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ON THEOREMS OF WHITNEY AND TUTTE

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Short proofs of two theorems are given: (i) Whitney's 2-isomorphism theorem characterizing all graphs with the same cycle matroid, and (ii) Tutte's excluded minor characterization of those binary matroids that are graphic. Graph connectivity plays an important role in both proofs.

1. Introduction

Familiarity with graph and matroid theory is assumed; see [1] and [10]. Where G is a graph with $S \subseteq E(G)$, $G[S]$ denotes the subgraph induced by S. A partition $\{S, T\}$ of $E(G)$ is a *k-separation* of G, for k a positive integer, if $|S| \geqslant k \leqslant |T|$ and $|V(G[S]) \cap V(G[T])| \leqslant k$. A graph is *$n$-connected*, for n a positive integer, if it has no k-separation for $k < n$; a 2-connected graph is *nonseparable*.

Let G be a nonseparable graph with 2-separation $\{S, T\}$ and let $V(G[S]) \cap V(G[T]) = \{x, y\}$. Let G' be the graph obtained from G by interchanging in $G[S]$ the incidences of the edges at x and y. Then G' is obtained from G by *reversing* $G[S]$. A graph obtainable from G by a sequence of reversals is *2-isomorphic* to G.

Let $M(G)$ denote the cycle matroid of G. Whitney [12] proved the following result.

(1.1) Let G and G' be nonseparable graphs. Then $M(G) = M(G')$ if and only if G and G' are 2-isomorphic.

Let K_5 and $K_{3,3}$ denote the Kuratowski graphs and let F_7 denote the Fano matroid. Denote the dual of a matroid M by M^*.

Tutte [8] proved the following result.

(1.2) Let M be a binary matroid. Then M is graphic if and only if M has no F_7, F_7^*, $M^*(K_5)$ or $M^*(K_{3,3})$ minor.

The purpose of this paper is to provide new short proofs of (1.1) and (1.2). It is hoped the present proofs will be more accessible and will provide additional insight into the results. Graph connectivity plays an important role in both proofs.

The paper is outlined as follows: (1.1) is proved in Section 2, some preliminary

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results needed to prove (1.2) are contained in Section 3, and the proof of (1.2) is in Section 4.

2. Whitney's theorem

Let $\mathrm{st}_G(u)$, called the *star* of u in G, be the set of edges in G incident to a vertex u. If a graph G' is obtained from a graph G by simply renaming the vertices, then these graphs are regarded as equal, denoted $G = G'$. Deletion in both matroids and graphs is denoted by '\' and contraction by '/'. The edge-sets of forests and cycles are equated with the subgraphs they induce.

Tutte generalized graph connectivity to matroids. Given a matroid M on E, a k-*separation* of M, for k a positive integer, is a partition $\{S, T\}$ of E such that $|S| \geqslant k \leqslant |T|$ and $r(S)+r(T)-r(E) \leqslant k-1$; M is n-*connected*, for n a positive integer, if it has no k-separation for $k<n$. Tutte [9] proved the following result; for a simpler proof see Cunningham [2].

(2.1) For a connected graph G, G is n-connected if and only if $M(G)$ is n-connected.

The following result is due to Whitney [11]; the proof is essentially that of Sachs [4].

(2.2) Let G and G' be graphs with $M(G)=M(G')$. If G is 3-connected, then $G=G'$.

Proof. Let $u \in V(G)$. Since G is 3-connected, $G[E(G)-\mathrm{st}_G(u)]$ is nonseparable. Since $M(G)=M(G')$, $\mathrm{st}_G(u)$ is a cocircuit of $M(G')$ and by (2.1) $G'[E(G)-\mathrm{st}_G(u)]$ is nonseparable. If $\mathrm{st}_G(u)$ is not the star of some vertex of G', then $G'[E(G)-\mathrm{st}_G(u)]$ has a 1-separation, a contradiction. Thus G and G' have exactly the same stars, i.e., $G=G'$. □

Let C be a cycle of G. The edge-sets of the blocks (i.e., the maximal nonseparable subgraphs) of G/C are the *bridges* of C. Evidently the set of bridges is a partition of $E(G)-C$. Equivalently, bridges are defined by the equivalence relation on $E(G)-C$: $e_1 \sim e_2$ if and only if $e_1 = e_2$ or there exists a path P of G containing e_1 and e_2, and no internal vertex of P is a vertex of C. A bridge B is a k-*bridge* if $\{B, E(G)-B\}$ is a k-separation. Thus, if B is a k-bridge of C and G is k-connected, then $|V(G[B]) \cap V(C)| = k$.

