Chapter 44

Submodular functions and polymatroids

In this chapter we describe some of the basic properties of a second main object of the present part, the submodular function. Each submodular function gives a polymatroid, which is a generalization of the independent set polytope of a matroid. We prove as a main result the theorem of Edmonds [1970b] that the vertices of a polymatroid are integer if and only if the associated submodular function is integer.

44.1. Submodular functions and polymatroids

Let f be a *set function* on a set S, that is, a function defined on the collection $\mathcal{P}(S)$ of all subsets of S. The function f is called *submodular* if

$$(44.1)\qquad f(T) + f(U) \geq f(T \cap U) + f(T \cup U)$$

for all subsets T, U of S. Similarly, f is called *supermodular* if $-f$ is submodular, i.e., if f satisfies (44.1) with the opposite inequality sign. f is *modular* if f is both submodular and supermodular, i.e., if f satisfies (44.1) with equality.

A set function f on S is called *nondecreasing* if $f(T) \leq f(U)$ whenever $T \subseteq U \subseteq S$, and *nonincreasing* if $f(T) \geq f(U)$ whenever $T \subseteq U \subseteq S$.

As usual, denote for each function $w : S \to \mathbb{R}$ and for each subset U of S,

$$(44.2)\qquad w(U) := \sum_{s \in U} w(s).$$

So w may be considered also as a set function on S, and one easily sees that w is modular, and that each modular set function f on S with $f(\emptyset) = 0$ may be obtained in this way. (More generally, each modular set function f on S satisfies $f(U) = w(U) + \gamma$ (for $U \subseteq S$), for some unique function $w : S \to \mathbb{R}$ and some unique real number γ.)

In a sense, submodularity is the discrete analogue of convexity. If we define, for any $f : \mathcal{P}(S) \to \mathbb{R}$ and any $x \in S$, a function $\delta f_x : \mathcal{P}(S) \to \mathbb{R}$ by: $\delta f_x(T) := f(T \cup \{x\}) - f(T)$, then f is submodular if and only if δf_x is nonincreasing for each $x \in S$.

In other words:

Theorem 44.1. *A set function f on S is submodular if and only if*

$$f(U \cup \{s\}) + f(U \cup \{t\}) \geq f(U) + f(U \cup \{s,t\}) \tag{44.3}$$

for each $U \subseteq S$ and distinct $s, t \in S \setminus U$.

Proof. Necessity being trivial, we show sufficiency. We prove (44.1) by induction on $|T \triangle U|$, the case $|T \triangle U| \leq 2$ being trivial (if $T \subseteq U$ or $U \subseteq T$) or being implied by (44.3). If $|T \triangle U| \geq 3$, we may assume by symmetry that $|T \setminus U| \geq 2$. Choose $t \in T \setminus U$. Then, by induction,

$$f(T \cup U) - f(T) \leq f((T \setminus \{t\}) \cup U) - f(T \setminus \{t\}) \leq f(U) - f(T \cap U), \tag{44.4}$$

(as $|T \triangle ((T \setminus \{t\}) \cup U)| < |T \triangle U|$ and $|(T \setminus \{t\}) \triangle U| < |T \triangle U|$). This shows (44.1). ∎

Define two polyhedra associated with a set function f on S:

$$\begin{aligned} P_f &:= \{x \in \mathbb{R}^S \mid x \geq \mathbf{0}, x(U) \leq f(U) \text{ for each } U \subseteq S\}, \\ EP_f &:= \{x \in \mathbb{R}^S \mid x(U) \leq f(U) \text{ for each } U \subseteq S\}. \end{aligned} \tag{44.5}$$

Note that P_f is nonempty if and only if $f \geq \mathbf{0}$, and that EP_f is nonempty if and only if $f(\emptyset) \geq 0$.

If f is a submodular function, then P_f is called the *polymatroid associated with* f, and EP_f the *extended polymatroid associated with* f. A polyhedron is called an (extended) polymatroid if it is the (extended) polymatroid associated with some submodular function. A polymatroid is bounded (since $0 \leq x_s \leq f(\{s\})$ for each $s \in S$), and hence is a polytope.

The following observation presents a basic technique in proofs for submodular functions, which we often use without further reference:

Theorem 44.2. *Let f be a submodular set function on S and let $x \in EP_f$. Then the collection of sets $U \subseteq S$ satisfying $x(U) = f(U)$ is closed under taking unions and intersections.*

Proof. Suppose $x(T) = f(T)$ and $x(U) = f(U)$. Then

$$\begin{aligned} f(T) + f(U) &\geq f(T \cap U) + f(T \cup U) \geq x(T \cap U) + x(T \cup U) \\ &= x(T) + x(U) = f(T) + f(U), \end{aligned} \tag{44.6}$$

implying that equality holds throughout. So $x(T \cap U) = f(T \cap U)$ and $x(T \cup U) = f(T \cup U)$. ∎

A vector x in EP_f (or in P_f) is called a *base vector* of EP_f (or of P_f) if $x(S) = f(S)$. A *base vector* of f is a base vector of EP_f. The set of all base vectors of f is called the *base polytope* of EP_f or of f. It is a face of EP_f, and denoted by B_f. So

$$B_f = \{x \in \mathbb{R}^S \mid x(U) \leq f(U) \text{ for all } U \subseteq S,\ x(S) = f(S)\}. \tag{44.7}$$

(It is a polytope, since $x_s = x(S) - x(S \setminus \{s\}) \geq f(S) - f(S \setminus \{s\})$ for each $s \in S$.)

Let f be a submodular set function on S and let $a \in \mathbb{R}^S$. Define the set function $f|a$ on S by

(44.8) $$(f|a)(U) := \min_{T \subseteq U}(f(T) + a(U \setminus T))$$

for $U \subseteq S$. It is easy to check that $f|a$ again is submodular and that

(44.9) $$EP_{f|a} = \{x \in EP_f \mid x \leq a\} \text{ and } P_{f|a} = \{x \in P_f \mid x \leq a\}.$$

It follows that if P is an (extended) polymatroid, then also the set $P \cap \{x \mid x \leq a\}$ is an (extended) polymatroid, for any vector a. In fact, as Lovász [1983c] observed, if $f(\emptyset) = 0$, then $f|a$ is the unique largest submodular function f' satisfying $f'(\emptyset) = 0$, $f' \leq f$, and $f'(U) \leq a(U)$ for each $U \subseteq V$.

44.1a. Examples

Matroids. Let $M = (S, \mathcal{I})$ be a matroid. Then the rank function r of M is submodular and nondecreasing. In Theorem 39.8 we saw that a set function r on S is the rank function of a matroid if and only if r is nonnegative, integer, nondecreasing and submodular with $r(U) \leq |U|$ for all $U \subseteq S$. (This last condition may be replaced by: $r(\emptyset) = 0$ and $r(\{s\}) \leq 1$ for each s in S.) Then the polymatroid P_r associated with r is equal to the independent set polytope of M (by Corollary 40.2b).

A generalization is obtained by partitioning S into sets $S_1, \ldots, S_k$, and defining

(44.10) $$f(J) := r(\bigcup_{i \in J} S_i)$$

for $J \subseteq \{1, \ldots, k\}$. It is not difficult to show that each integer nondecreasing submodular function f with $f(\emptyset) = 0$ can be constructed in this way (see Section 44.6b).

As another generalization, if $w : S \to \mathbb{R}_+$, define $f(U)$ to be the maximum of $w(I)$ over $I \in \mathcal{I}$ with $I \subseteq U$. Then f is submodular. (To see this, write $w = \lambda_1 \chi^{T_1} + \cdots + \lambda_n \chi^{T_n}$, with $\emptyset \neq T_1 \subset T_2 \subset \cdots \subset T_n \subseteq S$. Then by (40.3), $f(U) = \sum_{i=1}^n \lambda_i r(U \cap T_i)$, implying that f is submodular.)

