3

Unimodular Matroids

NEIL WHITE

3.1. Equivalent Conditions for Unimodularity

Unimodular matroids were defined in Chapter 1 as the class of matroids which may be coordinatized over every field. In Theorem 3.1.1 we give a number of equivalent characterizations of this class. Certainly the two most striking and powerful of these are Tutte's excluded minor characterization and Seymour's decomposition [conditions (8) and (9) of Theorem 3.1.1]. We first need some definitions and notation.

A coordinatization of $M(S)$ over $\mathbb{Q}$ given by $n \times N$ matrix A with integer entries, and $n < N$, is said to be *totally unimodular* if every $k \times k$ submatrix has determinant equal to 0 or ± 1, for all k, $1 \leqslant k \leqslant n$, and is said to be *locally unimodular* if every $n \times n$ submatrix has determinant equal to 0 or ± 1.

Let D be the bond-element incidence matrix of $M(S)$. That is, if $R_1, R_2, \ldots, R_m$ are the bonds of M and $S = \{x_1, x_2, \ldots, x_N\}$, then $D = (b_{ij})$, with $b_{ij} = 1$ if $x_j \in R_i$, and $b_{ij} = 0$ otherwise. Similarly, let E be the circuit-element incidence matrix of M. Suppose that it is possible to change some of the entries of D from 1 to -1 to get a matrix D', and similarly, change E to E', so that $D'(E')^t = 0$ over $\mathbb{Q}$ (where t denotes transpose). Then we say that M is *signable*. [This is closely related to the notion of orientability, considered in a chapter of White (1988).]

In Section 7.6 of White (1986) 1-sums, 2-sums, or (for binary matroids) 3-sums of two matroids $M_1(E_1)$ and $M_2(E_2)$ were defined as $P_x(M_1, M_2) - x$, where $P_x(M_1, M_2)$ is the generalized parallel connection across a flat x, and x is empty, a point, or a 3-point line (respectively). To avoid triviality we insist that $P_x(M_1, M_2) - x$ have larger cardinality than M_1 or M_2. For binary matroids, with which we are concerned here, an equivalent definition is to say that each of these three sums is the matroid $M_1 \triangle M_2$ on the symmetric difference $E_1 \triangle E_2$ which has as its cycles (i.e., disjoint unions of circuits) all subsets of the form $C_1 \triangle C_2$, where C_i is a cycle of M_i. Then

(A) $M_1 \triangle M_2$ is the 1-sum of M_1 and M_2 if $E_1 \cap E_2 = \varnothing$ and $E_1 \neq \varnothing$, $E_2 \neq \varnothing$.
(B) $M_1 \triangle M_2$ is the 2-sum of M_1 and M_2 if $E_1 \cap E_2 = \{e\}$, e is neither a loop nor an isthmus of M_1 or of M_2, and $|E_1| \geqslant 3$, $|E_2| \geqslant 3$.
(C) $M_1 \triangle M_2$ is the 3-sum of M_1 and M_2 if $E_1 \cap E_2 = L$, where $|L| = 3$, L is a line (and therefore L is a circuit) in each of M_1 and M_2, L includes no bond of M_1 or M_2, and $|E_1| \geqslant 7$, $|E_2| \geqslant 7$.

In fact, the 1-sum is just direct sum. The 2-sum is just pasting together of M_1 and M_2 at the common element e, followed by the deletion of e, so that the rank of $M_1 \Delta M_2$ is as large as possible, namely $rM_1 + rM_2 - 1$. The 3-sum is a similar pasting together along a common line, again keeping the rank as large as possible, namely $rM_1 + rM_2 - 2$.

The matroid R_{10} in the following theorem is given in Exercise 1.7. $U_{2,4}$ is the 4-point line, F_7 the 7-point Fano plane, and F_7^* the orthogonal dual of F_7.

A matroid is called *unimodular* (or *regular*) if it satisfies any of the conditions of the following theorem.

3.1.1. Theorem. *The following conditions are equivalent, for a matroid $M(S)$.*
(1) *M has a totally unimodular coordinatization over $\mathbb{Q}$.*
(2) *M has a locally unimodular coordinatization over $\mathbb{Q}$.*
(3) *The brackets for M may be assigned the values $0, \pm 1$ in $\mathbb{Q}$ so that the syzygies of Proposition 1.6.1 are satisfied.*
(4) *M may be coordinatized over K, for every field K.*
(5) *M may be coordinatized over $GF(2)$ and over K, for some K with char $K \neq 2$.*
(6) *M is signable.*
(7) *For every hyperplane H of M there exists a function $F_H : S \to \mathbb{Q}$ such that kernel $F_H = H$ for every H, image $F_H \subseteq \{0, 1, -1\}$, and for every three hyperplanes H_1, H_2, and H_3 containing a common coline, there exists α_1, α_2, and $\alpha_3 \in \{1, -1\}$ such that $\alpha_1 F_{H_1} + \alpha_2 F_{H_2} + \alpha_3 F_{H_3} = 0$.*
(8) *M has no minor isomorphic to $U_{2,4}$, F_7,* or *F_7^*.*
(9) *M may be constructed by 1-, 2-, and 3-sums from graphic matroids, cographic matroids, and matroids isomorphic to R_{10}.*