A *bond* is a connected loopless graph on two vertices and a *polygon* is a connected graph every vertex of which has degree 2.

(2.3) Let G be a nonseparable graph that is not a bond or polygon. If G has a 2-separation, then some cycle of G has a 2-bridge.

Proof. Let $\{S, T\}$ be a 2-separation of G. Since G is not a polygon either $G[S]$ or $G[T]$ has a block H that contains a cycle. Further, $\{E(H), E(G)-E(H)\}$ is a 2-separation of G. Let $K=G[E(G)-E(H)]$ and let $V(H)\cap V(K)=\{x, y\}$. Since H is nonseparable, it contains a cycle C containing x and y. Further, there exists bridges of C, say $B_1, \ldots, B_t$, such that $E(K)=B_1\cup\cdots\cup B_t$. If $|B_i|\geqslant 2$ for some i, then B_i is the desired 2-bridge. Thus, $|B_i|=1$ for $1\leqslant i\leqslant t$. It follows that K is a bond.

Since K is a bond and G is not a bond, H is not a bond. Thus, by the above argument, there exists a (two-edge) cycle of K that has a 2-bridge. □

One half of (1.1) is easy. The hard half is:

(2.4) If G and G' are nonseparable graphs with $M(G)=M(G')$, then G and G' are 2-isomorphic.

Proof. Clearly the result is true if $|E(G)|=1$. The proof proceeds by induction. By (2.2) it may be assumed that G has 2-separation. Further, if G is a bond or polygon, then the result is easily verified. Thus, by (2.3) G has a cycle C that has 2-bridge B.

Since $M(G)=M(G')$, C is a cycle of G'. In addition, B is a bridge of C in G'. (This follows from the first definition of a bridge and (2.1).) Moreover, it is claimed that B is a 2-bridge of C in G'. All of $G[B]$, $G'[B]$, $G[E(G)-B]$ and $G'[E(G)-B]$ are connected. (This follows easily from the second definition of a bridge.) Therefore

$$|V(G[B])|=|V(G'[B])| \quad \text{and} \quad |V(G[E(G)-B])|=|V(G'[E(G)-B])|,$$

since $|V(G)|=|V(G')|$, it follows that B is a 2-bridge of C in G'.

Now $\{B, E(G)-B\}$ is a 2-separation of G and G'. Let $V(G[B])\cap V(G[E(G)-B])=\{x, y\}$. Add an edge $e\notin E(G)$ to both $G[B]$ and $G[E(G)-B]$ joining x and y to create graphs H and K, respectively. Then H and K are nonseparable and have fewer edges than G. Let H' and K' be obtained from $G'[B]$ and $G'[E(G)-B]$, respectively, in the same way. Then $M(H)=M(H')$ and $M(K)=M(K')$. By induction H is 2-isomorphic to H' and K is 2-isomorphic to K'. It follows that G is 2-isomorphic to G'. □

Truemper [7] has strengthened (2.4) by showing G can be obtained from G' by at most $|V(G)|-2$ reversals. Such a bound is easily obtained from the above method of proof.

A set of P of edges of a nonseparable graph G is a *hypopath* of G if P is a path in some graph 2-isomorphic to G. An easy consequence of (1.1) is a result due to Löfgren [3].

(2.5) Let M be a 2-connected binary matroid such that for some element e, $M\backslash e=M(G)$ for some graph G. Let C be a circuit of M containing e and let $P=C-\{e\}$. Then M is graphic if and only if P is a hypopath of G.