For more on the relation between submodular functions and matroids, see Sections 44.6a and 44.6b.

Matroid intersection. Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be matroids, with rank functions r_1 and r_2 respectively. Then the function f given by

(44.11) $$f(U) := r_1(U) + r_2(S \setminus U)$$

for $U \subseteq S$, is submodular. By the matroid intersection theorem (Theorem 41.1), the minimum value of f is equal to the maximum size of a common independent set.

Set unions. Let $T_1, \ldots, T_n$ be subsets of a finite set T and let $S = \{1, \ldots, n\}$. Define

(44.12) $$f(U) := |\bigcup_{i\in U} T_i|$$

for $U \subseteq S$. Then f is nondecreasing and submodular. More generally, for $w : T \to \mathbb{R}_+$, the function f defined by

(44.13) $$f(U) := w(\bigcup_{i\in U} T_i)$$

for $U \subseteq S$, is nondecreasing and submodular.

More generally, for any nondecreasing submodular set function g on T, the function f defined by

(44.14) $$f(U) := g(\bigcup_{i\in U} T_i)$$

for $U \subseteq S$, again is nondecreasing and submodular.

Let $G = (V, E)$ be the bipartite graph corresponding to $T_1, \ldots, T_n$. That is, G has colour classes S and T, and $s \in S$ and $t \in T$ are adjacent if and only if $t \in T_s$. Then we have: $x \in P_f$ if and only if there exist $z \in P_g$ and $y : E \to \mathbb{Z}_+$ such that

(44.15) $$\begin{aligned} y(\delta(v)) &= x(v) \quad \text{for all } v \in S,\\ y(\delta(v)) &= z(v) \quad \text{for all } v \in T. \end{aligned}$$

So y may be considered as an ‘assignment’ of a ‘supply’ z to a ‘demand’ x. If g and x are integer we can take also y and z integer.

Directed graph cut functions. Let $D = (V, A)$ be a directed graph and let $c : A \to \mathbb{R}_+$ be a ‘capacity’ function on A. Define

(44.16) $$f(U) := c(\delta^{\text{out}}(U))$$

for $U \subseteq V$ (where $\delta^{\text{out}}(U)$ denotes the set of arcs leaving U). Then f is submodular (but in general not nondecreasing). A function f arising in this way is called a *cut function*.

Hypergraph cut functions. Let $(V, \mathcal{E})$ be a hypergraph. For $U \subseteq V$, let $f(U)$ be the number of edges $E \in \mathcal{E}$ split by U (that is, with both $E \cap U$ and $E \setminus U$ nonempty). Then f is submodular.

Directed hypergraph cut functions. Let V be a finite set and let $(E_1, F_1), \ldots, (E_m, F_m)$ be pairs of subsets of V. For $U \subseteq V$, let $f(U)$ be the number of indices i with $U \cap E_i \neq \emptyset$ and $F_i \not\subseteq U$. Then f is submodular. (In proving this, we can assume $m = 1$, since any sum of submodular functions is submodular again.)

More generally, we can choose $c_1, \ldots, c_m \in \mathbb{R}_+$ and define

(44.17) $$f(U) = \sum (c_i \mid U \cap E_i \neq \emptyset, F_i \not\subseteq U)$$

for $U \subseteq V$. Again, f is submodular. This generalizes the previous two examples (where $E_i = F_i$ for each i or $|E_i| = |F_i| = 1$ for each i).

Maximal element. Let V be a finite set and let $h : V \to \mathbb{R}$. For nonempty $U \subseteq V$, define

(44.18) $$f(U) := \max\{h(u) \mid u \in U\},$$

and define $f(\emptyset)$ to be the minimum of $h(v)$ over $v \in V$. Then f is submodular.

Subtree diameter. Let $G = (V, E)$ be a forest (a graph without circuits), and for each $X \subseteq E$ define

(44.19) $$f(X) := \sum_K \text{diameter}(K),$$

where K ranges over the components of the graph (V, X). Here diameter(K) is the length of a longest path in K. Then f is submodular (Tamir [1993]); that is:

(44.20) $$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$$

for $X, Y \subseteq E$.

To see this, denote, for any $X \subseteq E$, the set of vertices covered by X by VX. We first show (44.20) for $X, Y \subseteq E$ with (VX, X) and (VY, Y) connected and $VX \cap VY \neq \emptyset$. Note that in this case $X \cap Y$ and $X \cup Y$ give connected subgraphs again.

The proof of (44.20) is based on the fact that for all $s, t, u, v \in V$ one has:

(44.21) $$\text{dist}(s, u) + \text{dist}(t, v) \geq \text{dist}(s, t) + \text{dist}(u, v)$$
$$\text{or } \text{dist}(t, u) + \text{dist}(s, v) \geq \text{dist}(s, t) + \text{dist}(u, v),$$

where dist denotes the distance in G.

To prove (44.20), let P and Q be longest paths in $X \cap Y$ and $X \cup Y$ respectively. If EQ is contained in X or in Y, then (44.20) follows, since P is contained in X and in Y. So we can assume that EQ is contained neither in X nor in Y. Let Q have ends u, v, with $u \in VX$ and $v \in VY$. Let P have ends s, t. So $s, t, u \in VX$ and $s, t, v \in VY$. Hence (44.21) implies (44.20).

We now derive (44.20) for all $X, Y \subseteq E$. Let $\mathcal{X}$ and $\mathcal{Y}$ be the collections of edge sets of the components of (V, X) and of (V, Y) respectively. Let $\mathcal{F}$ be the family made by the union of $\mathcal{X}$ and $\mathcal{Y}$, taking the sets in $\mathcal{X} \cap \mathcal{Y}$ twice. Then

(44.22) $$f(X) + f(Y) \geq \sum_{Z \in \mathcal{F}} f(Z).$$

We now modify $\mathcal{F}$ iteratively as follows. If $Z, Z' \in \mathcal{F}$, $Z \not\subseteq Z' \not\subseteq Z$, and $VZ \cap VZ' \neq \emptyset$, we replace Z, Z' by $Z \cap Z'$ and $Z \cup Z'$. By (44.20), (44.22) is maintained. By Theorem 2.1, these iterations stop. We delete the empty sets in the final $\mathcal{F}$.

Then the inclusionwise maximal sets in $\mathcal{F}$ have union equal to $X \cup Y$ and form the nonempty edge sets of the components of $(V, X \cup Y)$. Similarly, the inclusionwise minimal sets in $\mathcal{F}$ form the nonempty edge sets of the components of $(V, X \cap Y)$. So

(44.23) $$\sum_{Z \in \mathcal{F}} f(Z) = f(X \cap Y) + f(X \cup Y),$$

and we have (44.20).

Further examples. Choquet [1951,1955] showed that the classical Newtonian capacity in $\mathbb{R}^3$ is submodular. Examples of submodular functions based on probability are given by Fujishige [1978b] and Han [1979], and other examples by Lovász [1983c].

44.2. Optimization over polymatroids by the greedy method

Edmonds [1970b] showed that one can optimize a linear function $w^{\mathsf{T}}x$ over an (extended) polymatroid by an extension of the greedy algorithm. The submodular set function f on S is given by a *value giving oracle*, that is, by an oracle that returns $f(U)$ for any $U \subseteq S$.

Let f be a submodular set function on S, and suppose that we want to maximize $w^{\mathsf{T}}x$ over EP_f, for some $w : S \to \mathbb{R}$. We can assume that $EP_f \neq \emptyset$, that is $f(\emptyset) \geq 0$, and hence that $f(\emptyset) = 0$ (since decreasing $f(\emptyset)$ maintains submodularity). We can also assume that $w \geq \mathbf{0}$, since if some component of w is negative, the maximum value is unbounded.