Proof of the equivalent of (1) through (5). (1)$\Rightarrow$(2) and (4)$\Rightarrow$(5) are trivial, and (2)$\Rightarrow$(3)$\Rightarrow$(4) are immediate from Proposition 1.6.1, where the bracket values $0, \pm 1 \in \mathbb{Q}$ are simply regarded as elements of the field K. Since the syzygies hold over $\mathbb{Q}$, they also hold mod p, where $p = \text{char } K$.

We now have only to prove (5)$\Rightarrow$(1). This proof is due to Brylawski (1975). Let $A = (I_n | L)$ be a coordinatization of $M(S)$ over K, where $M(S)$ is binary and char $K \neq 2$. We assume that A is in (B, T)-canonical form, where T is a spanning tree of the bipartite graph Γ whose adjacency matrix is determined by L (see Section 1.2). We now claim that each entry in L (and hence in A) is 0 or ± 1. Let w be a non-zero entry of L, other than one of the entries corresponding to T. Then w corresponds to an edge of $\Gamma - T$, and hence has a basic circuit C in Γ. We will prove that $w = \pm 1$ by induction on the size of C.

It is not difficult to see that the edges of C correspond to a cyclic sequence of $2k$ non-zero entries of L, for some $k \geqslant 2$, with the property that each odd-numbered entry in the sequence is in the same column as its predecessor, and each even-numbered entry in the same row as its predecessor. For example, the submatrix containing the sequence of entries may look like the following:

$$\begin{matrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & w \end{matrix}.$$

Now either these $2k$ entries are the only non-zero entries in a $k \times k$ submatrix of L, or if there are other entries, w forms a circuit of size less than $2k$ with entries which are either in T or themselves have basic circuits of size $< 2k$. By the induction hypothesis, these other entries are all ± 1, hence in any case we get a $j \times j$ submatrix J of L having exactly $2j$ non-zero entries, with 2 in each row and column, and each entry except w equal to ± 1. But then J is uniquely the sum of two permutation matrices, so $\det J = \pm 1 \pm w$. But since $M(S)$ is binary, it may also be coordinatized over $GF(2)$ by replacing each non-zero entry in A by 1 in $GF(2)$, since basic circuits of M must be preserved. But then over $GF(2)$, $\det J = 0$, hence we must also have $\det J = 0$ over K to preserve dependence. Therefore $w = \pm 1$.

The proof of (1)–(5) will now be complete if we prove the following lemma, by regarding A as a matrix over $\mathbb{Q}$, since the operations in the proof of the lemma do not depend on the characteristic.

3.1.2. Lemma. *Let $M(S)$ be a binary matroid which is coordinatized by a matrix A in echelon form over $\mathbb{Q}$, with every entry of A equal to 0 or ± 1. Then A is totally unimodular.*

Proof. Let W be a square submatrix of A. We now do row operations on W to reduce it to echelon form. Given $w_{ij} \neq 0$, since $w_{ij} = \pm 1$, we add $-w_{ij}w_{hj}$ times row i to row h for each h, to get w_{ij} to be the only non-zero entry in column j. Now consider an entry w_{hk} in the original submatrix W, where $h \neq i, k \neq j$. Then the above row operations replace w_{hk} by $w_{hk} - w_{ij}w_{hj}w_{ik}$, which is 0 or ± 1 unless $w_{hk} = -w_{ij}w_{hj}w_{ik} \neq 0$. But then the following 2×4 submatrix existed in the matrix A:

$$\begin{bmatrix} 1 & 0 & w_{ij} & w_{ik} \\ 0 & 1 & w_{hj} & w_{hk} \end{bmatrix}.$$

This submatrix coordinatizes a minor of $M(S)$ which is isomorphic to L_4, a contradiction to the assumption that $M(S)$ is binary.

Thus the reduction of W to echelon form may be completed while keeping all entries 0 or ± 1. Thus det $W = 0$ or ± 1, and A is totally unimodular

□

Proof of equivalence of (1) *through* (7). First we will show $(1) \Rightarrow (6) \Rightarrow (5)$.