3. Graphic minors of a nongraphic matroid

Where e is in a two-element cocircuit of a matroid M, M/e is a *series contraction* of M. If N is a matroid obtained by a sequence of series contractions from M and N has no two-element cocircuit, then define $N = \lfloor M \rfloor$. Define $\lceil M \rceil$ by replacing 'cocircuit' by 'circuit', 'series' by 'parallel' and 'contraction' by 'deletion'. (Thus, $\lceil M \rceil = \lfloor M^* \rfloor^*$.) The graphs $\lfloor G \rfloor$ and $\lceil G \rceil$ are defined in the obvious way. Seymour [5] and Cunningham [2] have the following result.

(3.1) If M is a 3-connected matroid, then there exists an element e such that $\lfloor M \backslash e \rfloor$ is 3-connected.

Throughout the remainder of the section the following notation is used. M is a 3-connected nongraphic binary matroid. For every element f, $M\backslash f$ and M/f are graphic, and for a fixed element e, $\lfloor M\backslash e \rfloor$ is 3-connected. Also C is a circuit of M containing e and $P = C - \{e\}$. Finally, G is a graph such that $M(G) = M\backslash e$. By (2.1) either G or $\lfloor G \rfloor$ is 3-connected. In graphic minors of M subsets of C are equated with the subgraphs they induce.

By (2.5) P is not a hypopath of G. Clearly P is acyclic in G and so P must have at least four vertices of odd degree in any graph 2-isomorphic to G.

(3.2) For any $f \in E(G)$ there exists a graph 2-isomorphic to G/f in which P' has exactly two vertices of odd degree, where $P' = P - \{f\}$. (Note P need not contain $\{f\}$.)

Proof. For any $f \in F(G)$, M/f is graphic. Since M is binary, $P' \cup \{e\}$ is the edge-disjoint union of circuits of M/f. Let G' be a graph such that $M(G') = M/f$. Then every vertex of $P' \cup \{e\}$ is of even degree. Thus P' has exactly two vertices of odd degree in $G'\backslash e$. By (2.4) $G'\backslash e$ and G/f are 2-isomorphic. $\square$

An *arc* S of a graph H is a maximal path with at least two edges such that the set $V(S) \cap V(H[E(H) - S])$ is precisely the set of ends of S. If G is not 3-connected, but $\lfloor G \rfloor$ is, then it is straightforward to show that every two-edge cocycle (i.e., a cocircuit of $M(G)$) is contained in some arc of G, and so to obtain $\lfloor G \rfloor$ from G one needs only to replace each arc by a single edge. Further, every graph 2-isomorphic to G can be obtained by reversing subgraphs contained in arcs of G, and if f is an edge of an arc of G, then $\lfloor G/f \rfloor$ is 3-connected.

(3.3) If G is not 3-connected, then G contains exactly three arcs, each one containing exactly two edges, exactly one of which is from P. Moreover, these are all the edges of P.

Proof. It is first shown that every arc contains exactly two edges, exactly one of which is from P. Let S be an arc of G containing more than one edge from P and let $f \in S \cap P$. Since, in every graph 2-isomorphic to G, P has at least four vertices of odd degree, it follows that in every graph 2-isomorphic to G/f, $P-\{f\}$ has at least four vertices of odd degree, a contradiction to (3.2). A similar argument shows that S has at most one edge not from P.

Next it is shown that G has at most three arcs. Suppose G has more than three arcs, and let $f \in S \cap P$, where S is an arc of G. Then G/f has at least three arcs, and the degree 2 vertex of each of these arcs is a vertex of odd degree of $P-\{f\}$ in every graph 2-isomorphic to G/f, a contradiction to (3.2).

Now, if G has at most two arcs, and these arcs contain all the edges of P, then it is easy to see that either P is a hypopath of G, which is a contradiction, or in every graph 2-isomorphic to G/f, where $f \in S-P$ and S is an arc of G, P has four vertices of odd degree, which is a contradiction to (3.2).

Thus it may be assumed that G has at most three arcs and that there exists an edge of P not in an arc of G. Let $f \in S \cap P$, where S is an arc of G. Then $P-\{f\}$ is a hypopath of G/f. It follows that the edges of P not in an arc of G/f form a path P' of G/f, and thus of G. Moreover, every arc of G/f contains a vertex that is an end of P'. It follows that every arc of G contains such a vertex. Since P is not a hypopath of G, G has at least two arcs, and at least two of the arcs of G contain the same end, say v, of P'. Now it is easy to check that by contracting an edge not in P, but contained in some arc containing v, that a contradiction to (3.2) is obtained. □

(3.4) If G is not 3-connected, then G is either 2-isomorphic to the graph of Fig. 1(a) or to the graph of Fig. 1(b).

