so a set of independent incidences satisfies

$$|I'| \leqslant |V'|+3|F'|-3 \quad \text{on non-empty subsets } I'.$$

Figure 1.17.



1.6.3. Example. We consider the picture in Figure 1.17a. This incidence structure of six vertices, three faces, and 12 incidences, creates the picture matrix

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & x_1 & y_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & x_2 & y_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & x_4 & y_4 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & x_5 & y_5 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_2 & y_2 & 1 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & x_3 & y_3 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & x_5 & y_5 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & x_6 & y_6 & 1 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 & y_1 & 1\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_3 & y_3 & 1\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_4 & y_4 & 1\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & x_6 & y_6 & 1
\end{bmatrix}$$

This has rank $12 = |V| + 3|F| - 3$ for generic points – and the picture has only trivial scenes.

For a special position, such as Figure 1.17b, with vertices

$$\mathbf{p}_1 = (0, 0),\ \mathbf{p}_2 = (1, 1),\ \mathbf{p}_3 = (2, 0),\ \mathbf{p}_4 = (0, 3),\ \mathbf{p}_5 = (1, 2),\ \mathbf{p}_6 = (2, 3),$$

the matrix has rank 11 – and has non-trivial scenes, as the three quadrilateral faces of a 'triangular prism'.

Each of these three faces has exactly four vertices – and we can return to the elementary picture matroid by row reducing on the final nine columns, provided no face has all vertices collinear:

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & x_1 & y_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & x_2 & y_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & x_4 & y_4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_2 & y_2 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & x_3 & y_3 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & x_5 & y_5 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 & y_1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_3 & y_3 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & x_4 & y_4 & 1 \\
[\bar{p}_2\bar{p}_4\bar{p}_5] & -[\bar{p}_1\bar{p}_4\bar{p}_5] & 0 & [\bar{p}_1\bar{p}_2\bar{p}_5] & -[\bar{p}_1\bar{p}_2\bar{p}_4] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & [\bar{p}_3\bar{p}_5\bar{p}_6] & -[\bar{p}_2\bar{p}_5\bar{p}_6] & 0 & [\bar{p}_2\bar{p}_3\bar{p}_6] & -[\bar{p}_2\bar{p}_3\bar{p}_5] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
[\bar{p}_3\bar{p}_4\bar{p}_5] & 0 & -[\bar{p}_1\bar{p}_4\bar{p}_6] & [\bar{p}_1\bar{p}_3\bar{p}_6] & 0 & -[\bar{p}_1\bar{p}_3\bar{p}_4] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

A similar reduction, applied to all picture matrices, is the foundation for the next theorem.

1.6.4. Theorem. *The incidences I of an independence structure $S = (V, F; I)$ are independent in a generic plane picture if and only if $|I'| \leqslant |V'| + 3|F'| - 3$ on all non-empty subsets I'.*

Proof. For a generic picture, faces with less than four vertices impose no conditions, and will not change the independence of the structure. Faces of $n > 4$ vertices can be replaced by $(n-3)$ 4-valent faces, choosing three fixed vertices and adding the others one at a time. This converts the problem to a problem on an elementary plane picture $H^\wedge(\mathbf{P})$, and the interested reader can check that the counts $|I'| \leqslant |V'| + 3|F'| - 3$ correspond to the counts $F^\wedge \leqslant |V^\wedge| - 3$ on this elementary picture. □

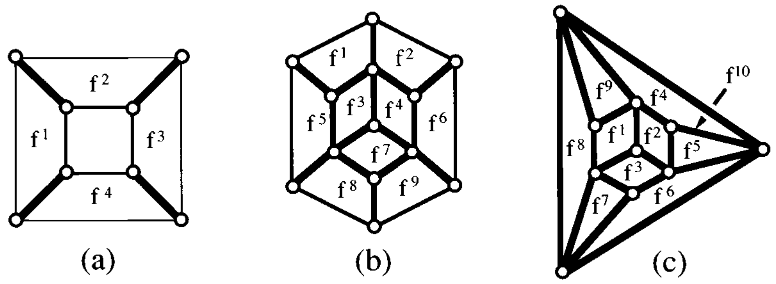
A more central matroid for computer science arises from one more truncation of the picture matroid. A *sharp scene* is a spatial scene in which any two faces lie in distinct planes. A sharp scene has a 4-space of trivial changes over the same picture: with one face fixed in the horizontal, we can change the vertical scale, in addition to the 3-space of scenes that change the plane of the selected initial face.

The following corollary is used in computer algorithms for recognizing pictures of sharp scenes.

1.6.5. Corollary. *For an incidence structure $S = (V, F; I)$ a generic picture has sharp scenes if and only if $|I'| \leqslant |V'| + 3|F'| - 4$ for all subsets I' with at least two faces.*

Figure 1.18 shows some generically sharp pictures (pictures with sharp scenes).

Figure 1.18. $|I| = |V| + 3|F| - 4$. (a) $|F| = 4, |V| = 8, |I| = 16$; (b) $|F| = 9, |V| = 13, |I| = 36$; (c) $|F| = 10,\ |V| = 10,\ |I| = 36$.



1.6.6. Example. To highlight the similarity to rigidity matroids, consider *simple pictures* in which each vertex is incident with exactly two faces. With this assumption, we have $|V'| = 2|I'|$ on relevant subsets and the count for sharp scenes $|I'| \leqslant |V'| + 3|F'| - 4$ simplifies to the count $|V'| \leqslant 3|F'| - 4$. This assumption also means that the incidence structure describes a graph, or multi-graph – with 'vertices' F and an 'edge' for each element of V connecting its two faces.

If we row reduce the picture matrix on the columns for the vertices, the matrix assumes a rigidity-like pattern. For example, the matrix from Example 1.6.3 becomes

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & x_1 & y_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & x_2 & y_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & x_3 & y_3 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & x_4 & y_4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & x_5 & y_5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & x_6 & y_6 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & x_1 & y_1 & 1 & 0 & 0 & 0 & -x_1 & -y_1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & x_2 & y_2 & 1 & -x_2 & -y_2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_3 & y_3 & 1 & -x_3 & -y_3 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & x_4 & y_4 & 1 & 0 & 0 & 0 & -x_4 & -y_4 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & x_5 & y_5 & 1 & -x_5 & -y_5 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_6 & y_6 & 1 & -x_6 & -y_6 & -1
\end{bmatrix}$$

The block in the lower right corner is a '3-frame matrix' analogous to the 2-frame used in section 1.2, and studied in the exercises. It does not directly correspond to a problem of rigid frameworks, although it does have a subtle interpretation in terms of bar and body frameworks in the plane (Exercise 1.45).