Let A be a totally unimodular matrix over $\mathbb{Q}$ coordinatizing $M(S)$ and in echelon form with respect to the basis B. Then the i-th row of A is non-zero on precisely the elements of the basic bond $S - \overline{B - \{b_i\}}$ corresponding to the i-th element of B. Furthermore, by row operations, we may bring A into echelon form A' with respect to any other basis B', thus obtaining a row for any bond of M. Furthermore, A' must still be totally unimodular, since $n \times n$ determinants are preserved by the row operations, and any $k \times k$ determinant of A' may be augmented by columns from B' to obtain an $n \times n$ determinant, at most changing the sign of the determinant.

Now let D' be a matrix obtained by taking such a row for each bond of M. D' is then just the bond-element incidence matrix with some 1's changed to -1's, and the row-space of D' is the same as the row-space of A. By Proposition 1.3.1, $M^*(S)$ also has a coordinatization A^* obtained from A by transposing. It is very easy to check that A^* is also totally unimodular. Letting E' be the matrix obtained for the bonds of M^* as D' was for M, we see that E' is just the circuit-element incidence matrix of M with some 1's changed to -1's. Furthermore, the rows of D' and the rows of E' are orthogonal, again by Proposition 1.3.1, hence $D'(E')^t = 0$, proving $(1) \Rightarrow (6)$.

Now suppose that we are given D' and E' as above, with $D'(E')^t = 0$. Since each row of D' is orthogonal to each row of E', we see immediately that if R is a bond and C a circuit of $M(S)$, then $|R \cap C|$ is even. Thus from Theorem 2.2.1, M is binary.

Let B be a basis of $M(S)$ and assume the elements of S have been ordered so that the elements of B come first. The basic bonds $S - \overline{B - \{b_i\}}$ for $b_i \in B$ give us a submatrix D'' of D',

$$D'' = (I' \mid U),$$

where I' is the matrix of columns corresponding to the elements of B', and I' is an $n \times n$ identity matrix with some of the entries possibly changed from 1 to -1. Now, the dimension of the row-space of D' is at least n, the dimension of the row-space of D''.

Similarly, by taking the rows of E' corresponding to the basic circuits of B, we have

$$E'' = (V \mid I''),$$

where I'' is the matrix of columns corresponding to $S - B$, and I'' is an $(N - n) \times (N - n)$ identity matrix with some of the entries possibly changed from

1 to -1. The dimension of the row-space of E' is at least $N-n$, the dimension of the row-space of E''. Since the row-spaces of D' and E' are orthogonal subspaces of an N-dimensional vector-space over $\mathbb{Q}$, we have equality in both cases, that is, row-rank $(D')=n$ and row-rank $(E')=N-n$.

We will now show that D'' is a totally unimodular matrix coordinatizing $M(S)$. Let B' be any basis of M. If we construct D''' from the basic bonds of B' as we did D'' from B (but keeping the ordering of the elements of S fixed), we see that D''' and D'' are row-equivalent, since the rows of each are a basis of the row space of D'. Thus the columns of D'' corresponding to B' are linearly independent.

Let C be a circuit of $M(S)$. Then C corresponds to a row e_c of E' which is orthogonal to the rows of D'', and hence the entries of e_c are the coefficients of a linear dependence of the columns of D'' corresponding to the elements of C. Thus D'' is a coordinatization of M, and by Lemma 3.1.2 it is also unimodular.

Thus (1)–(6) are equivalent. The equivalence of these with (7) now follows easily by noting that the functions f_H correspond to rows of the signed bond-element matrix D', with f_H in particular corresponding to the row for the bond $S-H$.

Proof of conditions (8) and (9). The implication (5)$\Rightarrow$(8) is easy, since L_4 cannot be a minor if M is binary, and F_7 or F_7^* cannot be coordinatized over any field whose characteristic is not 2 (see Exercise 1.9). The converse was proved by Tutte using his very deep Homotopy Theorem (Tutte 1958), and is certainly one of the most beautiful and important results in matroid theory. We state the Homotopy Theorem and sketch the proof of (8)$\Rightarrow$(7) in the next section.