Now order the elements in S as $s_1, \ldots, s_n$ such that $w(s_1) \geq \cdots \geq w(s_n)$. Define

(44.24) $$U_i := \{s_1, \ldots, s_i\} \text{ for } i = 0, \ldots, n,$$

and define $x \in \mathbb{R}^S$ by

(44.25) $$x(s_i) := f(U_i) - f(U_{i-1}) \text{ for } i = 1, \ldots, n.$$

Then x maximizes $w^{\mathsf{T}}x$ over EP_f, as will be shown in the following theorem.

To prove it, consider the following linear programming duality equation:

(44.26) $$\begin{aligned} &\max\{w^{\mathsf{T}}x \mid x \in EP_f\} \\ &= \min\{\sum_{T \subseteq S} y(T)f(T) | y \in \mathbb{R}_+^{\mathcal{P}(S)}, \sum_{T \in \mathcal{P}(S)} y(T)\chi^T = w\}. \end{aligned}$$

Define:

(44.27) $$\begin{aligned} y(U_i) &:= w(s_i) - w(s_{i+1}) && (i = 1, \ldots, n-1), \\ y(S) &:= w(s_n), \\ y(T) &:= 0 && (T \neq U_i \text{ for each } i). \end{aligned}$$

Theorem 44.3. *Let f be a submodular set function on S with $f(\emptyset) = 0$ and let $w : S \to \mathbb{R}_+$. Then x and y given by* (44.25) *and* (44.27) *are optimum solutions of* (44.26).

Proof. We first show that x belongs to EP_f; that is, $x(T) \leq f(T)$ for each $T \subseteq S$. This is shown by induction on $|T|$, the case $T = \emptyset$ being trivial. Let $T \neq \emptyset$ and let k be the largest index with $s_k \in T$. Then by induction,

(44.28) $$x(T \setminus \{s_k\}) \leq f(T \setminus \{s_k\}).$$

Hence

(44.29) $$x(T) \leq f(T \setminus \{s_k\}) + x(s_k) = f(T \setminus \{s_k\}) + f(U_k) - f(U_{k-1}) \leq f(T)$$

(the last inequality follows from the submodularity of f). So $x \in EP_f$.

Also, y is feasible for (44.26). Trivially, $y \geq \mathbf{0}$. Moreover, for any i we have by (44.27):

(44.30) $$\sum_{T \ni s_i} y(T) = \sum_{j \geq i} y(U_j) = w(s_i).$$

So y is a feasible solution of (44.26).

Optimality of x and y follows from:

(44.31) $$w^{\mathsf{T}} x = \sum_{s \in S} w(s) x_s = \sum_{i=1}^{n} w(s_i)(f(U_i) - f(U_{i-1})) \\ = \sum_{i=1}^{n-1} f(U_i)(w(s_i) - w(s_{i+1})) + f(S) w(s_n) = \sum_{T \subseteq S} y(T) f(T).$$

The third equality follows from a straightforward reordering of the terms, using that $f(\emptyset) = 0$. ∎

Note that if f is integer, then x is integer, and that if w is integer, then y is integer. Moreover, if f is nondecreasing, then x is nonnegative. Hence, in that case, x and y are optimum solutions of

(44.32) $$\max\{w^{\mathsf{T}} x \mid x \in P_f\} \\ = \min\{\sum_{T \subseteq S} y(T) f(T) \mid y \in \mathbb{R}_+^{\mathcal{P}(S)}, \sum_{T \in \mathcal{P}(S)} y(T) \chi^T \geq w\}.$$

Therefore:

Corollary 44.3a. *Let f be a nondecreasing submodular set function on S with $f(\emptyset) = 0$ and let $w : S \to \mathbb{R}_+$. Then x and y given by* (44.25) *and* (44.27) *are optimum solutions for* (44.32).

Proof. Directly from Theorem 44.3, using the fact that $x \geq \mathbf{0}$ if f is nondecreasing. ∎

As for complexity we have:

Corollary 44.3b. *Given a submodular set function f on a set S (by a value giving oracle) and a function $w \in \mathbb{Q}^S$, we can find an $x \in EP_f$ maximizing $w^{\mathsf{T}} x$ in strongly polynomial time. If f is moreover nondecreasing, then $x \in P_f$ (and hence x maximizes $w^{\mathsf{T}} x$ over P_f).*

Proof. By the extension of the greedy method given above. ∎

The greedy algorithm can be interpreted geometrically as follows. Let w be some linear objective function on S, with $w(s_1) \geq \ldots \geq w(s_n)$. Travel via the vertices of P_f along the edges of P_f, by starting at the origin, as follows: first go from the origin as far as possible (in P_f) in the positive s_1-direction, say to vertex x_1; next go from x_1 as far as possible in the positive s_2-direction, say to x_2, and so on. After n steps one reaches a vertex x_n maximizing $w^{\mathsf{T}} x$

over P_f. In fact, the effectiveness of this algorithm characterizes polymatroids (Dunstan and Welsh [1973]).

44.3. Total dual integrality

Theorem 44.3 implies the box-total dual integrality of the following system:

(44.33) $x(U) \leq f(U)$ for $U \subseteq S$.

Corollary 44.3c. *If f is submodular, then* (44.33) *is box-totally dual integral.*

Proof. Consider the dual of maximizing $w^{\mathsf{T}}x$ over (44.33), for some $w \in \mathbb{Z}^S_+$. By Theorem 44.3, it has an optimum solution $y : \mathcal{P}(S) \to \mathbb{R}_+$ with the sets $U \subseteq S$ having $y(U) > 0$ forming a chain. So these constraints give a totally unimodular submatrix of the constraint matrix (by Theorem 41.11). Therefore, by Theorem 5.35, (44.33) is box-TDI. ∎

This gives the integrality of polyhedra:

Corollary 44.3d. *For any integer submodular set function f, the polymatroid P_f and the extended polymatroid EP_f are integer.*

Proof. Directly from Corollary 44.3c. (In fact, integer optimum solutions are explicitly given by Theorem 44.3 and Corollary 44.3a.) ∎

44.4. f is determined by EP_f

Theorem 44.3 implies that for any extended polymatroid P there is a unique submodular function f satisfying $f(\emptyset) = 0$ and $EP_f = P$, since:

Corollary 44.3e. *Let f be a submodular set function on S with $f(\emptyset) = 0$. Then*

(44.34) $f(U) = \max\{x(U) \mid x \in EP_f\}$

for each $U \subseteq S$.

Proof. Directly from Theorem 44.3 by taking $w := \chi^U$. ∎

So there is a one-to-one correspondence between nonempty extended polymatroids and submodular set functions f with $f(\emptyset) = 0$. The correspondence relates integer extended polymatroids with integer submodular functions.

There is a similar correspondence between nonempty polymatroids and *nondecreasing* submodular set functions f with $f(\emptyset) = 0$. For any (not necessarily nondecreasing) nonnegative submodular set function f, define $\bar{f}$ by:

(44.35) $\bar{f}(\emptyset) = 0,$
$\bar{f}(U) = \min_{T \supseteq U} f(T)$ for nonempty $U \subseteq S$.

It is easy to see that $\bar{f}$ is nondecreasing and submodular and that $P_{\bar{f}} = P_f$ (Dunstan [1973]). In fact, $\bar{f}$ is the unique nondecreasing submodular set function associated with P_f, with $\bar{f}(\emptyset) = 0$, as (Kelley [1959]):

Corollary 44.3f. *If f is a nondecreasing submodular function with $f(\emptyset) = 0$, then*

(44.36) $f(U) = \max\{x(U) \mid x \in P_f\}$

for each $U \subseteq S$.