The picture matrix has an important geometric dual interpretation – describing *parallel redrawings* of faces in space. Each vertex (a, b) in the picture corresponds to a normal $(a, b, 1)$ of a non-vertical plane

$$ax + by + z + d = 0.$$

Each face containing (a, b) in the scene corresponds to a point (x_0, y_0, z_0) in space satisfying the equation

$$ax_0 + by_0 + z_0 + d = 0.$$

Thus the dual realization of the picture has a plane corresponding to each vertex and a point corresponding to each face.

Different dual realizations have corresponding faces parallel. A trivial realization has all vertices at the same point, corresponding to a solution to the picture equation with all faces at the same place. A non-trivial realization has distinct vertices, with faces parallel to the original faces (same normal). A trivial realization has a three-dimensional space of trivial parallel redrawings – the translations. A non-trivial realization has a four-dimensional space of *trivial parallel redrawings* generated by the translations and the dilations.

1.6.7. Example. Consider the dual of Example 1.6.6 (Figure 1.17c). This is, essentially, a triangular bipyramid: the top and bottom 'vertices' are implicit in the intersection of three planes. For general choices for the normals of six faces, the only realizations are trivial, with all vertices at the same point (Figure 1.17d). The 'general' bipyramid has 'general vertices' – but not general face normals – giving a lower rank for the parallel redrawing matrix, and an extra parallel redrawing: the similarity map.

Formally, this example has only three vertices, and six faces which meet in pairs along the three sides of a spatial triangle. As such, this becomes an example of the parallel redrawing polymatroid on spatial graphs (see Exercises 1.54).

1.7. Special Positions

Graphs which are isostatic (or independent) in generic realizations as a bar framework in the plane or in 3-space also have special *critical forms* that are dependent (or stressed) because of the underlying geometry of the framework.

Similar critical forms arise for conic-rigidity and for plane pictures. There is some combinatorics, and a lot of geometry, in the study of these special positions.

1.7.1. Example. Recall the frameworks of Example 1.2.2. The general rigidity matrix has the form

$$\begin{bmatrix}
x_4-x_1 & y_4-y_1 & 0 & 0 & 0 & 0 & x_1-x_4 & y_1-y_4 & 0 & 0 & 0 & 0 \\
x_5-x_1 & y_5-y_1 & 0 & 0 & 0 & 0 & 0 & 0 & x_1-x_5 & y_1-y_5 & 0 & 0 \\
x_6-x_1 & y_6-y_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1-y_6 & y_1-y_6 \\
0 & 0 & x_4-x_2 & y_4-y_2 & 0 & 0 & x_2-x_4 & y_2-y_4 & 0 & 0 & 0 & 0 \\
0 & 0 & x_5-x_2 & y_5-y_2 & 0 & 0 & 0 & 0 & x_2-x_5 & y_2-y_5 & 0 & 0 \\
0 & 0 & x_6-x_2 & y_6-y_2 & 0 & 0 & 0 & 0 & 0 & 0 & x_2-x_6 & y_2-y_6 \\
0 & 0 & 0 & 0 & x_4-x_3 & y_4-y_3 & x_3-x_4 & y_3-y_4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & x_5-x_3 & y_5-y_3 & 0 & 0 & x_3-x_5 & y_3-y_5 & 0 & 0 \\
0 & 0 & 0 & 0 & x_6-x_3 & y_6-y_3 & 0 & 0 & 0 & 0 & x_3-x_6 & y_3-y_6
\end{bmatrix}$$

We previously saw that this has generic rank 9, but has low rank, corresponding to non-trivial infinitesimal motions, for six points on a conic section. A lower rank corresponds to all 9×9 minors being zero; thus a set of 84 conditions has to be checked. Each minor gives a polynomial equation in the coordinates of the six points:

$$p(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4, \mathbf{p}_5, \mathbf{p}_6) = 0.$$

What do these polynomials look like? They are of degree 9 in the points. We have already seen that each polynomial is zero for six points on a conic section. This condition corresponds to an irreducible polynomial of degree 12 in the points (degree 2 in each point) that must be a factor of each of the minors. What is left in each minor after this polynomial has been factored out? Just a simple linear *factor* that turns out to be a trivial statement about distinct coordinates for the points.

For example, if we delete the first two columns and the seventh column, the remaining minor can be calculated by Laplace expansion, using the first row. This gives the trivial factor $(y_1 - y_4)$ times a minor that must be (and is) the irreducible polynomial for the conic. If we delete the first two columns and the eighth column, the linear factor is $(x_1 - x_4)$. These inessential linear by-products of the deleted columns are called *tie down factors*, and a basic theorem is stated as follows.

1.7.2. Theorem. (*White & Whiteley*, 1983) *For any graph G that is generically isostatic in the plane there is a polynomial pure condition* $C_G(\mathbf{p}_1, \ldots, \mathbf{p}_{|V|})$, *of degree* $\mathrm{val}_i - 1$ *in each point of valence* val_i *in the graph, such that all* $|E| \times |E|$ *minors of the ridigity matrix are trivial linear factors times the pure condition.*

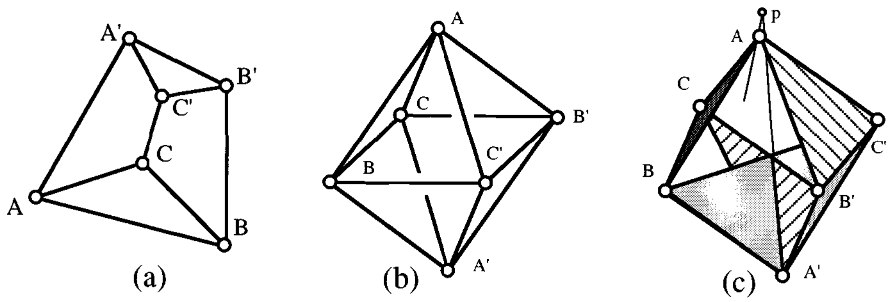
1.7.3. Example. We have seen that the graph of Figure 1.19a is generically isostatic in the plane (section 1.2), and, by Maxwell's Theorem of section 1.3, that this framework is dependent if and only if it is the projection of a spatial configuration with at least two distinct faces. We found three conditions which guarantee such a projection:

(i) if the three points A, B, and C form a collinear triangle;
(ii) if the three points A', B', and C' form a collinear triangle;
(iii) if the three lines AA', BB', and CC' are concurrent.

These three conditions give three irreducible polynomial factors for the pure condition of the graph which are of total degree 12 in the points. Thus they are the complete pure condition of the graph. This pure condition is also equivalent to the single geometric condition (which is algebraically reducible): the two triangles ABC and $A'B'C'$ are perspective from a line (i.e. the usual conclusion of Desargues' theorem holds).