The implication (9)$\Rightarrow$(5) is easy by observing that 1-sums, 2-sums, and 3-sums preserve coordinatizability over $GF(2)$ and $GF(3)$ [see p. 186 of White (1986)]. Seymour's Theorem (1980) is (8)$\Rightarrow$(9). The proof is much too long to be included here. One advantage of this result is that it includes Tutte's Theorem as a corollary. □

3.2 Tutte's Homotopy Theorem and Excluded Minor Characterization

We now give a careful statement of Tutte's Homotopy Theorem, and sketch its use to prove Tutte's excluded minor characterization of unimodular matroids. There are several reasons why we choose to do so. The first is the historical importance of Tutte's work, despite the fact that his excluded minor characterization can now also be proved by Seymour's method. The second is the importance of the ideas involved for further work in coordinatizations. This importance seems restricted by Tutte's heavy use of the crucial property of binary matroids that coline is contained in at most three distinct hyperplanes (or copoints). Nevertheless, both Reid (unpublished) and Bixby (1979) were able to extend Tutte's methods to obtain the excluded minor characterization of ternary matroids. The third reason is that such a sketch of Tutte's ideas is not available in accessible form elsewhere, except in Tutte's own writing. Although Tutte's terminology and notation are perhaps suitable for someone who is interested primarily in the graph-theoretical aspects of matroid theory, they are quite confusing to the large majority of matroid theorists who use terminology similar to that used in these volumes. For example, what Tutte calls a point is in our terminology a bond, and for our purposes is best complemented to get a hyperplane. It is hoped that the translation provided here will be useful not only as an overview of Tutte's methods, but also as an entry point to Tutte's papers for those who wish to study them in detail.

We first need some definitions. A *copoint* (or hyperplane), *coline*, or *coplane* in a matroid $M(E)$ of rank n is a flat of rank $n-1, n-2$, or $n-3$ (respectively). A flat Y is *T-connected* if $M(E)/Y$ is connected. A *path* in M is a sequence $(X_1, X_2, \ldots, X_k)$ of copoints such that for $1 \leqslant i \leqslant k-1$, $X_i \cap X_{i+1}$ is a T-connected coline. Thus each such coline $X_i \cap X_{i+1}$ is contained in a third copoint distinct from X_i and X_{i+1}. A collection $\mathscr{C}$ of copoints of M is a *linear subclass of copoints* (see White 1986, Exercise 7.8) if whenever X_1, X_2, and X_3 are distinct copoints all containing a common coline, and $X_1 \in \mathscr{C}$ and $X_2 \in \mathscr{C}$, then $X_3 \in \mathscr{C}$. A path is *off* $\mathscr{C}$ if no copoint of the path is a member of $\mathscr{C}$. A path is *closed* if the first and last copoints in the path are identical. We now describe four types of closed paths which will be called *elementary paths off* $\mathscr{C}$, for a particular linear subclass $\mathscr{C}$.

(1) (X, Y, X), an arbitrary closed path of length 2 off $\mathscr{C}$.

(2) (X, Y, Z, X), a closed path of length 3 off $\mathscr{C}$ such that $X \cap Y \cap Z$ is either a coline or a coplane.

(3) (X, Y, Z, T, X), a closed path of four distinct copoints off $\mathscr{C}$, where $X \cap Y \cap Z \cap T$ is a coplane P, $X \cap Y$ and $Z \cap T$ span a copoint A, $X \cap T$ and $Y \cap Z$ span a copoint B, $A \in \mathscr{C}$, $B \in \mathscr{C}$, and every T-connected coline containing P is contained either in A or in B.

(4) (A, X, B, Y, A), a closed path of four distinct copoints off $\mathscr{C}$ where $A \cap X \cap B \cap Y = D$ and the contraction $M(E)/D$ is a matroid of rank 4 containing six distinct points $P_1, P_2, \ldots, P_6$ with A/D spanned by $\{P_2, P_3, P_5, P_6\}$, B/D by $\{P_1, P_3, P_4, P_6\}$, X/D by $\{P_2, P_3, P_4\}$, Y/D by $\{P_1, P_2, P_6\}$, and with $\{P_1, P_2, P_4, P_5\}$ spanning another copoint off $\mathscr{C}/D$, where $\mathscr{C}/D = \{X/D : X \in \mathscr{C}\}$. Furthermore, $\{P_1, P_2, P_3\}$, $\{P_1, P_5, P_6\}$, $\{P_2, P_4, P_6\}$, and $\{P_3, P_4, P_5\}$ all span copoints which are in $\mathscr{C}/D$, and all other points of M/D are on the three lines P_1P_4, P_2P_5, and P_3P_6.

Now, if $P = (X_1, X_2, \ldots, X_k)$ and $R = (X_k, X_{k+1}, \ldots, X_m)$ are two paths, we define their product PR as the path $(X_1, X_3, \ldots, X_k, \ldots X_m)$. If $Q = (X_k, \ldots, X_k)$ is one of the elementary paths defined above, we say that PQR and PR are *elementary deformations* of each other with respect to $\mathscr{C}$. Two paths P and P' off $\mathscr{C}$ are *homotopic* with respect to $\mathscr{C}$ if one may be obtained from the other by a finite sequence of elementary deformations with respect to $\mathscr{C}$. Homotopy is clearly an equivalence relation.