Proof. This follows from Corollary 44.3a by taking $w := \chi^T$. ∎

This one-to-one correspondence between polymatroids and nondecreasing submodular set functions f with $f(\emptyset) = 0$ relates integer polymatroids to integer such functions:

Corollary 44.3g. *For each integer polymatroid P there exists a unique integer nondecreasing submodular function f with $f(\emptyset) = 0$ and $P = P_f$.*

Proof. By Corollary 44.3d and (44.36). ∎

By (44.36) we have for any nonnegative submodular set function f that $\bar{f}(U) = \max\{x(U) \mid x \in P_f\}$. Since we can optimize over EP_f in polynomial time (with the greedy algorithm described above), with the ellipsoid method we can optimize over $P_f = EP_f \cap \mathbb{R}^S_+$ in polynomial time. Hence we can calculate $\bar{f}(U)$ in polynomial time. Alternatively, calculating $\bar{f}(U)$ amounts to minimizing the submodular function $f'(T) := f(T \cup U)$.

In fact $\bar{f}$ is the largest among all nondecreasing submodular set functions g on S with $g(\emptyset) = 0$ and $g \leq f$, as can be checked straightforwardly.

44.5. Supermodular functions and contrapolymatroids

Similar results hold for supermodular functions and the associated contrapolymatroids. Associate the following polyhedra with a set function g on S:

(44.37) $Q_g := \{x \in \mathbb{R}^S \mid x \geq \mathbf{0}, x(U) \geq g(U)$ for each $U \subseteq S\}$,
$EQ_g := \{x \in \mathbb{R}^S \mid x(U) \geq g(U)$ for each $U \subseteq S\}$.

If g is supermodular, then Q_g and EQ_g are called the *contrapolymatroid* and the *extended contrapolymatroid associated with* g, respectively. A vector $x \in EQ_g$ (or Q_g) is called a *base vector* of EQ_g (or Q_g) if $x(S) = g(S)$. A *base vector* of g is a base vector of EQ_g.

Since $EQ_g = -EP_{-g}$, we can reduce most problems on (extended) contrapolymatroids to (extended) polymatroids. Again we can minimize a linear function $w^{\mathsf{T}}x$ over EQ_g with the greedy algorithm, as described in Section 44.2. (In fact, we can apply the same formulas (44.25) and (44.27) for g instead of f.) If g is nondecreasing, it yields a nonnegative optimum solution, and hence a vector x minimizing $w^{\mathsf{T}}x$ over Q_g.

Similarly, the system

(44.38) $\quad x(U) \geq g(U)$ for $U \subseteq S$

is box-TDI, as follows directly from the box-total dual integrality of

(44.39) $\quad x(U) \leq -g(U)$ for $U \subseteq S$.

Let EP_f be the extended polymatroid associated with the submodular function f with $f(\emptyset) = 0$. Let B_f be the face of base vectors of EP_f, i.e.,

(44.40) $\quad B_f = \{x \in EP_f \mid x(S) = f(S)\}$.

A vector $y \in \mathbb{R}^S$ is called *spanning* if there exists an x in B_f with $x \leq y$. Let Q be the set of spanning vectors.

A vector y belongs to Q if and only if $(f|y)(S) = f(S)$, that is (by (44.8) and (44.9)) if and only if

(44.41) $\quad y(U) \geq f(S) - f(S \setminus U)$

for each $U \subseteq S$. So Q is equal to the contrapolymatroid EQ_g associated with the submodular function g defined by $g(U) := f(S) - f(S \setminus U)$ for $U \subseteq S$. Then B_f is equal to the face of minimal elements of EQ_g.

There is a one-to-one correspondence between submodular set functions f on S with $f(\emptyset) = 0$ and supermodular set functions g on S with $g(\emptyset) = 0$, given by the relations

(44.42) $\quad g(U) = f(S) - f(S \setminus U)$ and $f(U) = g(S) - g(S \setminus U)$

for $U \subseteq S$.

Then the pair $(-g, -Q)$ is related to the pair (f, P) by a relation similar to the duality relation of matroids (cf. Section 44.6f).

44.6. Further results and notes

44.6a. Submodular functions and matroids

Let P be the polymatroid associated with the nondecreasing integer submodular set function f on S, with $f(\emptyset) = 0$. Then the collection

(44.43) $\quad \mathcal{I} := \{I \subseteq S \mid \chi^I \in P\}$

forms the collection of independent sets of a matroid $M = (S, \mathcal{I})$ (this result was announced by Edmonds and Rota [1966] and proved by Pym and Perfect [1970]). By Corollary 40.2b, the subpolymatroid (cf. Section 44.6c)

(44.44) $$P|\mathbf{1} = \{x \in P \mid x \leq \mathbf{1}\}$$

is the convex hull of the incidence vectors of the independent sets of M. By (44.8), the rank function r of M satisfies

(44.45) $$r(U) = \min_{T \subseteq U}(|U \setminus T| + f(T))$$

for $U \subseteq S$.

As an example, if f is the submodular function given in the set union example in Section 44.1a, we obtain the transversal matroid on $\{1,\ldots,n\}$ with $I \subseteq \{1,\ldots,n\}$ independent if and only if the family $(T_i \mid i \in I)$ has a transversal (Edmonds [1970b]).

44.6b. Reducing integer polymatroids to matroids

In fact, each integer polymatroid can be derived from a matroid as follows (Helgason [1974]). Let f be a nondecreasing submodular set function on S with $f(\emptyset) = 0$. Choose for each s in S, a set X_s of size $f(\{s\})$, such that the sets X_s ($s \in S$) are disjoint. Let $X := \bigcup_{s \in S} X_s$, and define a set function r on X by

(44.46) $$r(U) := \min_{T \subseteq S}(|U \setminus \bigcup_{s \in T} X_s| + f(T))$$

for $U \subseteq X$. One easily checks that r is the rank function of a matroid M (by checking the axioms (39.38)), and that for each subset T of S

(44.47) $$f(T) = r(\bigcup_{s \in T} X_s).$$

Therefore, f arises from the rank function of M, as in the Matroids example in Section 44.1a. The polymatroid P_f associated with f is just the convex hull of all vectors x for which there exists an independent set I in M with $x_s = |I \cap X_s|$ for all s in S.

Given a nondecreasing submodular set function f on S with $f(\emptyset) = 0$, Lovász [1980a] called a subset $U \subseteq S$ a *matching* if

(44.48) $$f(U) = \sum_{s \in U} f(\{s\}).$$

If $f(\{s\}) = 1$ for each s in S, f is the rank function of a matroid, and U is a matching if and only if U is independent in this matroid. If $f(\{s\}) = 2$ for each s in S, the elements of S correspond to certain flats of rank 2 in a matroid. Now determining the maximum size of a matching is just the matroid matching problem (cf. Chapter 43).

44.6c. The structure of polymatroids

Vertices of polymatroids (Edmonds [1970b], Shapley [1965,1971]). Let f be a submodular set function on a set $S = \{s_1,\ldots,s_n\}$ with $f(\emptyset) = 0$. Let P_f be the polymatroid associated with f. It follows immediately from the greedy algorithm, as in the proof of Corollary 44.3a, that the vertices of P_f are given by (for $i = 1,\ldots,n$):

(44.49) $$x(s_{\pi(i)}) = \begin{cases} f(\{s_{\pi(1)},\ldots,s_{\pi(i)}\}) - f(\{s_{\pi(1)},\ldots,s_{\pi(i-1)}\}) & \text{if } i \leq k, \\ 0 & \text{if } i > k, \end{cases}$$

where π ranges over all permutations of $\{1,\ldots,n\}$ and where k ranges over $0,\ldots,n$.

Similarly, for any submodular set function f on S with $f(\emptyset) = 0$, the vertices of the extended polymatroid EP_f are given by

(44.50) $$x(s_{\pi(i)}) = f(\{s_{\pi(1)},\ldots,s_{\pi(i)}\}) - f(\{s_{\pi(1)},\ldots,s_{\pi(i-1)}\})$$

for $i = 1,\ldots,n$, where π ranges over all permutations of $\{1,\ldots,n\}$.