1.7.4. Example. In 3-space, the graph of an octahedron is generically isostatic as a triangulated convex sphere (Figure 1.19b). A similar analysis of the minors shows that all $|E| \times |E|$ minors of the rigidity matrix are either simple multiples of a basic pure condition that is of degree $\mathrm{val}_i - 2$ in each point, or are identically zero. A classical theorem shows that the pure condition is the projective condition that four opposite faces ABC, $AB'C'$, $A'BC'$, and $A'B'C$ are on planes through a common point $\mathbf{p}$ (Figure 1.19c).

Figure 1.19.



Since a rigidity matrix for these special positions represents a specialization of the variables for a generic position, the corresponding matroid is a weak-map image of the generic rigidity matroid. The new circuits created in these specializations are *geometric circuits* rather than the *generic circuits* of the generic realizations (see Exercises 1.56, 1.57, and 1.58).

As the examples have hinted, the critical forms for infinitesimal rigidity are given by *projective geometric conditions*. Engineers in the last century realized that the statics and infinitesimal mechanics of a framework are invariant under projective transformations and Exercise 1.10 asks the reader to confirm that the rank of the plane rigidity matrix is projectively invariant. There is also projective geometry buried in the special positions of the spline matroid, the scene analysis matroid, and the parallel redrawing matroid (Exercises 1.39, 1.51, and 1.53).

Example 1.7.1 and Theorem 1.7.2 indicated that single polynomial 'pure conditions' control the lowering of the rank of matrix of a generic basis in the rigidity matrix. A similar analysis leading to polynomial 'pure conditions' applies to the other matrices from splines, scene analysis, etc. There is a rich field of research in this interplay of geometry and combinatorics arising in the matroids of critical forms.

1.8. Conclusions

The families of rigidity matroids, spline matroids, and scene analysis matroids represent only the best known and more accessible examples of the large collection of matroids that arise in discrete applied geometry. A number of related matroids are presented in the exercises.

We underline some common themes of these examples from discrete applied geometry.

(1) The matroid is defined by a sparse matrix. The non-zero entries correspond to 'coordinates' of geometric objects in the real plane (3-space, etc.), and follow a combinatorial pattern based on a graph or incidence structure.

For sections 1.1–1.5, this incidence structure was the underlying graph. We then replaced each non-zero entry by a d-vector (drawn from the geometry) and each zero entry by the zero d-vector. The result was an expanded pattern which passes through d copies of graphic matroid (with variables), and then an algebraic (or geometric) specialization. Our task was to trace the effects of this specialization of the entries.

More generally, as in section 1.6, we begin with an incidence matrix, and make vector substitutions under one 'type' of vertex (e.g. the faces). In other applications, we even replace entries by rectangular matrices rather than single rows, creating a polymatroid. However, all of these examples carry properties inherited from the incidence matrix.

(2) Analysis of the matroid is based on the dual concepts of row dependences (self-stresses) and column dependences (infinitesimal motions).

(3) If variables are used for these entries, we find the 'generic' matroid for the class of geometric realizations. Independent sets, and bases, in the generic matroid can often, but not always, be characterized by simple counts (semimodular functions) on the underlying combinatorial structures.

(4) The cases where simple counts fail (rigidity in 3-space, conic-rigidity, etc.) point to a fundamental problem in defining matroids by semimodular functions that are negative on small sets. The inductive techniques being developed represent matroidal approaches to a matroid problem.
(5) There are fundamental similarities in apparently disparate geometric problems. These similarities become accessible in the patterns of the sparse matrices, and in the matroidal properties. In new examples, the patterns of the sparse matrix suggest the appropriate analog among our basic patterns, and permit an easy transfer of methods among the examples.
(6) The behaviour of non-generic examples creates polynomial conditions that give interesting interactions of projective geometry, algebraic geometry, and combinatorics in the specialized matroids.

We conclude as we began. Engineering and computer science examples create a set of represented matroids for which the combinatorics of even generic examples are currently unsolved. The existing work applies, and illustrates, a number of matroid methods, including circuit elimination, matroid union, and truncation. The examples also suggest some new approaches to represented matroids.

1.9. Historical Notes

The results of this chapter are drawn from a number of scattered papers stretching over the last 130 years. The explicit connections among the various concepts of rigidity appear in Gluck (1975) and Asimow & Roth (1978) and are presented in great detail in Connelly (1987). The explicit use of matroid methods for generic bar frameworks structures is drawn from Crapo (1985), Graver (1984), Lovász & Yemini (1982), Recski (1984, 1989), and Whiteley (1988a).

Theorem 1.2.3 is a combination of Laman's theorem (1970) and Lovasz & Yemini (1982). The proof follows Whiteley (1988a) and the 2-frame represents a matroid union of two copies of the graph. The inductive results in the last century are summarized in Henneberg (1911), and are developed in Tay & Whiteley (1985). The analysis of the semimodular functions in terms of matroid unions and truncations is related to classical matroid results of Edmonds (1970), Pym & Perfect (1970), and Brylawski (1985a, b).

Maxwell's theorem dates to the development of 'graphical statics' by engineers and geometers in the last century (Maxwell, 1864; Cremona 1890). The classical geometric approach is presented anew in Crapo & Whiteley (1988), along with the converse and its extensions. The approach to Maxwell's theorem and its converse used in section 1.2 is based on Whiteley (1982).

The infinitesimal rigidity of convex triangulated spherical polyhedra dates back implicitly to Cauchy (1831), and explicitly to Dehn (1916). The generic results were extracted by Gluck (1975) and simplified in Graver (1984), and in Tay & Whiteley (1985). The topological development of triangulated surfaces by vertex splitting is found in Barnette (1982), and greatly extended in Fogelsanger (1988). That vertex splitting preserves infinitesimal rigidity and conic-independence is found in Whiteley (1991a, 1991c).

The problem of the generic dimension of the space of C^1_2-splines was raised by Strang (1973), and studied in Schumaker (1979) and in Alfeld (1987). The basic result was proven in Billera (1986) and in Whiteley (1991b). The approach presented here is related to techniques of Chui & Wang (1983) and was developed by explicit analogy to spatial frameworks in Whiteley (1986c, 1990a, 1991b).

The explicit matroid for general scene analysis was presented in Sugihara (1984), with a conjecture for Corollary 1.6.5. Whiteley (1988b) gave the first proof of the theorem, implicitly re-proving Crapo (1985) for the result on elementary pictures. Sugihara (1986) presents this proof anew, with a full background on the motivation and related problems in computer reconstruction of spatial objects. The dual parallel redrawing matroids were developed in Whiteley (1986a, 1988b). Parallel redrawing of spatial polyhedra and convex polytopes is basic to the study of Minkowski decomposition of polytopes (Shephard, 1963; Kallay, 1982; Smilansky, 1987).

