Chapter 41

Matroid intersection

Edmonds discovered that matroids have even more algorithmic power than just that of the greedy method. He showed that there exist efficient algorithms also for *intersections* of matroids. That is, a maximum-weight common independent set in *two* matroids can be found in strongly polynomial time. Edmonds also found good min-max characterizations for matroid intersection.

Matroid intersection yields a motivation for studying matroids: we may apply it to two matroids from different classes of examples of matroids, and thus we obtain methods that exceed the bounds of any particular class.

We should note here that if $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ are matroids, then $(S, \mathcal{I}_1 \cap \mathcal{I}_2)$ need not be a matroid. (An example with $|S| = 3$ is easy to construct.)

Moreover, the problem of finding a maximum-size common independent set in *three* matroids is NP-complete (as finding a Hamiltonian circuit in a directed graph is a special case; also, finding a common transversal of three partitions is a special case).

41.1. Matroid intersection theorem

Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be two matroids, on the same set S. Consider the collection $\mathcal{I}_1 \cap \mathcal{I}_2$ of *common independent sets*. The pair $(S, \mathcal{I}_1 \cap \mathcal{I}_2)$ is generally *not* a matroid again.

Edmonds [1970b] showed the following formula, for which he gave two proofs — one based on linear programming duality and total unimodularity (see the proof of Theorem 41.12 below), and one reducing it to the matroid union theorem (see Corollary 42.1a and the remark thereafter). We give the direct proof implicit in Brualdi [1971e].

Theorem 41.1 (matroid intersection theorem)**.** *Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be matroids, with rank functions r_1 and r_2, respectively. Then the maximum size of a set in $\mathcal{I}_1 \cap \mathcal{I}_2$ is equal to*

(41.1) $$\min_{U \subseteq S} (r_1(U) + r_2(S \setminus U)).$$

Proof. Let k be equal to (41.1). It is easy to see that the maximum is not more than k, since for any common independent set I and any $U \subseteq S$:

$$(41.2) \qquad |I| = |I \cap U| + |I \setminus U| \leq r_1(U) + r_2(S \setminus U).$$

We prove equality by induction on $|S|$, the case $|S| \leq 1$ being trivial. So assume that $|S| \geq 2$.

If minimum (41.1) is attained only by $U = S$ or $U = \emptyset$, choose $s \in S$. Then $r_1(U) + r_2(S \setminus (U \cup \{s\})) \geq k$ for each $U \subseteq S \setminus \{s\}$, since otherwise both U and $U \cup \{s\}$ would attain (41.1), whence $\{U, U \cup \{s\}\} = \{\emptyset, S\}$, contradicting the fact that $|S| \geq 2$. Hence, by induction, $M_1 \setminus s$ and $M_2 \setminus s$ have a common independent set of size k, implying the theorem.

So we can assume that (41.1) is attained by some U with $\emptyset \neq U \neq S$. Then $M_1|U$ and $M_2 \cdot U$ have a common independent set I of size $r_1(U)$. Otherwise, by induction, there exists a subset T of U with

$$(41.3) \qquad r_1(U) > r_{M_1|U}(T) + r_{M_2 \cdot U}(U \setminus T) = r_1(T) + r_2(S \setminus T) - r_2(S \setminus U),$$

contradicting the fact that U attains (41.1). Similarly, $M_1 \cdot (S \setminus U)$ and $M_2|(S \setminus U)$ have a common independent set J of size $r_2(S \setminus U)$.

Now $I \cup J$ is a common independent set of M_1 and M_2. Indeed, $I \cup J$ is independent in M_1, as I is independent in $M_1|U$ and J is independent in $M_1 \cdot (S \setminus U) = M_1/U$ (cf. (39.10)). Similarly, $I \cup J$ is independent in M_2. As $|I \cup J| = r_1(U) + r_2(S \setminus U)$, this proves the theorem. ∎

This implies a characterization of the existence of a common base in two matroids:

Corollary 41.1a. *Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be matroids, with rank functions r_1 and r_2, respectively, such that $r_1(S) = r_2(S)$. Then M_1 and M_2 have a common base if and only if $r_1(U) + r_2(S \setminus U) \geq r_1(S)$ for each $U \subseteq S$.*

Proof. Directly from Theorem 41.1. ∎

It is easy to derive from the matroid intersection theorem a similar min-max relation for the minimum size of a common spanning set:

Corollary 41.1b. *Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be matroids, with rank functions r_1 and r_2, respectively. Then the minimum size of a common spanning set of M_1 and M_2 is equal to*

$$(41.4) \qquad \max_{U \subseteq S} (r_1(S) - r_1(U) + r_2(S) - r_2(S \setminus U)).$$

Proof. The minimum is equal to the minimum of $|B_1 \cup B_2|$ where B_1 and B_2 are bases of M_1 and M_2 respectively. Hence the minimum is equal to $r_1(S) + r_2(S)$ minus the maximum of $|B_1 \cap B_2|$ over such B_1, B_2. This last maximum is characterized in the matroid intersection theorem, yielding the present corollary. ∎

The following result of Rado [1942] (a generalization of Hall's marriage theorem (Theorem 22.1), and therefore sometimes called the Rado-Hall theorem) may be derived from the matroid intersection theorem, applied to M and the transversal matroid M_2 induced by $\mathcal{X}$.

Corollary 41.1c (Rado's theorem)**.** *Let $M = (S, \mathcal{I})$ be a matroid, with rank function r, and let $\mathcal{X} = (X_1, \ldots, X_n)$ be a family of subsets of S. Then $\mathcal{X}$ has a transversal which is independent in M if and only if*

$$(41.5) \qquad r(\bigcup_{i \in I} X_i) \geq |I|$$

for each $I \subseteq \{1, \ldots, n\}$.

Proof. Let r_2 be the rank function of the transversal matroid M_2 induced by $\mathcal{X}$. By the matroid intersection theorem, M and M_2 have a common independent set of size n if and only if

$$(41.6) \qquad r(U) + r_2(S \setminus U) \geq n \text{ for each } U \subseteq S.$$

Now for each $T \subseteq S$ one has (by Kőnig's matching theorem (cf. Corollary 22.2a)):

$$(41.7) \qquad r_2(T) = \min_{I \subseteq \{1, \ldots, n\}} (\big|\bigcup_{i \in I} X_i \cap T\big| + n - |I|).$$

So (41.6) is equivalent to:

$$(41.8) \qquad r(U) + \big|\bigcup_{i \in I} X_i \setminus U\big| + n - |I| \geq n$$

for all $U \subseteq S$ and $I \subseteq \{1, \ldots, n\}$. We can assume that $U = \bigcup_{i \in I} X_i$, since replacing U by $\bigcup_{i \in I} X_i$ does not increase the left-hand side in (41.8). So the condition is equivalent to (41.5), proving the corollary. ∎

Notes. Mirsky [1971a] gave an alternative proof of Rado's theorem. Welsh [1970] showed that, in turn, Rado's theorem implies the matroid intersection theorem. Las Vergnas [1970] gave an extension of Rado's theorem. Rado [1942] (and also Welsh [1971]) showed that Rado's theorem in fact characterizes matroids. Perfect [1969a] generalized Rado's theorem to characterizing the maximum size of an independent partial transversal. Related results are in Perfect [1971].

41.1a. Applications of the matroid intersection theorem

In this section we mention a number of applications of the matroid intersection theorem. Further applications will be given in the next chapter on matroid union.

Kőnig's theorems. Let $G = (V, E)$ be a bipartite graph, with colour classes U_1 and U_2. For $i = 1, 2$, let $M_i = (E, \mathcal{I}_i)$ be the matroid with $F \subseteq E$ independent if and only if each vertex in U_i is covered by at most one edge in F.

So M_1 and M_2 are partition matroids. The common independent sets in M_1 and M_2 are the matchings in G, and the common spanning sets are the edge covers in G. For $i = 1, 2$ and $F \subseteq E$, the rank $r_i(F)$ of F in M_i is equal to the number of vertices in U_i covered by F.

By the matroid intersection theorem, the maximum size of a matching in G is equal to the minimum of $r_1(F) + r_2(E \setminus F)$ taken over $F \subseteq E$. This last is equal to the minimum size of a vertex cover in G. So we have Kőnig's matching theorem (Theorem 16.2).

Similarly, by Corollary 41.1b, the minimum size of an edge cover in G (assuming G has no isolated vertices), is equal to the maximum of $|V| - r_1(F) - r_2(E \setminus F)$ taken over $F \subseteq E$. This last is equal to the maximum size of a stable set in G. So we have the Kőnig-Rado edge cover theorem (Theorem 19.4).