Topkis [1984] characterized adjacency of the vertices of a polymatroid, while Bixby, Cunningham, and Topkis [1985] and Topkis [1992] gave further results on vertices of and paths on a polymatroid and on related partial orders of S.

Facets of polymatroids. Let f be a nondecreasing submodular set function on S with $f(\emptyset) = 0$. One easily checks that P_f is full-dimensional if and only if $f(\{s\}) > 0$ for all s in S. If P_f is full-dimensional there is a unique minimal collection of linear inequalities defining P_f (clearly, up to scalar multiplication). They correspond to the facets of P_f. Edmonds [1970b] found that this collection is given by the following theorem. A subset $U \subseteq S$ is called an *f-flat* if $f(U \cup \{s\}) > f(U)$ for all $s \in S \setminus U$, and U is called *f-inseparable* if there is no partition of U into nonempty sets U_1 and U_2 with $f(U) = f(U_1) + f(U_2)$. Then:

Theorem 44.4. *Let f be a nondecreasing submodular set function on S with $f(\emptyset) = 0$ and $f(\{s\}) > 0$ for each $s \in S$. The following is a minimal system determining the polymatroid P_f:*

(44.51) $$\begin{array}{ll} x_s \geq 0 & (s \in S), \\ x(U) \leq f(U) & (U \text{ is a nonempty } f\text{-inseparable } f\text{-flat}). \end{array}$$

Proof. It is easy to see that (44.51) determines P_f, as any other inequality $x(U) \leq f(U)$ follows from (44.51). The irredundancy of collection (44.51) can be seen as follows.

Clearly, each inequality $x_s \geq 0$ determines a facet. Next consider a nonempty f-inseparable f-flat U. Suppose that the face determined by U is not a facet. Then it is contained in another face, say determined by T. Let x be a vertex of P_f with $x(U \setminus T) = f(U \setminus T)$, $x(U) = f(U)$, and $x(S \setminus U) = 0$. Such a vertex exists by the greedy algorithm (cf. (44.49)).

Since x is on the face determined by U, it is also on the face determined by T. So $x(T) = f(T)$. Hence $f(T) = x(T) = x(T \cap U) = f(U) - f(U \setminus T)$. So we have equality throughout in:

(44.52) $$f(U \setminus T) + f(T) \geq f(U \setminus T) + f(T \cap U) \geq f(U).$$

This implies that $U \setminus T = \emptyset$ or $T \cap U = \emptyset$ (as U is f-inseparable), and that $f(T) = f(T \cap U)$. If $U \setminus T = \emptyset$, then $U \subset T$, and hence (as U is an f-flat) $f(T) > f(U) \geq f(T \cap U)$, a contradiction. If $T \cap U = \emptyset$, then $f(T) = f(T \cap U) = 0$, implying that $T = \emptyset$, again a contradiction. ∎

It follows that the face $\{x \in P_f \mid x(S) = f(S)\}$ of maximal vectors in P_f is a facet if and only if $f(U) + f(S \setminus U) > f(S)$ for each proper nonempty subset U of S. More generally, its codimension is equal to the number of inclusionwise minimal nonempty sets U with $f(U) + f(S \setminus U) = f(S)$ (cf. Fujishige [1984a]).

Faces of polymatroids (Giles [1975]). We now extend the characterizations of vertices and facets of polymatroids given above to arbitrary faces. Let P be the polymatroid associated with the nondecreasing submodular set function f on S with $f(\emptyset) = 0$. Suppose that P is full-dimensional. If $\emptyset \neq S_1 \subset \cdots \subset S_k \subseteq T \subseteq S$, then

(44.53) $$F = \{x \in P \mid x(S_1) = f(S_1), \ldots, x(S_k) = f(S_k), x(S \setminus T) = 0\}$$

is a face of P of dimension at most $|T| - k$. (Indeed, F is nonempty by the characterization (44.49) of vertices, while $\dim(F) \leq |T| - k$, as the incidence vectors of $S_1, \ldots, S_k$ are linearly independent.)

In fact, each face has a representation (44.53). Indeed, let F be a face of P. Define $T = \{s \in S \mid x_s > 0$ for some x in $F\}$, and let $S_1 \subset \cdots \subset S_k$ be any maximal chain of nonempty subsets of T with the property that

(44.54) $$F \subseteq \{x \in P \mid x(S_1) = f(S_1), \ldots, x(S_k) = f(S_k), x(S \setminus T) = 0\}.$$

Then we have equality in (44.54), and $\dim(F) = |T| - k$. (Here a maximal chain is a chain which is contained in no larger chain satisfying (44.54) — since the empty chain satisfies (44.54), there exist maximal chains.)

In order to prove this assertion, suppose that F has dimension d. As the right-hand side of (44.54) is a face of P of dimension at most $|T| - k$, it suffices to show that $d = |T| - k$. Therefore, suppose $d < |T| - k$. Then there exists a subset U of S such that $x(U) = f(U)$ for all x in F, and such that the incidence vector of $U \cap T$ is linearly independent of the incidence vectors of $S_1, \ldots, S_k$. That is, $U \cap T$ is not the union of some of the sets $S_i \setminus S_{i-1}$ $(i = 1, \ldots, k)$. Since $x(U \cap T) = x(U) = f(U) \geq f(U \cap T)$ for all x in F, we may assume that $U \subseteq T$. Since the collection of subsets U of S with $x(U) = f(U)$ is closed under taking unions and intersections, we may assume moreover that U is comparable with each of the sets in the chain $S_1 \subset \cdots \subset S_k$. Hence U could be added to the chain to obtain a larger chain, contradicting our assumption. So $d = |T| - k$.

Note that a chain $S_1 \subset \cdots \subset S_k$ of nonempty subsets of T is a maximal chain satisfying (44.54) if and only if there is equality in (44.54) and (setting $S_0 := \emptyset$):

(44.55) $f(S_k \cup \{s\}) > f(S_k)$ for all s in $T \setminus S_k$, and each of the sets $S_i \setminus S_{i-1}$ is f_i-inseparable, where f_i is the submodular set function on $S_i \setminus S_{i-1}$ given by $f_i(U) := f(U \cup S_{i-1}) - f(S_{i-1})$ for $U \subseteq S_i \setminus S_{i-1}$.

This may be derived straightforwardly from the existence, by (44.49), of appropriate vertices of F.

It is not difficult to show that if F has a representation (44.53), then F is the direct sum of $F_1, \ldots, F_k$ and Q, where F_i is the face of maximal vectors in the polymatroid associated with f_i $(i = 1, \ldots, k)$, and Q is the polymatroid associated with the submodular set function g on $T \setminus S_k$ given by $g(U) := f(U \cup S_k) - f(S_k)$ for $U \subseteq T \setminus S_k$. Since $\dim(F_i) \leq |S_i \setminus S_{i-1}| - 1$ and $\dim(Q) \leq |T \setminus S_k|$, this yields that $\dim(F) = |T| - k$ if and only if $\dim(F_i) = |S_i \setminus S_{i-1}| - 1$ $(i = 1, \ldots, k)$ and $\dim(Q) = |T \setminus S_k|$. From this, characterization (44.55) can be derived again. It also yields that if F, represented by (44.53), has dimension $|T| - k$, then the unordered partition $\{S_1, S_2 \setminus S_1, \ldots, S_k \setminus S_{k-1}, T \setminus S_k\}$ is the same for all maximal chains $S_1 \subset \cdots \subset S_k$.

For a characterization of the faces of a polymatroid, see Fujishige [1984a].