Particular examples of geometric 'critical forms' date back to the last century. The explicit study of 'pure conditions' was carried out in White & Whiteley (1983, 1987). The general behaviour of bipartite frameworks appears for statics in Bolker & Roth (1980) and for infinitesimal mechanics in Whiteley (1984a). The projective invariance of statics and infinitesimal mechanics dates back at least to Rankine (1863), and is modernized in Roth & Whiteley (1981). The projective invariance of parallel redrawing appears in Kallay (1982) and is proven for C^1_2-splines in Whiteley (1986b, 1991b).

Exercises

A difficult problem is indicated by *, an unsolved one by **.

1.1. Show that for any graph $G = (V, E)$ and any $m > 0$, the function $f(E) = m|V| + k$ defines a non-decreasing semimodular function on subsets of the edges.

1.2. If, for $k > -2m$, we define circuits $|E'|$ on the edges of a graph by $|E'| = m|V'| + k + 1$ and $|E''| \leqslant m|V'| + k$, verify directly that the circuit elimination axiom for matroids is satisfied.

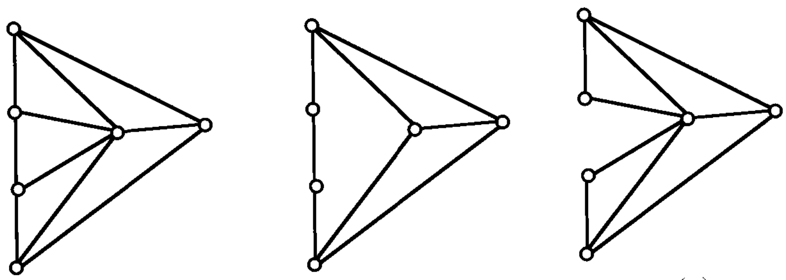
1.3. Show that the semimodular function $f(E) = |V| - 1$ defines the cycle matroid of a graph or multi-graph.

1.4. Show that the semimodular function $f(E) = |V|$ defines the bicycle matroid of a graph.

1.5. (Whiteley, 1988a) Show that the semimodular function $f(E) = 2|V| - 1$ defines the matroid union of the bicycle matroid and the cycle matroid of a graph. (This is the generic matroid for bar frameworks on the surface of a cone).

1.6. Show that the frameworks in Figures 1.20b and 1.20c are each minimal rigid plane frameworks, i.e. removing any one bar leaves a finite motion. Conclude that 'rigidity' does not induce a matroid on the plane framework in Figure 1.20a.

Figure 1.20.



(a) (b) (c)

1.7. (a) Show that if $G(\mathbf{p})$ and $G(\mathbf{q})$ are two frameworks with the same bar lengths for all edges, then $\mathbf{p}-\mathbf{q}$ is an infinitesimal motion of the framework $G((\mathbf{p}+\mathbf{q})/2)$.
(b) Show that $\mathbf{p}-\mathbf{q}$ is a non-trivial infinitesimal motion of $G((\mathbf{p}+\mathbf{q})/2)$ if $G(\mathbf{p})$ is not congruent to $G(\mathbf{q})$.
(c) Give an example where $G(\mathbf{p})$ is congruent to $G(\mathbf{q})$ and $\mathbf{p}-\mathbf{q}$ is a non-trivial infinitesimal motion of $G((\mathbf{p}+\mathbf{q})/2)$.
(d) Show that if $G(\mathbf{p}_i)$ is a sequence of frameworks all with the same bar lengths but not congruent to $\mathbf{p}$, such that $\lim \mathbf{p}_i = \mathbf{p}$, then $G(\mathbf{p})$ has a non-trivial infinitesimal motion.
(e) (Local uniqueness) Show that if $G(\mathbf{p})$ is infinitesimally rigid, then there is an open neighborhood $N(\mathbf{p})$ in $\mathbb{R}^{2|V|}$ such that if $G(\mathbf{q})$ has the same edge lengths as $G(\mathbf{p})$ and $\mathbf{q}\in N(\mathbf{p})$, then $G(\mathbf{q})$ is congruent to $G(\mathbf{p})$.

1.8. Show that an infinitesimal motion $\mathbf{u}$ of a non-collinear plane framework $G(\mathbf{p})$ is non-trivial if and only if there is a pair of joints $\mathbf{p}_h$, $\mathbf{p}_k$ (not joined by a bar) such that

$$(\mathbf{p}_h-\mathbf{p}_k)\cdot(\mathbf{p}_h'-\mathbf{p}_k')\neq 0.$$

1.9. (Whiteley, 1984a) Recall that a plane conic centered at the origin (an ellipse, an hyperbola, or two lines) can be written $[\mathbf{p}_1][Q][\mathbf{p}_1]^t=r$ for a symmetric matrix $[Q]$ and a constant r. Show that any bipartite graph $K_{A,B}$, a framework with all its joints on such a conic has a non-trivial infinitesimal motion

$$\mathbf{a}_i'=[\mathbf{a}_i][Q] \quad \text{and} \quad \mathbf{b}_j'=[\mathbf{b}_j][Q]$$

for $\mathbf{a}_i\in A$ and $\mathbf{b}_j\in B$. Show that this is a non-trivial infinitesimal motion unless the conic contains all lines joining $(\mathbf{a}_h, \mathbf{a}_i)$ and $(\mathbf{b}_j, \mathbf{b}_k)$.

1.10. (Projective invariance) Show that if a non-singular projective transformation in the plane, T, takes the points $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_{|V|})$ to the points $\mathbf{q} = (T(\mathbf{p}_1), \ldots, T(\mathbf{p}_{|V|}))$ then the bar frameworks $G(\mathbf{p})$ and $G(\mathbf{q})$ define the same matroid for any graph G. (Hint: prove that T takes a dependent set to a dependent set, using self-stresses and the projective weights of the transformed points.)

1.11. (Crapo, 1988) Show that a graph $G = (V, E)$ is generically isostatic in the plane if and only if the set E of edges is the disjoint union of three trees T_i such that each vertex of G is incident with exactly two of the trees T_i, and with the further property that no two distinct subtrees of the trees T_i have the same span.

1.12. Given a graph $G = (V, E)$, the *cone graph with* v_0 is the graph $G * v_0$ formed by adding one new vertex v_0 having edges to all vertices of G (Figure 1.15a). Show, from the counts, that the cone graph $G * v_0$ is generically isostatic in the plane if and only if G is generically isostatic on the line.