Common transversals. Let $\mathcal{X} = (X_1, \dots, X_m)$ and $\mathcal{Y} = (Y_1, \dots, Y_m)$ be families of subsets of a finite set S. Then the matroid intersection theorem implies Theorem 23.1 of Ford and Fulkerson [1958c]: $\mathcal{X}$ and $\mathcal{Y}$ have a common transversal if and only if

(41.9) $$|X_I \cap Y_J| \geq |I| + |J| - m$$

for all subsets I and J of $\{1, \dots, m\}$, where $X_I := \bigcup_{i \in I} X_i$ and $Y_J := \bigcup_{j \in J} Y_j$.

To see this, let M_1 and M_2 be the transversal matroids induced by $\mathcal{X}$ and $\mathcal{Y}$ respectively, with rank functions r_1 and r_2 say. So $\mathcal{X}$ and $\mathcal{Y}$ have a common transversal if and only if M_1 and M_2 have a common independent set of size m. By Theorem 41.1, this last holds if and only if $r_1(Z) + r_2(S \setminus Z) \geq m$ for each $Z \subseteq S$. Using Kőnig's matching theorem, this is equivalent to:

(41.10) $$\min_{I \subseteq \{1,\dots,m\}} (m - |I| + |X_I \cap Z|) + \min_{J \subseteq \{1,\dots,m\}} (m - |J| + |Y_J \setminus Z|) \geq m$$

for each $Z \subseteq S$. Equivalently, for all $I, J \subseteq \{1, \dots, m\}$:

(41.11) $$\min_{Z \subseteq S} (m - |I| + |X_I \cap Z| + m - |J| + |Y_J \setminus Z|) \geq m.$$

As this minimum is attained by $Z := Y_J$, this is equivalent to (41.9).

Coloured trees. Let $G = (V, E)$ be a graph and let the edges of G be coloured with k colours. That is, we have partitioned E into sets $E_1, \dots, E_k$, called *colours*. Then there exists a spanning tree with all edges coloured differently if and only if $G - F$ has at most $t + 1$ components, for any union F of t colours, for any $t \geq 0$. This follows from the matroid intersection theorem applied to the cycle matroid $M(G)$ of G and the partition matroid N induced by $E_1, \dots, E_k$.

Indeed, $M(G)$ and N have a common independent set of size $|V| - 1$ if and only if $r_{M(G)}(E \setminus F) + r_N(F) \geq |V| - 1$ for each $F \subseteq E$. Now $r_N(F)$ is equal to the number of E_i intersecting F. So we can assume that F is equal to the union of t of the E_i, with $t := r_N(F)$. Moreover, $r_{M(G)}(E \setminus F)$ is equal to $|V| - \kappa(G - F)$, where $\kappa(G - F)$ is the number of components of $G - F$. So the requirement is that $|V| - \kappa(G - F) + t \geq |V| - 1$. In other words, $\kappa(G - F) \leq t + 1$.

Detachments. The following is a special case of a theorem of Nash-Williams [1985], which he derived from the matroid intersection theorem — in fact it is a consequence of the result on coloured trees given above.

Let $G = (V, E)$ be a graph and let $b : V \longrightarrow \mathbb{Z}_+$. Call a graph $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ a *b-detachment* of G if there is a function $\phi : \widetilde{V} \longrightarrow V$ such that $|\phi^{-1}(v)| = b(v)$ for each $v \in V$, and such that there is a one-to-one function $\psi : \widetilde{E} \longrightarrow E$ with $\psi(e) = \{\phi(u), \phi(v)\}$ for each edge $e = uv$ of $\widetilde{G}$.

Then there exists a connected b-detachment if and only if

(41.12) $$b(U) + \kappa(G - U) \leq |E_U| + 1 \text{ for each } U \subseteq V,$$

where $\kappa(G')$ denotes the number of components of graph G' and where E_U denotes the set of edges intersecting U.

To see this, let $H = (\widetilde{V}, E')$ be the graph obtained from G by replacing each vertex v by $b(v)$ new vertices, and by connecting for each edge $e = uv$ of G, the $b(u)$ new vertices associated with u with the $b(v)$ new vertices associated with v. We assign to these $b(u)b(v)$ edges the 'colour' e.