3.2.1 Proposition. *Let $\mathscr{C}$ be a linear subclass of copoints in a connected matroid $M(E)$, and let X and Y be copoints of M such that $Y \notin \mathscr{C}$. Then there exists a path from X to Y which is off $\mathscr{C}$ with the possible exception of the first copoint X.*

A proof of this in our notation may be found in Crapo & Rota (1970).

3.2.2. Proposition. (*Tutte's Homotopy Theorem*). *Let $\mathscr{C}$ be any linear subclass of the matroid $M(E)$, and let P be any closed path off $\mathscr{C}$. Then P is homotopic to a trivial path with respect to $\mathscr{C}$.*

We omit the proof of Proposition 3.2.2 since it is fairly long and technical. We prefer instead to show how it is applied to prove the excluded minor characterization for unimodular matroids.

3.2.3. Theorem. *A matroid M is unimodular if and only if M is binary and has no minor isomorphic to the Fano plane F_7 or the orthogonal matroid F_7^*.*

Proof. We have already observed that the necessity is easy. To prove the sufficiency, suppose that M is a minimal matroid such that M is binary with no minor isomorphic to F_7 or F_7^* and yet M is not unimodular. Then for arbitrary $a \in E$, $M - a = M'$ is unimodular. Let $\mathscr{C}$ be the linear subclass of copoints X of M' such that $a \in \mathrm{cl}(X)$ in M.

Now we fix a unimodular coordinatization of M', given by $f_X: E \to \{0, \pm 1\} \subseteq \mathbb{Q}$ for every copoint X of M', as in Proposition 1.5.5. Our task is to construct such an f_X for every copoint X of M.

Let X and Y be copoints of M' on a T-connected coline, with X and Y off $\mathscr{C}$. Then there exists $x \in E - (X \cup Y \cup \{a\})$. Let $t(X, Y) = f_X(x) f_Y(x)$. Then $t(X, Y)$ is independent of the choice of x, for if $y \in E - (X \cup Y \cup \{a\})$ and $f_X(x) f_Y(x) \neq f_X(y) f_Y(y)$, then the coordinatizing matrix can easily be shown to have a submatrix

$$\begin{matrix} 1 & 0 & f_X(x) & f_X(y) \\ 0 & 1 & f_Y(x) & f_Y(y) \end{matrix}$$

which implies a minor L_4 of M', a contradiction.

Now let $R = (X_1, X_2, \ldots, X_k)$ be any path in M' off $\mathscr{C}$. We define $u(R) = \prod_{i=1}^{k-1} t(X_i, X_{i+1}) = \pm 1$, and claim that $u(R) = 1$ for every closed path off $\mathscr{C}$. To prove this claim, it suffices by the Homotopy Theorem to prove that $u(R) = 1$ for each of the four elementary paths off $\mathscr{C}$.

(1) Let $R=(X,Y,X)$, then $u(R)=t(X,Y)^2=1$.

(2) Let $R=(X,Y,Z,X)$. Then X,Y,Z cannot contain a common coline L, since none of them contains the point a, and the binary matroid M cannot have four copoints on L. Therefore X, Y, and Z intersect in a coplane P. If there is a point $x \notin X \cup Y \cup Z$, then $u(R) = f_X(x)f_Y(x)f_Y(x)f_Z(x)f_Z(x)f_X(x)=1$. If there is no such point x, then for R to be a path, we must have $e \in X-(Y \cup Z), f \in Y-(X \cup Z), g \in Z-(X \cup Y)$. Since $Y \cap Z$ is a coline, there must also be $b \in (Y \cap Z)-X$, and similarly $c \in (X \cap Z)-Y, d \in (X \cap Y)-Z$. Then these six points together with the point a induce a Fano configuration in M/P, a contradiction.

(3) In this case, we have f_X, f_Y, f_Z, f_T and it is easy to see from Lemma 1.5.6 that these four functionals are linearly dependent, since $X \cap Y \cap Z \cap T$ *is* a coplane P. This dependence implies that the following determinant is zero, where $b \in (X \cap Y)-P$, $c \in (Y \cap Z)-P$, $d \in (Z \cap T)-P$, $e \in (T \cap X)-P$:

$$\begin{vmatrix} 0 & 0 & f_Z(b) & f_T(b) \\ f_X(c) & 0 & 0 & f_T(c) \\ f_X(d) & f_Y(d) & 0 & 0 \\ 0 & f_Y(e) & f_Z(e) & 0 \end{vmatrix} = 0$$

which implies $u(R)=1$.