1.13. (a) Show that adding a 2-valent joint to an isostatic plane framework gives an isostatic plane bar framework, if the new bars are not collinear.
(b) Show that deleting a bar (1, 2) from an isostatic plane framework, and inserting a 3-valent joint $\mathbf{p}_0$, with non-collinear bars (0, 1), (0, 2), and (0, 3) gives an isostatic plane bar framework for generic positions of $\mathbf{p}_0$.
(c) (Tay & Whiteley, 1985) Show that every generically 2-isostatic graph is generated from a single edge by a *Henneberg* 2-*sequence* of steps (a) and (b), and that every graph with such a sequence is generically 2-isostatic.

1.14. Give a Henneberg 2-sequence for each graph in Figure 1.4a.

1.15. Show that a set of edges E' is a circuit in the plane generic rigidity matroid if and only if $|E'| = 2|V'| - 2$ and $|E''| \leqslant 2|V''| - 3$ for all proper subsets.

*1.16. (Lovász & Yemini, 1982) If a graph is 6-connected in a vertex sense, then it has realizations as an infinitesimally rigid plane bar framework.

1.17. (a) Show that all resolved loads on a plane framework are equilibrium loads.
(b) Show that for a plane framework with at least 2 distinct joints the equilibrium loads form a space of dimensions $2|V| - 3$.

*1.18. (a) Extend the definition of polyhedra to 2-connected planar graphs.
(b) (Crapo & Whiteley, 1988) Extend Theorems 1.3.3 and 1.3.5 to these 2-connected polyhedra.
(c) Show that every self-stress on a plane framework is the sum of self-stresses on subframeworks with 2-connected graphs.
(d) Conclude that a plane framework with a planar graph has a non-trivial self-stress if and only if it contains the projection of a spatial polyhedron with at least two distinct face planes.

*1.19. (a) (Crapo & Whiteley, 1988) Extend Theorem 1.3.3 to projections of arbitrary oriented polyhedra from 3-space.
(b) Show that all self-stresses from oriented projected polyhedra satisfy the cycle condition

$$\sum_{\Pi} \omega_{jk}(y_j - y_k, x_k - y_j, x_j x_k - x_k y_j) = (0, 0, 0)$$

for each face–edge cycle in the oriented polyhedron.
(c) (Crapo & Whiteley, 1988) Assume that a self-stress on the graph of an

abstract oriented polyhedron satisfies the cycle condition:

$$\sum_{\Pi} \omega_{jk}(y_j - y_k,\ x_k - x_j,\ x_j y_k - x_k y_j) = (0, 0, 0)$$

for each face–edge cycle in the oriented polyhedron. Show that this framework is the projection of a spatial polyhedron, with distinct face planes over any edge with a non-zero scalar in the self-stress.

(d) Give an example of the graph of a toroidal polyhedron with a self-stress that violates the cycle condition and hence is not the projection of a toroidal polyhedron.

1.20. (a) (Whiteley, 1984a) Extend Exercise 1.9 to give an infinitesimal motion of any bipartite framework in 3-space with all joints on a quadric surface centered at the origin: $[\mathbf{p}_1][Q][\mathbf{p}_1]^t = r$.

(b) When is this infinitesimal motion non-trivial?

1.21. (a) A *simple Henneberg 3-sequence* is a sequence of graphs beginning with a triangle, and the following steps:

(1) adding a 3-valent vertex;

(2) removing one edge $\{1, 2\}$ and adding a 4-valent vertex $\mathbf{p}_0$, with edges $\{0, 1\}$, $\{0, 2\}$, $\{0, 3\}$, and $\{0, 4\}$.

Show that any graph with a simple Henneberg 3-sequence is generically 3-rigid.

(b) Show that the graph of an octahedron is generically 3-isostatic, by constructing a simple Henneberg 3-sequence.

(c) Show that the bipartite graph $K_{4,6}$ is generically 3-isostatic, by constructing a simple Henneberg 3-sequence.

1.22. (a) Show that the graph of Figure 1.9c is generically non-rigid in 3-space.

(b) Show that this graph is a generic circuit in 3-space.

1.23. Show that the cone graph $G * v_0$ is generically 3-isostatic if and only if G is generically isostatic in the plane.

1.24. (a) Show that a 5-valent vertex, in an isostatic framework in 3-space with no four joints coplanar, can be replaced by one of the two patterns in Figure 1.11a, producing smaller isostatic framework(s).

(b) Give an example of a 5-valent vertex that requires the second replacement principle.

(c) Give an example of a graph, with all vertices 5-valent, which is generically isostatic in 3-space.

1.25. (a) Give an example of a graph that is 11-connected in a vertex sense, but for which no spatial realizations are infinitesimally rigid in 3-space.

** (b) (Lovász & Yemini, 1982) Show that if a graph is 12-connected in a vertex sense, then it has realizations as an infinitesimally rigid bar framework in 3-space.

**1.26. (a) Show that the graph of any triangulated sphere has realizations as an isostatic framework with all joints on the unit sphere.

(b) Characterize the graphs that give isostatic frameworks for some realization with joints on the unit sphere.

*1.27. (Whiteley, 1987) Show that the graph of a triangulated sphere that is 4-connected in a vertex sense, plus one edge, is a circuit in the generic rigidity matroid for 3-space (Figure 1.15b). (Hint: use vertex splitting.)

*1.28. (Fogelsanger, 1988) Show that the graph of a triangulated surface in 3-space spans the generic rigidity matroid for its vertices in 3-space.

1.29. Give the plane analog of the vertex splitting theorem 1.4.6.

1.30. (a) Give the definition of infinitesimal rigidity for a bar framework in d-space. Show that an isostatic bar framework $G(\mathbf{p})$ in d-space with $|V| \geqslant d$ satisfies

$$|E| = d|V| - \frac{d(d+1)}{2}$$

and

$$|E'| \leqslant d|V'| - \frac{d(d+1)}{2}$$

for all subsets $|V'| \geqslant d$.

(b) Give the definition of static rigidity for a bar framework in d-space. Show that infinitesimal rigidity and static rigidity are equivalent for bar frameworks in d-space.

(c) Give an example, for each $d > 2$, of a framework that is not generically rigid, but satisfies $|E| = d|V| - d(d+1)/2$ and $|E'| \leqslant d|V'| - d(d+1)/2$ for all subsets $|V'| \geqslant d$.

1.31. (Bolker & Roth, 1980) Show that generic realizations of the graph $K_{7,7}$ in 4-space are not infinitesmially rigid. (Hint: 14 points lie on a quadric in 4-space.) What is the dimension of its space of self-stresses?

*1.32. (a) Show that the cone framework $G{*}v_0(\mathbf{p})$ is infinitesimally rigid in d-space if and only if the central projection $G(\Pi(\mathbf{p}))$ from apex $\mathbf{p}_0$ is infinitesimally rigid in $(d-1)$-space.