44.6d. Characterization of polymatroids

Let P be the polymatroid associated with the nondecreasing submodular set function f on S with $f(\emptyset) = 0$. The following three observations are easily derived from the representation (44.49) of vertices of P. (a) If x_0 is a vertex of P, there exists a vertex x_1 of P such that $x_1 \geq x_0$ and x_1 has the form (44.49) with $k = n$. (b) A vertex x_1 of P can be represented as (44.49) with $k = n$ if and only if $x_1(S) = f(S)$. (c) The convex hull of the vertices x_1 of P with $x_1(S) = f(S)$ is the face $\{x \in P \mid x(S) = f(S)\}$ of P. It follows directly from (a), (b) and (c) that $x \in P$ is a maximal element of P (with respect to $\leq$) if and only if $x(S) = f(S)$. So for each vector y in P there is a vector x in P with $y \leq x$ and $x(S) = f(S)$.

Applying this to the subpolymatroids $P|a = P \cap \{x \mid x \leq a\}$ (cf. Section 44.1), one finds the following property of polymatroids:

(44.56) for each $a \in \mathbb{R}^S_+$ there exists a number $r(a)$ such that each maximal vector x of $P \cap \{x \mid x \leq a\}$ satisfies $x(S) = r(a)$.

Here *maximal* is maximal in the partial order $\leq$ on vectors. The number $r(a)$ is called the *rank* of a, and any x with the properties mentioned in (44.56) is called a *base* of a.

Edmonds [1970b] (cf. Dunstan [1973], Woodall [1974b]) noticed the following (we follow the proof of Welsh [1976]):

Theorem 44.5. *Let $P \subseteq \mathbb{R}^S_+$. Then P is a polymatroid if and only if P is compact, and satisfies* (44.56) *and*

(44.57) *if $\mathbf{0} \leq y \leq x \in P$, then $y \in P$.*

Proof. Necessity was observed above. To see sufficiency, let f be the set function on S defined by

(44.58) $f(U) := \max\{x(U) \mid x \in P\}$

for $U \subseteq S$. Then f is nonnegative and nondecreasing. Moreover, f is submodular. To see this, consider $T, U \subseteq S$. Let x be a maximal vector in P satisfying $x_s = 0$ if $s \notin T \cup U$, and let y be a maximal vector in P satisfying $y(s) = 0$ if $s \notin T \cap U$ and $x \leq y$. Note that (44.56) and (44.58) imply that $x(T \cap U) = f(T \cap U)$ and $y(T \cup U) = f(T \cup U)$. Hence

(44.59) $$f(T)+f(U) \geq y(T)+y(U) = y(T\cap U)+y(T\cup U) \geq x(T\cap U)+y(T\cup U) = f(T \cap U) + f(T \cup U),$$

that is, f is submodular.

We finally show that P is equal to the polymatroid P_f associated to f. Clearly, $P \subseteq P_f$, since if $x \in P$ then $x(U) \leq f(U)$ for each $U \subseteq S$, by definition (44.58) of f.

To see that $P_f = P$, suppose $v \in P_f \setminus P$. Let u be a base of v (that is, a maximal vector $u \in P$ satisfying $u \leq v$). Choose u such that the set

(44.60) $U := \{s \in S \mid u_s < v_s\}$

is as large as possible. Since $v \notin P$, we have $u \neq v$, and hence $U \neq \emptyset$. As $v \in P_f$, we know

(44.61) $u(U) < v(U) \leq f(U)$.

Define

(44.62) $w := \frac{1}{2}(u + v)$.

So $u \leq w \leq v$. Hence u is a base of w, and each base of w is a base of v.

For any $z \in \mathbb{R}^S$, define z' as the projection of z on the subspace $L := \{x \in \mathbb{R}^S \mid x_s = 0 \text{ if } s \in S \setminus U\}$. That is:

(44.63) $z'(s) := z(s)$ if $s \in U$, and $z'(s) := 0$ if $s \in S \setminus U$.

By definition of f, there is an $x \in P$ with $x(U) = f(U)$. We may assume that $x \in L$. Choose $y \in L$ with $x \leq y$ and $u' \leq y$. Then

(44.64) $x(S) = x(U) = f(U) > u(U) = u'(U) = u'(S)$.

So $r(y) > u'(S)$. Hence, by (44.56), there exists a base z of y with $u' \leq z$ and $z(S) > u'(S)$. So $u'_s < z_s$ for at least one $s \in U$. This implies, since $u'_s < w'_s$ for each $s \in S$, that there is an $a \in P$ with $u' \leq a \leq w'$ and $a \neq u'$, hence $a(U) > u'(U)$.

Since $a \leq w' \leq w$, there is a base b of w with $a \leq b$. Then $b(S) = u(S)$ (since also u is a base of w) and $b(U) \geq a(U) > u'(U) = u(U)$. Hence $b_s < u_s = v_s$ for some $s \in S \setminus U$. Moreover, $b_s \leq w_s < v_s$ for each $s \in U$. So U is properly contained in $\{s \in S \mid b_s < v_s\}$, contradicting the maximality of U. ∎

(For an alternative characterization, see Welsh [1976].)

By (44.8) and (44.9) the rank of a is given by

(44.65) $r(a) = \min_{U \subseteq S}(a(S \setminus U) + f(U))$

(from this one may derive a 'submodular law' for r: $r(a \wedge b) + r(a \vee b) \leq r(a) + r(b)$, where $\wedge$ and $\vee$ are the meet and join in the lattice $(\mathbb{R}^S, \leq)$ (Edmonds [1970b])).

Since if P has integer vertices and a is integer, the intersection $P|a = \{x \in P \mid x \leq a\}$ is integer again, we know that for integer polymatroids (44.56) also holds if we restrict a and x to integer vectors. So if a is integer, then there exists an integer vector $x \leq a$ in P with $x(S) = r(a)$.

Theorem 44.5 yields an analogous characterization of extended polymatroids. Let f be a submodular set function on S with $f(\emptyset) = 0$. Choose $c \in \mathbb{R}^S_+$ such that

(44.66) $g(U) := f(U) + c(U)$

is nonnegative for all $U \subseteq S$. Clearly, g again is submodular, and $g(\emptyset) = 0$. Then the extended polymatroid EP_f associated with f and the polymatroid P_g associated with g are related by:

(44.67) $P_g = \{x \mid x \geq \mathbf{0}, x - c \in EP_f\} = (c + EP_f) \cap \mathbb{R}^S_+$.

Since P_g is a polymatroid, by (44.56) we know that EP_f satisfies:

(44.68) for each a in $\mathbb{R}^S$ there exists a number $r(a)$ such that each maximal vector x in $EP_f \cap \{x \in \mathbb{R}^S \mid x \leq a\}$ satisfies $x(S) = r(a)$.

One easily derives from Theorem 44.5 that (44.68) together with

(44.69) if $y \leq x \in EP_f$, then $y \in EP_f$,

characterizes the class of all extended polymatroids among the closed subsets of $\mathbb{R}^S$.

44.6e. Operations on submodular functions and polymatroids

The class of submodular set functions on a given set is closed under certain operations. Obviously, the sum of two submodular functions is submodular again. In particular, adding a constant t to all values of a submodular function maintains submodularity. Also the multiplication of a submodular function by a nonnegative scalar maintains submodularity. Moreover, if f is a nondecreasing submodular set function on S, and q is a real number, then the function f' given by $f'(U) := \min\{q, f(U)\}$ for $U \subseteq S$, is submodular again. (Monotonicity cannot be deleted, as is shown by taking $S := \{a, b\}$, $f(\emptyset) = f(S) = 1$, $f(\{a\}) = 0$, $f(\{b\}) = 2$, and $q = 1$.)