(4) This case leads directly to F_7^* when we include the point a and contract by D, again a contradiction.

Now we are ready to construct the coordinatization of M, by defining f_X for every copoint X of M. For each copoint X, either

(A) $a \notin X$ and X is a copoint of M' (with $X \notin \mathscr{C}$),

(B) $a \in X$ and $X-a$ is a copoint of M' (with $X-a \in \mathscr{C}$), or

(C) $a \in X$ and $X-a$ is a coline of M'.

In cases (A) and (B), we already have f_X defined on $E-\{a\}$. We fix a copoint X_0 satisfying case (A), and set $f_{X_0}(a)=1$. Then for every copoint X in case (A), there must be a path R in M' from X_0 to X off $\mathscr{C}$, by Proposition 3.2.1. Let $f_X(a)=u(R)$. Since we have already shown that $u(R)=1$ when R is a closed path, we see that $f_X(a)$ is well-defined.

In case (B), set $f_X(a)=0$. In case (C), $X-a=L$ must be a disconnected coline of M' (since M is binary), that is, there are copoints Y and Z of M' containing L, with $E=Y \cup Z \cup \{a\}$. Simply define $f_X=f_Y \pm f_Z$, choosing the coefficient of f_Z so that $f_X(a)=0$.

To complete the proof, we need to show that for every three copoints X,Y,Z on a coline L, f_X, f_Y, and f_Z are linearly dependent. Suppose first that $a \notin L$. Then $a \in X$, without loss of generality. If $X-a=L$, then X is of type (C) above, and by the construction of f_X, we have the required linear dependence. If $X-a \supsetneq L$, then there exists $b \in X-L, b \neq a$. In M', we have $af_{X-a}+$

$\beta f_Y + \gamma f_Z = 0$. Since $f_X(a) = f_X(b) = 0$ by case (*B*), and $f_Y(a) f_Z(a) = u(R) = t(Y, Z)$ follows from case (*A*) using the path $R = (Y, Z)$ off $\mathscr{C}$, and since $t(Y, Z) = f_Y(b) f_Z(b)$, we have that $\alpha f_X(a) + \beta f_X(a) + \gamma f_Z(a) = \pm(\beta f_Y(b) + \gamma f_Z(b)) = 0$, hence $\alpha f_X + \beta f_Y + \gamma f_Z = 0$.

The remaining case is $a \in L$. If $L - a$ is still a coline in M', then f_X, f_Y, and f_Z are dependent on $E - \{a\}$, and take the value zero on a, hence are dependent on E. If $L - a$ is a coplane, it is necessary to construct some additional copoints and use dependences among their f's to deduce the desired dependency. We omit the details, which are in Tutte (1958). □

3.3. Applications of Unimodularity

An important application of Seymour's characterization of unimodular matroids [condition (9) in Theorem 3.1.1] is a polynomial algorithm for recognizing whether a matrix is totally unimodular, or more generally, whether an arbitrary matroid M is unimodular. In the general case, the number of independent sets in the matroid may be exponential compared to the rank and cardinality of the matroid, so for the problem to make sense we must assume that M is given by an *independence oracle*, a 'black box' that tells us in one step whether a given subset is independent in M. In the case of a vector matroid, for example, the independence oracle is simply a subroutine for checking linear independence. The algorithm proceeds roughly as follows:

3.3.1. Algorithm.

(1) *Check for decompositions into 1-sums, 2-sums, or 3-sums, using algorithms by Bixby and Cunningham (1981) and Cunningham and Edmonds (unpublished) for k-separations.*

(2) *Taking indecomposable matroids resulting from (1), check for graphicness by Bixby and Cunningham (1980), for cographicness by taking the orthogonal dual and checking for graphicness, and for isomorphism with* R_{10}.

This algorithm may be modified to check whether a given matrix A is unimodular as follows:

3.3.2. Algorithm.

(1) *Check that all entries of A are* 0, ± 1.

(2) *Letting M be the binary matroid on the columns of* A_1, *the binary matrix obtained by changing* -1*'s to 1's in A, apply Algorithm 3.31 to determine whether M is unimodular (where we note that Algorithm 3.3.1 is easier to implement for binary matroids).*

(3) *If M is unimodular, determine a unimodular signing* A_2 *of* A_1 *(which may be determined from such signings of the graphic, cographic, and* R_{10} *pieces, which are easy to sign).*

(4) *Applying Proposition 1.2.5, we check whether A_2 is projectively equivalent to A, using only scalar multiplications of ± 1.*

A second application of unimodularity is in linear programming.