1.33. A *simplicial* 4-*manifold* is a collection of tetrahedra such that each triangular face is shared by at most two tetrahedra, and the *link* of each vertex v_0 (the edges that form a triangular face of a tetrahedron with v_0) forms a triangulated 2-sphere. The manifold is *strongly connected* if each pair of tetrahedra is connected by a path of tetrahedra and shared triangles. Show that a strongly connected 2-manifold satisfies the lower bound (Barnette, 1973)

$$|E| \geqslant 4|V| - 10$$

by the following sequence of steps (Kalai, 1987):

(a) The star of each vertex (the edges of triangles sharing this vertex) is generically rigid in 3-space. (Hint: use Exercise 1.32.)

(b) If two infinitesimally rigid frameworks in 4-space share four vertices, their union is also infinitesimally rigid.

(c) The graph of a strongly connected simplicial 4-polytope is generically rigid in 4-space.

1.34. (a) (Gromov, 1986; Kalai, 1987) A *simplicial* 4-*pseudomanifold* allows the vertex link to be any triangulated 2-surface. Use Exercise 1.28 to extend the results of Exercise 1.33 to strongly connected pseudo-manifolds.

(b) (Barnette, 1973; Kalai, 1987) Generalize Exercise 1.33 to give the lower bound

$$|E| \geqslant d|V| - \frac{d(d+1)}{2}$$

for strongly connected simplicial d-manifolds.

1.35. (a) Show that if a conic-independent geometric graph $G(\mathbf{p})$ in the plane has three joints $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, then inserting a 3-valent joint $\mathbf{p}_0$ with no two of the lines $(\mathbf{p}_0, \mathbf{p}_1)$, $(\mathbf{p}_0, \mathbf{p}_2)$, or $(\mathbf{p}_0, \mathbf{p}_3)$ parallel creates a conic-independent geometric graph.

(b) Conversely, given a conic-independent geometric graph with a 3-valent vertex, show that deleting this joint with its bars leaves a conic-independent geometric graph.

1.36. (a) Show that if a conic-independent geometric graph has four joints $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, and $\mathbf{p}_4$, of which no three are collinear, and an edge $(\mathbf{p}_1, \mathbf{p}_4)$, then removing this edge and inserting a vertex $\mathbf{p}_0$ with edges $(\mathbf{p}_0, \mathbf{p}_1)$, $(\mathbf{p}_0, \mathbf{p}_2)$, $(\mathbf{p}_0, \mathbf{p}_3)$, and $(\mathbf{p}_0, \mathbf{p}_4)$ creates a conic-independent geometric graph for almost all positions of $\mathbf{p}_0$.

(b) Conversely, given a conic-independent geometric graph with a 4-valent vertex $\mathbf{p}_0$, show that there is an edge $(\mathbf{p}_i, \mathbf{p}_j)$ among the vertices adjacent to $\mathbf{p}_0$ such that deleting $\mathbf{p}_0$ and inserting $(\mathbf{p}_i, \mathbf{p}_j)$ leaves a conic-independent geometric graph.

1.37. Prove that if a conic-rigid (independent) geometric graph $G(\mathbf{p})$ contains two disjoint edges (1, 2) and (3, 4) and an additional vertex v_5, then the geometric graph with edges (1, 2), (3, 4) removed and an added point $\mathbf{p}_0$ joined by edges (0, 1), (0, 2), (0, 3), (0, 4), and (0, 5) is conic-rigid (independent) for generic positions $\mathbf{p}_0$.

1.38. (a) (Chui & Wang, 1983) Prove that two quadratic surfaces

$$z = (A^h x^2 + B^h xy + C^h y^2 + D^h x + E^h y + F^h)$$

and

$$z = (A^i x^2 + B^i xy + C^i y^2 + D^i x + E^i y + F^i)$$

meet over a line $px + qy + r = 0$, with common tangent planes, if and only if

$$\begin{aligned}(A^h x^2 + B^h xy + D^h x + E^h y + F^h) - (A^i x^2 + B^i xy + C^i y^2 + D^i x + E^i y + F^i)\\ = \omega^{hi}(p^2x^2 + 2pqxy + q^2y^2 + 2prx + 2qry + r^2).\end{aligned}$$

(b) Given any set of scalars ω_{ij} assigned to the interior edges of a triangulated plane disc, such that at each interior vertex

$$\sum_j \omega_{ij}(A_{ij}, B_{ij}, C_{ij}) = \mathbf{0},$$

show that there are unique scalars on the three edges of the exterior triangle such that $\sum_j \omega_{ij}(A_{ij}, B_{ij}, C_{ij}) = \mathbf{0}$ at the exterior vertices as well.

1.39. (Projective invariance of conic-rigidity; Whiteley, 1986b) Show that if a non-singular projective transformation in the plane, T, takes the points $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_{|V|})$ to the points $\mathbf{q} = (T(\mathbf{p}_1), \ldots, T(\mathbf{p}_{|V|}))$ then the geometric graphs $G(\mathbf{p})$ and $G(\mathbf{q})$ define the same conic-rigidity matroid for any graph G.

1.40. For a plane geometric graph $G(\mathbf{p})$ define the *triple-line vector* for a directed edge (i, j) as

$$\mathbf{D}_{ij}^3 = (p_{ij}^3, 3p_{ij}^2 q_{ij}, 3p_{ij} q_{ij}^2, q_{ij}^3)$$
$$= ((y_j - y_i)^3, 3(y_j - y_i)^2(x_i - x_j), 3(y_j - y_i)(x_i - x_j)^2, (x_i - x_j)^3) \quad \text{for } i < j$$

and

$$\mathbf{D}_{ji}^3 = -\mathbf{D}_{ij}^3 \quad \text{for } i < j.$$

Define $RCUB(G, \mathbf{p})$ as the $-|E| \times 4|V|$ *cubic-rigidity matrix* for a geometric graph with the following row for an edge (i, j):

$$[0\,0\,0 \;\ldots\; \mathbf{D}_{ij}^3 \;\ldots\; -\mathbf{D}_{ij}^3 \;\ldots\; 0\,0\,0]$$

(a) Prove that the solution space to $RCUB(G, \mathbf{p}) \times \mathbf{u}^t = \mathbf{0}$ has dimension at least 10.

(b) Show that the graph of a tetrahedron is cubic-rigid for almost all plane geometric graphs $G(\mathbf{p})$.

(c) Show that adding a general position 4-valent vertex to a cubic-rigid geometric graph produces a new cubic-rigid geometric graph.

(d) Conclude that for the generic geometric graph $K_m(\mathbf{p})$ in the plane, the cubic-rigidity matrix defines a matroid of rank $4|V| - 10$.

(e) Show that dependences in the cubic-rigidity matroid correspond to C_3^2-splines over a planar drawing of a 2-connected graph with triangular boundary.