Then there exists a connected b-detachment if and only if H has a spanning tree in which all edges have a different colour. By the previous example, such a spanning tree exists if and only if for each $F \subseteq E$, deleting from H the edges with colour in F gives a graph H' with at most $|F| + 1$ components.

Now the number of components of H' is equal to the $\kappa(G - F) + b(I_F) - |I_F|$, where I_F denotes the set of isolated (hence loopless) vertices of $G - F$. So the condition is equivalent to: $\kappa(G - F) - |F| + b(I_F) - |I_F| \leq 1$. As $\kappa(G - F) - |F|$ does not decrease by removing edges from F, we can assume that F is equal to the set of edges incident with I_F. So F is determined by $U := I_F$, namely $F = E_U$. Then $\kappa(G - F) - |I_F| = \kappa(G - U)$. So the condition is equivalent to (41.12).

41.1b. Woodall's proof of the matroid intersection theorem

P.D. Seymour attributed the following proof of the matroid intersection theorem to D.R. Woodall (cf. Seymour [1976a]):

Let k be the value of (41.1). Let $x \in S$ be such that $r_1(\{x\}) = r_2(\{x\}) = 1$. (If no such x exists the theorem is trivial, as in that case the minimum is 0.) Let $Y := S \setminus \{x\}$. Now we may assume that the restrictions $M_1 \setminus x$ and $M_2 \setminus x$ have no common independent set of size k. So, by induction,

(41.13) $$r_1(A_1) + r_2(A_2) \leq k - 1,$$

for some partition A_1, A_2 of Y. Moreover, the contractions M_1/x and M_2/x have no common independent set of size $k - 1$ (otherwise we can add x to obtain a common independent set of size k for M_1 and M_2). So, by induction,

(41.14) $$r_1(B_1 \cup \{x\}) - 1 + r_2(B_2 \cup \{x\}) - 1 \leq k - 2$$

(cf. (39.9) above), for some partition B_1, B_2 of Y. However,

(41.15) $$\begin{aligned} r_1(A_1 \cap B_1) + r_1(A_1 \cup B_1 \cup \{x\}) &\leq r_1(A_1) + r_1(B_1 \cup \{x\}), \\ r_2(A_2 \cap B_2) + r_2(A_2 \cup B_2 \cup \{x\}) &\leq r_2(A_2) + r_2(B_2 \cup \{x\}), \end{aligned}$$

by the submodularity (cf. (39.38)(ii)) of the rank functions. Moreover, by the definition of k,

(41.16) $$\begin{aligned} k &\leq r_1(A_1 \cap B_1) + r_2(A_2 \cup B_2 \cup \{x\}), \\ k &\leq r_1(A_1 \cup B_1 \cup \{x\}) + r_2(A_2 \cap B_2), \end{aligned}$$

as $A_1 \cap B_1, A_2 \cup B_2 \cup \{x\}$ and $A_1 \cup B_1 \cup \{x\}, A_2 \cap B_2$ form partitions of S. Adding the inequalities in (41.13), (41.14), (41.15), and (41.16) gives a contradiction.

41.2. Cardinality matroid intersection algorithm

A maximum-size common independent set can be found in polynomial time. This result follows from the matroid union algorithm of Edmonds [1968], since (as Edmonds [1970b] and Lawler [1970] observed) cardinality matroid intersection can be reduced to matroid union.

We describe below the direct algorithm given by Aigner and Dowling [1971] and Lawler [1975], based on finding paths in auxiliary graphs. A different algorithm was given by Edmonds [1979].

Note that the examples given in Section 41.1a provide applications for the matroid intersection algorithm. We should note that in the algorithm we require that in any matroid $M = (S, \mathcal{I})$, we can test in polynomial time if any subset of S belongs to $\mathcal{I}$ — no explicit list of all sets in $\mathcal{I}$ is required. Thus complexity results are all relative to the complexity of testing independence. As such a membership testing algorithm exists in each example mentioned, we obtain polynomial-time algorithms for these special cases.