It follows that the class of all submodular set functions on S forms a convex cone C in $\mathbb{R}^{\mathcal{P}(S)}$. This cone is polyhedral as the constraints (44.1) form a finite set of linear inequalities defining C. Edmonds [1970b] raised the problem of determining the extreme rays of the cone of all nonnegative nondecreasing submodular set functions f on S with $f(\emptyset) = 0$. It is not difficult to show that the rank function r of a matroid M determines an extreme ray of this cone if and only if r is not the sum of the rank functions of two other matroids, i.e., if and only if M is the sum of a connected matroid and a number of loops. But these do not represent all extreme rays: if $S = \{1, \ldots, 5\}$ and $w(1) = 2, w(s) = 1$ for $s \in S \setminus \{1\}$, let $f(U) := \min\{3, w(U)\}$ for $U \subseteq S$; then f is on an extreme ray, but cannot be decomposed as the sum of rank functions of matroids (L. Lovász's example; cf. also Murty and Simon [1978] and Nguyen [1978]).

Lovász [1983c] observed that if f_1 and f_2 are submodular and $f_1 - f_2$ is nondecreasing, then $\min\{f_1, f_2\}$ is submodular.

Let f be a nonnegative submodular set function on S. Clearly, for any $\lambda \geq 0$ we have $P_{\lambda f} = \lambda P_f$ (where $\lambda P_f = \{\lambda x \mid x \in P_f\}$). If $q \geq 0$, and f' is given by $f'(U) = \min\{q, f(U)\}$ for $U \subseteq S$, then f' is submodular and

(44.70) $$P_{f'} = \{x \in P_f \mid x(S) \leq q\},$$

as can be checked easily. So the class of polymatroids is closed under intersections with affine halfspaces of the form $\{x \in \mathbb{R}^S \mid x(S) \leq q\}$, for $q \geq 0$.

Let f_1 and f_2 be nondecreasing submodular set functions on S, with $f_1(\emptyset) = f_2(\emptyset) = 0$, and associated polymatroids P_1 and P_2 respectively. Let P be the polymatroid associated with $f := f_1 + f_2$. Then (McDiarmid [1975c]):

Theorem 44.6. $P_{f_1+f_2} = P_{f_1} + P_{f_2}$.

Proof. It is easy to see that $P_{f_1+f_2} \supseteq P_{f_1} + P_{f_2}$. To prove the reverse inclusion, let x be a vertex of $P_{f_1+f_2}$. Then x has the form (44.49). Hence, by taking the same permutation π and the same $k, x = x_1 + x_2$ for certain vertices x_1 of P_{f_1} and x_2 of P_{f_2}. Since $P_{f_1} + P_{f_2}$ is convex it follows that $P_{f_1+f_2} = P_{f_1} + P_{f_2}$. ∎

In fact, if f_1 and f_2 are integer, each *integer* vector in $P_{f_1} + P_{f_2}$ is the sum of *integer* vectors in P_{f_1} and P_{f_2} — see Corollary 46.2c. Similarly, if f_1 and f_2 are integer, each integer vector in $EP_{f_1} + EP_{f_2}$ is the sum of integer vectors in EP_{f_1} and EP_{f_2}.

Faigle [1984a] derived from Theorem 44.6 that, for any submodular function f, if $x, y \in P_f$ and $x = x_1 + x_2$ with $x_1, x_2 \in P_f$, then there exist $y_1, y_2 \in P_f$ with

$y = y_1 + y_2$ and $x_1+y_1, x_2+y_2 \in P_f$. (Proof: $y \in P_f \subseteq P_{2f-x} = P_{f-x_1} + P_{f-x_2}$.) An integer version of this can be derived from Corollary 46.2c and generalizes (42.13).

If $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ are matroids, with rank functions r_1 and r_2 and corresponding independent set polytopes P_1 and P_2, respectively, then by Section 44.6c above, $P_1 + P_2$ is the convex hull of sums of incidence vectors of independent sets in M_1 and M_2. Hence the 0,1 vectors in $P_1 + P_2$ are just the incidence vectors of the sets $I_1 \cup I_2$, for $I_1 \in \mathcal{I}_1$ and $I_2 \in \mathcal{I}_2$. Therefore, the polyhedron

(44.71) $$(P_1 + P_2)|\mathbf{1} = \{x \in P_1 + P_2 \mid x \leq \mathbf{1}\}$$

is the convex hull of the independent sets of $M_1 \vee M_2$. By Theorem 44.6 and (44.45), it follows that the rank function r of $M_1 \vee M_2$ satisfies

(44.72) $$r(U) = \min_{T \subseteq U}(|U \setminus T| + r_1(T) + r_2(T))$$

for $U \subseteq S$. Thus we have derived the matroid union theorem (Corollary 42.1a).

44.6f. Duals of polymatroids

McDiarmid [1975c] described the following duality of polymatroids. Let P be the polymatroid associated with the nondecreasing submodular set function f on S with $f(\emptyset) = 0$ and let a be a vector in $\mathbb{R}^S$ with $a \geq x$ for all x in P (i.e., $a(s) \geq f(\{s\})$ for all s in S). Define

(44.73) $$f^*(U) := a(U) + f(S \setminus U) - f(S)$$

for $U \subseteq S$. One easily checks that f^* again is nondecreasing and submodular, and that $f^*(\emptyset) = 0$. We call f^* the *dual* of f (with respect to a). Then $f^{**} = f$ taking the second dual with respect to the same a, as follows immediately from (44.73).

Let P^* be the polymatroid associated with f^*, and call P^* the *dual* polymatroid of P (with respect to a). Now the maximal vertices of P and P^* are given by (44.49) by choosing $k = n$. It follows that x is a maximal vertex of P if and only if $a - x$ is a maximal vertex of P^*. Since the maximal vectors of a polymatroid form just the convex hull of the maximal vertices, we may replace in the previous sentence the word 'vertex' by 'vector'. So the set of maximal vectors of P^* arises from the set of maximal vectors of P by reflection in the point $\frac{1}{2}a$.

Clearly, duals of matroids correspond in the obvious way to duals of the related polymatroids (with respect to the vector $\mathbf{1}$).

44.6g. Induction of polymatroids

Let $G = (V, E)$ be a bipartite graph, with colour classes S and T. Let f be a nondecreasing submodular set function on S with $f(\emptyset) = 0$, and define

(44.74) $$g(U) := f(N(U))$$

for $U \subseteq T$ (cf. Section 44.1a). (As usual, $N(U)$ denotes the set of vertices not in U adjacent to at least one vertex in U.)

The function g again is nondecreasing and submodular. Similarly to Rado's theorem (Corollary 41.1c), one may prove that a vector x belongs to P_g if and only if there exist $y \in \mathbb{R}^E_+$ and $z \in P_f$ such that

(44.75)
$$y(\delta(t)) = x_t \quad (t \in T),$$
$$y(\delta(s)) = z_s \quad (s \in S).$$

Moreover, if f and g are integer, we can take y and z to be integer. This procedure gives an 'induction' of polymatroids through bipartite graphs, and yields 'Rado's theorem for polymatroids' (cf. McDiarmid [1975c]).

In case f is the rank function of a matroid on S, a 0,1 vector x belongs to P_g if and only if there exists a matching in G whose end vertices in S form an independent set of the matroid, and the end vertices in T have x as incidence vector. So these 0,1 vectors determine a matroid on T, with rank function r given by

(44.76)
$$r(U) = \min_{W \subseteq U} (|U \setminus W| + f(N(W)))$$

for $U \subseteq T$ (cf. (44.45) and (44.74)).

Another extension is the following. Let $D = (V, A)$ be a directed graph and let V be partitioned into classes S and T. Let furthermore a 'capacity' function $c : A \to \mathbb{R}_+$ be given. Define the set function g on T by

(44.77)
$$g(U) := c(\delta^{\text{out}}(U))$$

for $U \subseteq T$, where $\delta^{\text{out}}(U)$ denotes the set of arcs leaving U. Then g is nonnegative and submodular, and it may be derived straightforwardly from the max-flow min-cut theorem (Theorem 10.3) that a vector x in $\mathbb{R}^T_+$ belongs to P_g if and only if there exist $T - S$ paths $Q_1, \ldots, Q_k$ and nonnegative numbers $\lambda_1, \ldots, \lambda_k$ (for some k), such that

(44.78)
$$\sum_{i=1}^{k} \lambda_i \chi^{AQ_i} \leq c \text{ and } \sum_{i=1}^{k} \lambda_i \chi^{b(Q_i)} = x,$$

where $b(Q_i)$ is the beginning vertex of Q_i. If the c and x are integer, we can take also the λ_i integer.