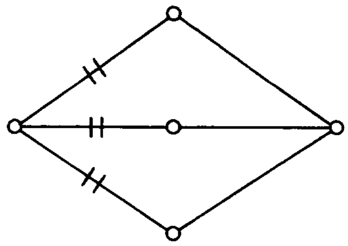
Proof. Let $f \in P$. Then by (2.5) $P-\{f\}$ is a hypopath of G/f. This together with (3.3) implies that each pair of arcs of G has at least one vertex in common. This leads to three cases.

First, two of the arcs could form a four-edge cycle. But this contradicts the fact that $\lfloor G \rfloor$ is 3-connected unless G is 2-isomorphic to the graph of Fig. 1(a).

Second, all three arcs could form a six-edge cycle. It is then straightforward to deduce that G has as a minor a graph H 2-isomorphic to the graph of Fig. 1(b). If H is a proper minor of G, that is, $H = G\backslash X/Y$ with $X \cup Y$ nonempty, then it is claimed that $M' = M\backslash X/Y$ is not graphic, which is a contradiction since all proper minors of M are assumed to be graphic. Suppose $M' = M(H')$ for some graph H'. By (2.4) H and $H'\backslash e$ are 2-isomorphic. Since M is binary, C is edge-disjoint union of cycles of H'. It follows that in $H'\backslash e$, P must have exactly two vertices of odd degree. But this is a contradiction since, clearly, H is not 2-isomorphic to a graph in which P has exactly two vertices of odd degree.

Finally, the three arcs could satisfy neither of the above but have exactly one vertex in common. In this case a contradiction to (3.2) is obtained by choosing $f \in S-P$, where S is an arc of G. □

An edge f of a 3-connected graph H is *deletable* if $\lfloor H\backslash f \rfloor$ is 3-connected and *contractable* if $\lceil H/f \rceil$ is 3-connected. Seymour [5] proved:


(a)

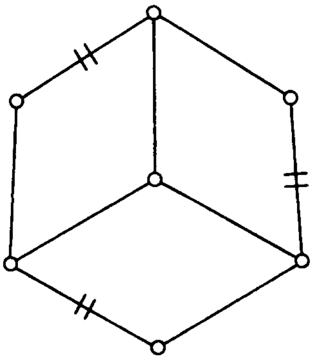

(b)

Fig. 1. The edges of P are marked.

(3.5) In a 3-connected graph every edge is either deletable or contractable.

Note that if G is 3-connected and f is a contractable edge of G, then every graph 2-isomorphic to G/f actually equals G/f. This fact together with (3.2) imply:

(3.6) If G is 3-connected, then every contractable edge of G joins two odd degree vertices of P.

A *triad* is the set of edges incident to a vertex of degree 3.

(3.7) If G is a 3-connected graph and f is a deletable edge of G not in P, then there exists a triad T of G, containing f, such that $|T \cap P| = 1$. Further, if $T = \{f, g, h\}$ (with $g \in P$) is the only such triad, then both ends of g and h are odd degree vertices of P.

Proof. Since f is deletable every arc of $G \backslash f$ has exactly two edges. The matroid $M \backslash f$ is graphic and so by (2.5) P is a hypopath of $G \backslash f$. Since P is not a hypopath of G, there exists an arc S of $G \backslash f$ such that $|S \cap P| = 1$. The first statement follows by taking $T = S \cup \{f\}$.

Now suppose $T = \{f, g, h\}$ is the only triad of G containing f and exactly one edge from P, say g. Then P is a path of the graph obtained from $G \backslash f$ by reversing $G[T - \{f\}]$, and so $P - \{g\}$ is a path of G with h incident to one end. The result follows easily. $\square$

For a circuit C' of M containing e, denote by $O_{C'}$ the set of odd degree vertices in G of (the subgraph induced by) $C' - \{e\}$. In particular, O_C is the set of odd degree vertices of P. Let N be a vertex-edge incidence matrix of G. Then a binary representation for M can be obtained from N by appending a column, corresponding to element e, having a 1 in precisely the rows corresponding to $O_{C'}$. As a consequence:

(3.8) For any circuit C' of M, $O_C = O_{C'}$.