3.3.3. Proposition. (*Heller 1957*). *The linear program*

$$\text{maximize } c^t x$$
$$\text{subject to } Ax \leqslant b,\ x \geqslant 0$$

has a solution x with integer coordinates, for every choice of a vector b with integer coordinates, if and only if A is totally unimodular.

In fact many of the most efficiently solved combinatorial optimization problems, such as matroid intersection and bipartite matching, may be realized as unimodular programming problems. Indeed, this proposition makes the distinction between integer programming and linear programming no longer an issue for such problems.

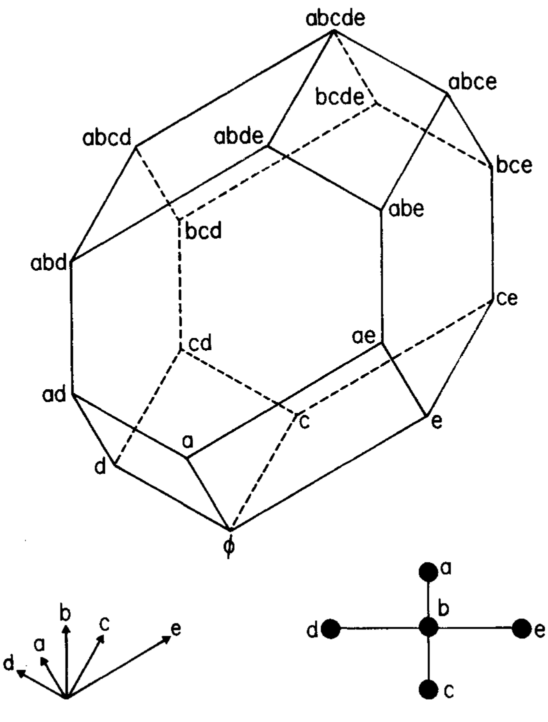
There is a polynomial algorithm for solving unimodular programming problems, according to Bland and Edmonds (unpublished); see Bixby and Cunningham (1980). This algorithm uses the Seymour decomposition to reduce to the case that A is graphic or cographic. However, this case is essentially a network flow problem or its dual. One might regard this algorithm to be of no interest because of the recent highly publicized polynomial algorithms for the general linear programming problem. However, network flow problems are so efficiently solved that one can still hope for more efficient algorithms for the unimodular case than the general one.

As a third application, we consider the integer max-flow-min-cut property. This is a well-known property of directed graphs (networks), but Seymour (1977) has characterized an interesting generalization to matroids. A special element e of $M(E)$ is singled out (corresponding to an auxiliary edge from sink to source in the network case). A capacity is assigned to each element of $M(E) - e$ and a flow is an assignment of a scalar to each circuit of M, such that the flow summed over all circuits containing an element x does not exceed the capacity of x. Then M has the integer max-flow-min-cut property if for every choice of e and an integer-valued capacity, there exists a non-negative integer-valued flow whose total value at e equals the minimum capacity of a cocircuit ('cut-set') of M containing e. Gallai (1959) and Minty (1966) proved independently that unimodular matroids have this property. However, Seymour (1977) has completely characterized the connected matroids with this property: they are the binary matroids with no minor isomorphic to F_7^*. This class of matroids is dual to that denoted by $\mathscr{R}'$ in Table 7.1 of White (1986). Thus they are either unimodular or contain an F_7 minor. But more is true. Matroids in $\mathscr{R}'$ must always be 2-sums of unimodular matroids and copies of F_7. This remarkable fact is an example of Seymour's concept of a *splitter*: a matroid N belonging to a hereditary class $\mathscr{F}$ which fits so tightly in $\mathscr{F}$ that any matroid M in $\mathscr{F}$ having N as a proper minor has a 1-sum or 2-sum decomposition. Thus any matroid in $\mathscr{F}$ is composed by 1-sums and 2-sums from copies of N and matroids in $\mathscr{F}$ having no minor isomorphic to N. This concept plays an important role in Seymour's proof of his characterization of unimodular matroids, in that R_{10} is a splitter for the class of unimodular matroids. Thus a stronger version of Seymour's theorem may be stated: a unimodular matroid may always be realized by 1-sums and 2-sums of copies of R_{10} and additional matroids which are 1-sums, 2-sums, and 3-sums of graphic and cographic matroids.