Here the function g in general is not nondecreasing, but the value

(44.79)
$$\bar{g}(U) = \min\{g(W) \mid U \subseteq W \subseteq T\}$$

of the associated nondecreasing submodular function (cf. (44.35)) is equal to the minimum capacity of a cut separating U and S, which is equal to the maximum amount of flow from U to S, subject to the capacity function c (by the max-flow min-cut theorem).

In an analogous way, one can construct polymatroids by taking vertex-capacities instead of arc-capacities. Moreover, the notion of induction of polymatroids through bipartite graphs can be extended in a natural way to the induction of polymatroids through directed graphs (cf. McDiarmid [1975c], Schrijver [1978]).

44.6h. Lovász's generalization of Kőnig's matching theorem

Lovász [1970a] gave the following generalization of Kőnig's matching theorem (Theorem 16.2).

For a graph $G = (V, E)$, $U \subseteq V$, and $F \subseteq E$, let $N_F(U)$ denote the set of vertices not in U that are adjacent in (V, F) to at least one vertex in U. Kőnig's matching theorem follows by taking $g(X) := |X|$ in the following theorem.

Theorem 44.7. *Let $G = (V, E)$ be a simple bipartite graph, with colour classes S and T. Let g be a supermodular set function on S, such that $g(\{v\}) \geq 0$ for each $v \in S$ and such that*

(44.80) $g(U \cup \{v\}) \leq g(U) + g(\{v\})$ *for nonempty* $U \subseteq S$ *and* $v \in S \setminus U$.

Then E has a subset F with $\deg_F(v) = g(\{v\})$ for each $v \in V$ and $|N_F(U)| \geq g(U)$ for each nonempty $U \subseteq S$ if and only if $|N_E(U)| \geq g(U)$ for each nonempty $U \subseteq S$.

Proof. Necessity being trivial, we show sufficiency. Choose $F \subseteq E$ such that

(44.81) $|N_F(U)| \geq g(U)$

for each nonempty $U \subseteq S$, with $|F|$ as small as possible. We show that F is as required.

Suppose to the contrary that $\deg_F(v) > g(\{v\})$ for some $v \in S$. By the minimality of F, for each edge $e = vw \in F$, there is a subset U_e of S with $v \in U_e$, $|N_F(U_e)| = g(U_e)$, and $w \notin N_F(U_e \setminus \{v\})$. Since the function $|N_F(U)|$ is submodular, the intersection U of the U_e over $e \in \delta(v)$ satisfies $|N_F(U)| = g(U)$ (using (44.81)). Then no neighbour w of v is adjacent to U. Hence $N_F(v)$ and $N_F(U \setminus \{v\})$ are disjoint. Moreover, $U \neq \{v\}$, since $N_F(U) = g(U)$ and $N_F(\{v\}) > g(v)$. This gives the contradiction

(44.82) $g(U) \leq g(U \setminus \{v\}) + g(\{v\}) < |N_F(U \setminus \{v\})| + |N_F(v)| = |N_F(U)|$. ∎

For a derivation of this theorem with the Edmonds-Giles method, see Frank and Tardos [1989].

44.6i. Further notes

Edmonds [1970b] and D.A. Higgs (as mentioned in Edmonds [1970b]) observed that if f is a set function on a set S, we can define recursively a submodular function $\bar{f}$ as follows:

(44.83) $\bar{f}(T) := \min\{f(T), \min(\bar{f}(S_1) + \bar{f}(S_2) - \bar{f}(S_1 \cap S_2))\}$,

where the second minimum ranges over all pairs S_1, S_2 of proper subsets of T with $S_1 \cup S_2 = T$.

Lovász [1983c] gave the following characterization of submodularity in terms of convexity. Let f be a set function on S and define for each $c \in \mathbb{R}^S_+$

(44.84) $\hat{f}(c) := \sum_{i=1}^{k} \lambda_i f(U_i)$,

where $\emptyset \neq U_1 \subset U_2 \subset \cdots \subset U_k \subseteq S$ and $\lambda_1, \ldots, \lambda_k > 0$ are such that $c = \sum_{i=1}^{k} \lambda_i \chi^{U_i}$. Then f is submodular if and only if $\hat{f}$ is convex. Similarly, f is supermodular if and only if $\hat{f}$ is concave. Related is the 'subdifferential' of a submodular function, investigated by Fujishige [1984d].

Korte and Lovász [1985c] and Nakamura [1988a] studied polyhedral structures where the greedy algorithm applies. Federgruen and Groenevelt [1986] extended the greedy method for polymatroids to 'weakly concave' objective functions (instead of linear functions). (Related work was reported by Bhattacharya, Georgiadis, and

Tsoucas [1992].) Nakamura [1993] extended polymatroids and submodular functions to Δ-polymatroids and Δ-submodular functions.

Gröflin and Liebling [1981] studied the following example of 'transversal polymatroids'. Let $G = (V, E)$ be an undirected graph, and define the submodular set function f on E by $f(F) := \left|\bigcup F\right|$ for $F \subseteq E$. Then the vertices of the associated polymatroid are all $\{0, 1, 2\}$ vectors x in $\mathbb{R}^E$ with the property that the set $F := \{e \in E \mid x_e \geq 1\}$ forms a forest each component of which contains at most one edge e with $x_e = 2$. If x is a maximal vertex, then each component contains exactly one edge e with $x_e = 2$.

Narayanan [1991] studied, for a given submodular function f on S, the lattice of all partitions $\mathcal{P}$ of S into nonempty sets such that there exists a $\lambda \in \mathbb{R}$ for which $\mathcal{P}$ attains $\min \sum_{U \in \mathcal{P}} (f(U) - \lambda)$ (taken over all partitions $\mathcal{P}$). Fujishige [1980b] studied minimum values of submodular functions.

For results on the (NP-hard) problems of *maximizing* a submodular function and of submodular set cover, see Fisher, Nemhauser, and Wolsey [1978], Nemhauser and Wolsey [1978,1981], Nemhauser, Wolsey, and Fisher [1978], Wolsey [1982a,1982b], Conforti and Cornuéjols [1984], and Fujito [1999].

Cunningham [1983], Fujishige [1983], and Nakamura [1988c] presented decomposition theories for submodular functions. Benczúr and Frank [1999] considered covering symmetric supermodular functions by graphs.

For surveys and books on polymatroids and submodular functions, see McDiarmid [1975c], Welsh [1976], Lovász [1983c], Lawler [1985], Nemhauser and Wolsey [1988], Fujishige [1991], Narayanan [1997], and Murota [2002]. For a survey on applications of submodular functions, see Frank [1993a].

Historically, submodular functions arose in lattice theory (Bergmann [1929], Birkhoff [1933]), while submodularity of the rank function of a matroid was shown by Bergmann [1929] and Whitney [1935]. Choquet [1951,1955] and Kelley [1959] studied submodular functions in relation to the Newton capacity and to measures in Boolean algebras. The relevance of submodularity for optimization was revealed by Edmonds [1970b].

Several alternative names have been proposed for submodular functions, like sub-valuation (Choquet [1955]), β-function (Edmonds [1970b]), and ground set rank function (McDiarmid [1975c]). The set of integer vectors in an integer polymatroid was called a hypermatroid by Helgason [1974] and Lovász [1977c]. A generalization of polymatroids (called supermatroids) was studied by Dunstan, Ingleton, and Welsh [1972].