For any two matroids $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ and any $I \in \mathcal{I}_1 \cap \mathcal{I}_2$, we define a directed graph $D_{M_1,M_2}(I)$, with vertex set S, as follows. For any $y \in I, x \in S \setminus I$,

(41.17) (y, x) is an arc of $D_{M_1,M_2}(I)$ if and only if $I - y + x \in \mathcal{I}_1$,
(x, y) is an arc of $D_{M_1,M_2}(I)$ if and only if $I - y + x \in \mathcal{I}_2$.

These are all arcs of $D_{M_1,M_2}(I)$. So this graph is the union of the graphs $D_{M_1}(I)$ and the *reverse* of $D_{M_2}(I)$ defined in Section 39.9.

The following is the base for finding a maximum-size common independent set in two matroids.

Cardinality common independent set augmenting algorithm

input: matroids $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ and a set $I \in \mathcal{I}_1 \cap \mathcal{I}_2$;
output: a set $I' \in \mathcal{I}_1 \cap \mathcal{I}_2$ with $|I'| > |I|$ (if any).
description of the algorithm: Consider the sets

(41.18) $X_1 := \{x \in S \setminus I \mid I \cup \{x\} \in \mathcal{I}_1\}$,
$X_2 := \{x \in S \setminus I \mid I \cup \{x\} \in \mathcal{I}_2\}$.

Moreover, consider the directed graph $D_{M_1,M_2}(I)$ defined above. There are two cases.

Case 1: $D_{M_1,M_2}(I)$ has an $X_1 - X_2$ path P. (Possibly of length 0 if $X_1 \cap X_2 \neq \emptyset$.) We take a shortest such path P (that is, with a minimum number of arcs). Now output $I' := I \triangle VP$.

Case 2: $D_{M_1,M_2}(I)$ has no $X_1 - X_2$ path. Then I is a maximum-size common independent set. ∎

This finishes the description of the algorithm. The correctness of the algorithm is given by the following two theorems.

Theorem 41.2. *If Case 1 applies, then* $I' \in \mathcal{I}_1 \cap \mathcal{I}_2$.

Proof. Assume that Case 1 applies. By symmetry it suffices to show that I' belongs to $\mathcal{I}_1$.

Let P start at $z_0 \in X_1$. The arcs in P leaving I form the only matching in $D_{M_1}(I)$ with union equal to $VP - z_0$, since otherwise P would have a shortcut. Moreover, for each $z \in VP \setminus I$ with $z \neq z_0$, one has $I + z \notin \mathcal{I}_1$, since otherwise $z \in X_1$, and hence P would have a shortcut. So by Corollary 39.13a, I' belongs to $\mathcal{I}_1$. ∎

Theorem 41.3. *If Case 2 applies, then* I *is a maximum-size common independent set.*

Proof. As Case 2 applies, there is no $X_1 - X_2$ path in $D_{M_1,M_2}(I)$. Hence there is a subset U of S with $X_1 \cap U = \emptyset$ and $X_2 \subseteq U$, and such that no arc enters U. We show

(41.19) $$r_{M_1}(U) + r_{M_2}(S \setminus U) \leq |I|.$$

To this end, we first show

(41.20) $$r_{M_1}(U) \leq |I \cap U|.$$

Suppose that $r_{M_1}(U) > |I \cap U|$. Then there exists an x in $U \setminus I$ such that $(I \cap U) \cup \{x\} \in \mathcal{I}_1$. Since $I \cup \{x\} \notin \mathcal{I}_1$ (as $x \notin X_1$), there is a $y \in I \setminus U$ with $I - y + x \in \mathcal{I}_1$. But then $D_{M_1}(I)$ has an arc from y to x, contradicting the facts that $x \in U$ and $y \notin U$ and that no arc enters U.

This shows (41.20). Similarly, $r_{M_2}(S \setminus U) \leq |I \setminus U|$. Hence we have (41.19). So by the matroid intersection theorem, I is a maximum-size common independent set. ∎

Clearly, the running time of the algorithm is polynomially bounded, since we can construct the auxiliary directed graph $D_{M_1,M_2}(I)$ and find the path P (if any), in polynomial time. Therefore:

Theorem 41.4. *A maximum-size common independent set in two matroids can be found in polynomial time.*

Proof. Directly from the above, as we can find a maximum-size common independent set after applying at most $|S|$ times the common independent set augmenting algorithm. ∎

The algorithm also yields a proof of the matroid intersection theorem (Theorem 41.1 above): if the algorithm stops with set I, we obtain a set U for which (41.19) holds.