1.41. (a) Prove that if a cubic-rigid (independent) geometric graph $G(\mathbf{p})$ contains two disjoint edges (1, 2) and (3, 4) and two additional vertice v_5, v_6, then the geometric graph with edges (1, 2), (3, 4) removed and with an added point $\mathbf{p}_0$ joined by edges (0, 1), (0, 2), (0, 3), (0, 4), (0, 5), and (0, 6) is cubic-rigid (independent) for generic positions $\mathbf{p}_0$.

(b) (Maehara, 1988) Show that the graph $K_{6,7}$ comes from the graph K_6 minus one edge, by a sequence of double-edges replacements of type (a).

(c) Show that $K_{6,7}$ is generically cubic-rigid.

(d) Show, by using a quadric surface in 4-space, that $K_{6,7}$ is not generically rigid in 4-space.

(e) Prove that the generic cubic-rigidity matroid and the generic 4-space rigidity are different.

** (f) Prove that each set of edges that is independent in the generic 4-space rigidity is independent in the generic cubic-rigidity matroid.

1.42. (a) Define the *d-fold line vector* for a directed edge (i, j) of a plane geometric graph $G(\mathbf{p})$, and the corresponding $\|E| \times (d+1)|V|$ *d-fold-rigidity matrix* for a geometric graph. Show that for the generic geometric graph $K_{|V|}(\mathbf{p})$ in the plane, with $|V| \geqslant d$, the d-fold-rigidity matrix defines a matroid of rank $(d+1)|V| - (d+2)(d+1)/2$.

(b) Prove that the generic d-fold-rigidity matroid and the generic $(d+1)$-space rigidity are different, for $d \geqslant 3$.

(c) Show that dependences in the d-fold-rigidity matroid correspond to C_d^{d+1}-splines over a planar drawing of a 2-connected graph with a triangular boundary.

1.43. (White & Whiteley, 1987) A *k-frame* $G(\mathbf{d})$ is a multi-graph $G = (V, E)$ (without loops) and an assignment $\mathbf{d}$ of vectors $\mathbf{d}_e \in \mathbb{R}^k$ to the edges. An *infinitesimal* *motion* of a k-frame is an assignment $\mathbf{u}$ of centers $\mathbf{u}_i \in \mathbb{R}^k$ to the vertices such that for each edge e joining v_i and v_j, $d_e \cdot (\mathbf{u}_i - \mathbf{u}_j) = 0$. A k-frame is *infinitesimally rigid* if every infinitesimal motion is *trivial*, with all $\mathbf{u}_i = \mathbf{c}$ for a fixed vector $\mathbf{c}$. A k-frame $G(\mathbf{d})$ is *isostatic* if $G(\mathbf{d})$ is a minimal infinitesimally rigid k-frame on its vertices (i.e. $|E| = k|V| - k$ and $G(\mathbf{d})$ is infinitesimally rigid).

Prove that for a multi-graph G, the following are equivalent:

(1) $G(\mathbf{d})$ is an isostatic k-frame for some $\mathbf{d} \in \mathbb{R}^{k|E|}$;

(2) G is the union of k edge-disjoint spanning trees;

(3) G satisfies $|E| = k|V| - k$ and $|E'| \leqslant k|V'| - k$ for all non-empty subsets E'.

1.44. (Whitely, 1988a) Given a graph and a partition of the vertices $(\ldots, V_j, \ldots)$ of a multi-graph, the *contracted graph* $G^*(V^*, E^*)$ identifies all vertices of each partition class V_j and removes all loops.

(a) Prove that a multi-graph has realizations as an infinitesimally rigid k-frame if and only if $|E^*| \geqslant k|V^*| - k$ for all contracted graphs G^* of partitions of the vertices.

(b) Show that a multi-graph G that is $2k$-connected in an edge sense is rigid as a generic k-frame.

1.45. A non-zero 3-vector $\mathbf{L}^i = (L_1^i, L_2^i, L_3^i)$ represents a *line* in the plane: $L_1^i x + L_2^i y + L_3^i = 0$. A *bar and body framework* in the plane is a multi-graph G, and an assignment $\mathbf{L}$ of lines $\mathbf{L}_{ij}$ to the edges of the graph. The *rigidity matrix* for the framework is the rigidity matric for the corresponding 3-frame. (Visualize the lines as bars joining large rigid bodies for the vertices joined by bars along the lines of the edges.)

(a) Prove that for a multi-graph G, the following conditions are equivalent:

(1) $G(\mathbf{L})$ is an isostatic bar and body framework in the plane for some $\mathbf{L} \in \mathbb{R}^{3|E|}$;

(2) G is the union of three edge-disjoint spanning trees;

(3) G satisfies $|E| = 3|V| - 3$ and $|E'| \leqslant 3|V'| - 3$ for all non-empty subsets E'.

(b) Show that a multi-graph G that is 6-connected in an edge sense has realizations as an infinitesimally rigid bar and body framework in the plane.

(c) (Whiteley, 1984c) Give an interpretation of the reduced simple picture matrix as a bar and body framework in the plane.

1.46. A 6-vector $\mathbf{L}^i = (L_1^i, L_2^i, L_3^i, L_4^i, L_5^i, L_6^i)$ represents a *line* in 3-space if and only if $L_1^i L_4^i + L_2^i L_5^i + L_3^i L_6^i = 0$. (These are the Plücker coordinates of the line.) A *bar and body framework* in 3-space is a multi-graph G, and an assignment $\mathbf{L}$ of lines $\mathbf{L}_{ij}$ to the edges of the graph. The *rigidity matrix* for the framework is the rigidity matrix for the corresponding 6-frame.

(a) (Tay, 1984) Prove that for a multi-graph G, the following conditions are equivalent:

(1) $G(\mathbf{L})$ is an isostatic bar and body framework in 3-space for some $\mathbf{L} \in \mathbb{R}^{6|E|}$;

(2) G is the union of 6 edge-disjoint spanning trees;

(3) (Tay, 1984) G satisfies $|E| = 6|V| - 6$ and $|E'| \leqslant 6|V'| - 6$ for all non-empty subsets E'.

(b) Give an example of a multi-graph G that is 11-connected in an edge sense, but has no realizations as an infinitesimally rigid bar and body framework in 3-space.

1.47. Define a *body and hinge framework* in 3-space as a graph G and an assignment of hinge lines $\mathbf{H}^{ij}$ to the edges of G. An *infinitesimal motion* of a body and hinge framework is an assignment of *screw centers* $\mathbf{S}^i = (S_1^i, S_2^i, S_3^i, S_4^i, S_5^i, S_6^i)$ to the vertices such that, for each hinge, $\mathbf{S}^i - \mathbf{S}^j = \omega^{ij}\mathbf{H}^{ij}$ for a scalar ω^{ij}. A body and hinge framework is infinitesmially rigid if the only infinitesimal motions are trivial, with all $\mathbf{S}^i$ the same.