Finally, we mention one more application of unimodular matroids, namely, the characterization of zonotopes which pack n-dimensional Euclidean space $\mathbb{E}^n$. Let $\mathscr{S} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_q\}$ be a set of vectors in $\mathbb{E}^n$. Without loss of generality we may assume that these vectors are non-zero and distinct up to scalar multiple, that is, that the vector matroid given by $\mathscr{S}$ is actually a combinatorial geometry. The *zonotope* determined by $\mathscr{S}$ is the set of vectors

$$Z = \left\{ \mathbf{v} : \mathbf{v} = \sum_{i=1}^{q} \alpha_i \mathbf{x}_i, \quad \text{where } -1 \leqslant \alpha_i \leqslant 1 \quad \text{for all } i \right\}.$$

Figure 3.1. A zonotope, its vector star, and its matroid.



Equivalently, we may say that Z is the vector sum of the q line segments $L_i =$ convex hull $(-\mathbf{x}_i, \mathbf{x}_i)$. We call $\mathscr{S}$ the *vector star* of Z. Zonotopes are convex, centrally symmetric polytopes with many interesting properties. A three-dimensional example is given in Figure 3.1. In this example, *abc* and *bde* are chosen to be collinear. The vertices of the zonotope are vectors with each $\alpha_i = \pm 1$, and we have labelled each vertex by the vectors having $\alpha_i = +1$ at that vertex.

An interesting question is whether Z *packs* $\mathbb{E}^n$, (where n is the dimension of Z), that is, whether translates of Z may be placed to fill up $\mathbb{E}^n$ while intersecting each other only on their exterior faces. Shephard (1974) and McMullen (1975) have completely answered this question, *via* the following proposition. We assume that $\mathscr{S}$ spans $\mathbb{E}^n$.

3.3.4. Proposition. *A zonotope packs* $\mathbb{E}^n$ *if and only if its vector star is a binary matroid.*

But, in fact, the vector star is given as a vector matroid over the field $\mathbb{R}$. Hence by Theorem 3.1.1. condition (5), the vector star is binary if and only if it is unimodular.

The zonotope pictured in Figure 3.1 does satisfy the conditions of Proposition 3.3.4. so it does pack $\mathbb{E}^3$.

Exercises

3.1. Show that graphic and cographic matroids are signable.

3.2. Prove that a matroid $M(E)$ may be decomposed as a 2-sum of two matroids if and only if M has a 2-separation, that is, a partition (X_1, X_2) of E with $|X_1| \geqslant 2, |X_2| \geqslant 2, rX_1 + rX_2 \leqslant rE + 1$.

3.3. Show that the class of unimodular matroids is not *filtered* in the sense of Brylawski and Kelly (1980), that is, that there exist unimodular matroids of the same rank n which are not both submatroids of any unimodular matroid of rank n.

3.4. (Aigner 1979) If A is a locally unimodular coordinatization (over $\mathbb{Q}$) of a unimodular matroid $M(E)$, A is $n \times N$ where $n = \text{rank } M, N = |E|$, then det $(AA)^t =$ the number of bases of M.

3.5. Prove that 1-, 2-, and 3-sums of unimodular matroids are unimodular. What are the corresponding operations on coordinatizing matrices?

3.6. Show that the binary coordinatization for R_{10} described in Exercise 1.7 is projectively equivalent to one in which each column has the same number of zeros. Thus determine that this matroid has a doubly transitive group of automorphisms. Use this information to show that R_{10} is unimodular, but neither graphic nor cographic.

3.7. If $M(E)$ is unimodular, $e \in E$, such that $M - e$ is isomorphic to R_{10}, show that e must be a loop, isthmus, or parallel element [i.e., M is the parallel extension of some element of $M-e$: see White (1986), p. 180)]. This is essentially all that is needed to check that R_{10} is a splitter for the class of unimodular matroids (see White 1986, Exercise 7.50).

3.8. Prove that for each vector **v** in the vector star of a zonotope Z, the set of edges of Z parallel to **v** form a 'zone', or minimal cut-set of the graph G determined by the edge-skeleton of Z, i.e., a bond in $M(G)$.

3.9. Prove that a zonotope in E^3 is space-filling if and only if all of its projections onto a plane orthogonal to a vector in its star yield tessalations (quadrilateral or hexagonal) of the plane.

3.10. Let $C_{k,n}$ denote the binary matroid determined by the binary matrix consisting of the $n \times n$ identity matrix next to an $n \times n$ matrix consisting of all cyclic shifts of a column of k ones followed by $n-k$ zeros. Show that $C_{2,n}$ is always graphic, that $C_{3,5}$ is R_{10}, and $C_{3,n}$ is not unimodular for all $n > 5$.

3.11. Is $C_{k,n}$ unimodular for any $k \geqslant 4, n > k$?

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