(3.9) If G is 3-connected, then every edge of G is incident to a vertex of O_C.

Proof. Suppose f is not incident to a vertex of O_C. Since M is 3-connected, there exists a circuit C' of M containing e but not f (for otherwise $\{e, f\}$ is a '2-separator'). By (3.5)–(3.7) (with C' replacing C) f is incident to a vertex of $O_{C'}$, a contradiction to (3.8). □

(3.10) If G is 3-connected, then G is one of the graphs of Fig. 2.

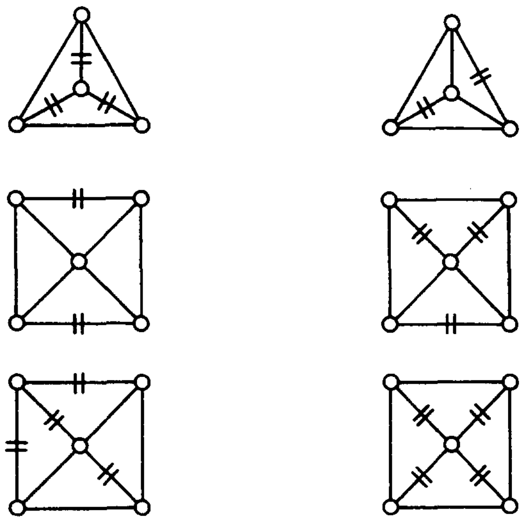


Fig. 2. The edges of P are marked.

Proof. By (3.1) and duality there exists a contractable edge of G. If P has more than four odd degree vertices in G, then $P-\{f\}$ has more than two odd degree vertices in G/f, a contradiction to (3.2). Thus $|O_C| = 4$.

Suppose u_1 and u_2 are distinct vertices not in O_C. By (3.9) and the 3-connectivity of G each of u_1 and u_2 is adjacent to at least three vertices of O_C. Thus there exist distinct vertices $v_1, v_2 \in O_C$ adjacent to both u_1 and u_2. Since P is acyclic, it may be assumed that $f = u_1v_1$ is not in P. By (3.5) and (3.6) f is deletable. By (3.7) v_1 has degree 3 and is adjacent (via edges g and h) to at least two vertices of O_C, a contradiction.

Thus $|V(G)| \leq 5$ and the result follows by case checking. □

4. Tutte's theorem

Since F_7, F_7^*, $M^*(K_5)$ and $M^*(K_{3,3})$ are nongraphic and minors of a graphic matroid are graphic, one half of (1.2) is easy. The converse is:

(4.1) If M is a nongraphic binary matroid, then M has as a minor one of F_7, F_7^*, $M(K_5)$ or $M^*(K_{3,3})$.

Proof. Let M be a counterexample with as few elements as possible. Then for any element f, $M\backslash f$ and M/f are graphic. Further, M is 3-connected, for if M has a 2-separation, then it is well known (see Seymour [5, 6], for example) that M has a proper minor that is not graphic, a contradiction.

By the results of Section 3 there exists an element e of M such that $M\backslash e = M(G)$, where G is one of the graphs of Figs. 1 or 2, and $P\cup\{e\}$ is a circuit of M. It is straightforward to check that M must be one of F_7, F_7^*, $M^*(K_5)$ or $M^*(K_{3,3})$. $\square$

It should be noted that Seymour [6] has given a similar proof of Tutte's Theorem. His proof makes use of (3.1) and his theory of 'grafts', which are defined in terms of vertices, whereas the present proof makes use of 'hypopaths', which are edge-sets. Because cycle matroids are defined in terms of edges, the latter approach seems more natural. Seymour also considers two cases implied by (3.1): (i) $M\backslash e$ is 3-connected, and (ii) $M\backslash e$ is not 3-connected and thus, $\lfloor M\backslash e\rfloor$ is 3-connected. The present treatment of (i) is similar to Seymour's, however, the present treatment of (ii) is different and simpler.

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