(a) Show that a hinge between two rigid bodies in 3-space is equivalent to five bars passing through the line of the hinge.

(b) (Tay & Whiteley, 1983; Whiteley, 1988a) Show that a body and hinge framework in 3-space is independent if and only if $5|E'| \leqslant 6|V'| - 6$ for all non-empty subsets of edges.

(c) (Tay & Whiteley, 1983; Whiteley, 1988a) Show that a body and hinge framework in 3-space is generically infinitesimally rigid if and only if $5|E^*| \geqslant 6|V^*| - 6$ for all contracted graphs G^* from partitions of the vertices.

1.48. (a) Show that the faces and edges of a spherical polyhedron define a graph that is infinitesimally rigid as a generic body and hinge framework in 3-space.

(b) (Crapo & Whiteley, 1982) Show that the faces and edges of a spherical polyhedron form an infinitesimally flexible body and hinge framework in 3-space if and only if the edges and vertices form a bar framework with a non-trivial self-stress.

1.49. (a) Show that a multi-graph $G = (F, E)$ has realizations as an independent picture of a simple picture if and only if the graph, with any added edge, is the union of three edge-disjoint spanning trees.

(b) Show that a multi-graph $G = (F, E)$ has realizations as an independent picture of a sharp simple picture if and only if the graph is the union of four edge-disjoint trees, and no three subtrees span the same vertices.

1.50. (a) Extend the definitions of section 1.6 to elementary pictures, and general pictures in $(d-1)$-space of scenes in d-space.

(b) Show that a hypergraph H of $(d+1)$-tuples has realizations as an independent $(d-1)$-picture if and only if $|F'| \leqslant |V'| - d$ for every non-empty subset of faces F'.

(c) Show that a generic picture in $(d-1)$-space of an incidence structure $S = (V, F; I)$ has sharp scenes in d-space if and only if $|I'| \leqslant |V'| + d|F'| - (d+1)$ for all subsets I' with at least two faces.

1.51. (Projective invariance of scene analysis) Show that the rank of the picture matrix is unchanged by a projective transformation of the picture.

1.52. (a) Show that for simple scenes in the plane, the independent sets are characterized by $|E'| \leqslant 2|V'| - 3$ for all non-empty subsets.

(b) Show that a plane geometric graph $G(\mathbf{p})$ at a generic point $\mathbf{p}$ is independent for parallel redrawings if and only if

$$|E| = 2|V| - 3 \quad \text{and} \quad |E'| \leqslant 2|V'| - 3 \quad \text{for all proper subsets.}$$

(c) Give an isomorphism between parallel redrawings of a plane geometric graph and infinitesimal motions of the corresponding bar framework, which takes trivial parallel redrawings to trivial infinitesimal motions.

1.53. (a) For a configuration of points and planes in space, satisfying a fixed incidence structure $S = (V, F; I)$, define the *parallel redrawing matrix*.

(b) (Projective invariance of parallel redrawing; Kallay, 1982) Show that the rank of the parallel redrawing matrix in 3-space is unchanged by a projective transformation of the points and planes of a configuration.

1.54. (a) (Whiteley, 1986a) Describe the parallel redrawing polymatroid for geometric graphs in 3-space, and characterize the independent sets for generic geometric graphs.

(b) (Whiteley, 1986a) Show that a non-trivial parallel redrawing of a geometric graph in 3-space induces a set of non-trivial infinitesimal motions of the corresponding bar framework.

(c) Give an example of a geometric graph in 3-space with only trivial parallel drawings, but with non-trivial infinitesimal motions as a bar framework.

(d) (Shephard, 1963) Show that the graph of a triangulated spherical polyhedron has only trivial parallel redrawings in 3-space.

(e) (Shephard, 1963) Generalize this to graphs of spherical polyhedra where any two vertices are joined by a path of triangular faces and edges.

1.55. (a) (Whiteley, 1990b) Choose an incidence structure $I \subset V \times F$ of vertices and faces that must be satisfied by spatial points and planes:

$$A^h x_j + B^h y_j + z_j + C^h = 0.$$

For each spatial realization, this creates a linear constraint on local changes to the face planes and the vertex points:

$$A^h(x_j)' + B^h(y_j)' + (z_j)' + (A^h)'x_j + (B^h)'y_j + (C^h)' = 0.$$

Over the set of incidences, this creates an $|I| \times (3|V| + 3|F|)$ matrix $RI(\mathbf{Q}, \mathbf{q})$. Show that for the incidences of a spherical polyhedron, and any realization with all faces and vertices distinct, the rows of $RI(\mathbf{Q}, \mathbf{q})$ are independent.

1.56. (a) Show that a collinear triangle is a geometric circuit for infinitesimal plane rigidity.

(b) A *simple plane framework* is a plane framework which can be created from a single bar by adding a sequence of 2-valent joints, with the two added bars not collinear. Show that a simple plane framework creates an independent rigidity matrix.

(c) Show that the plane framework on $K_{3,3}$ is a geometric circuit if the six points lie on a plane conic section, with no three collinear.

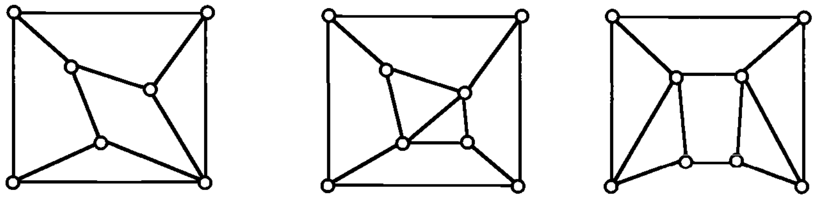
(d) Show that the projection of a triangular prism in 3-space, with all faces distinct non-vertical planes, is a geometric circuit as a plane framework.

1.57. Give the geometric conditions for frameworks on each of the graphs in Figure 1.21 to be geometric circuits.

1.58. (a) Show that for the graph of an octahedron with aa', bb', cc' not bars, the spatial framework $G(\mathbf{p})$ is dependent if and only if the six lines ab', $b'c$, ca', $a'b$, bc', and $c'a$ lie on a projective line complex in 3-space.

(b) Show that for the graph of an octahedron, the spatial framework $G(\mathbf{p})$ is dependent if and only if the four planes abc, $a'b'c$, $a'bc'$, and $ab'c'$ are concurrent in a point d.

Figure 1.21.



(a) (b) (c)

(c) (Whiteley, 1986b) Show that for the graph of an octahedron, the plane geometric graph $G(\mathbf{p})$ is conic-dependent if and only if the six lines ab', $b'c$, ca', $a'b$, bc', and $c'a$ are tangent to a conic section.

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