Notes. The above algorithm can be shown to take $O(n^2m(n+Q))$ time, where n is the maximum size of a common independent set, m is the size of the underlying set, and Q is the time needed to test if a given set is independent (in either matroid). Cunningham [1986] showed that if one chooses a shortest path as augmenting path, the sum of the lengths of all augmenting paths chosen is $O(n \log n)$, which gives an $O(n^{3/2}mQ)$-time algorithm. This algorithm extends several of the ideas behind the $O(n^{1/2}m)$ algorithm of Hopcroft, Karp, and Karzanov for cardinality bipartite matching (see Section 16.4). For more efficient algorithms, see Gabow and Tarjan [1984], Gusfield [1984], Gabow and Stallmann [1985], Frederickson and Srinivas [1989], Gabow and Xu [1989,1996], and Fujishige and Zhang [1995].

The problem of finding a maximum-size common independent set in *three* matroids is NP-complete, as finding a Hamiltonian circuit in a directed graph is a special case (as was observed by Held and Karp [1970]). Another special case is finding a common transversal of three collections of sets, which is also NP-complete (Theorem 23.16). In particular, the k-intersection problem can be reduced to the 3-intersection problem (cf. Lawler [1976b]).

Barvinok [1995] gave an algorithm for finding a maximum-size common independent set in k linear matroids, represented by given vectors over the rationals. The running time is linear in the cardinality of the underlying set and singly polynomial in the maximum rank of the matroids.

41.3. Weighted matroid intersection algorithm

Also a maximum-*weight* common independent set can be found in strongly polynomial time. This result was announced by Edmonds [1970b], who published an algorithm in Edmonds [1979]. An alternative algorithm (which we describe below) was announced by Lawler [1970] and described in Lawler [1975,1976b] — the correctness of this algorithm was proved by Krogdahl [1974,1976], using the results described in Section 39.9. A similar algorithm was described by Iri and Tomizawa [1976].

This algorithm is an extension of the cardinality matroid intersection algorithm given in Section 41.2. In each iteration, instead of finding a path P with a minimum number of arcs in $D_{M_1,M_2}(I)$, we will now require P to have minimum length with respect to some length function defined on $D_{M_1,M_2}(I)$.

To describe the algorithm, if matroids $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ and a weight function $w : S \to \mathbb{R}$ are given, call a set $I \in \mathcal{I}_1 \cap \mathcal{I}_2$ *extreme* if $w(J) \leq w(I)$ for each $J \in \mathcal{I}_1 \cap \mathcal{I}_2$ satisfying $|J| = |I|$.

Weighted common independent set augmenting algorithm

input: matroids $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$, a weight function $w : S \to \mathbb{Q}$, and an extreme common independent set I;
output: an extreme common independent set I' with $|I'| = |I| + 1$ (if any).
description of the algorithm: Consider again the sets X_1 and X_2 and the directed graph $D_{M_1,M_2}(I)$ on S, as in the cardinality case.

For any $x \in S$ define the 'length' $l(x)$ of x by:

(41.21) $$l(x) := \begin{cases} w(x) & \text{if } x \in I, \\ -w(x) & \text{if } x \notin I. \end{cases}$$

The *length* of a path P, denoted by $l(P)$, is equal to the sum of the lengths of the vertices traversed by P.

Case 1: $D_{M_1,M_2}(I)$ has an $X_1 - X_2$ path P. We choose P such that $l(P)$ is minimal and such that (secondly) P has a minimum number of arcs among all minimum-length $X_1 - X_2$ paths. Set $I' := I \triangle VP$.

Case 2: $D_{M_1,M_2}(I)$ has no $X_1 - X_2$ path. Then there is no common independent set larger than I. ∎

This finishes the description of the algorithm. The correctness of the algorithm if Case 2 applies follows directly from Theorem 41.3. In order to show the correctness if Case 1 applies, we first prove the following basic property of the length function l.

Lemma 41.5α. *Let C be a directed circuit in $D_{M_1,M_2}(I)$ and let $t \in VC$. Define $J := I \triangle VC$. If $J \notin \mathcal{I}_1 \cap \mathcal{I}_2$, then there exists a directed circuit C' with $VC' \subset VC$ such that $l(VC') < 0$, or $l(VC') \leq l(VC)$ and $t \in VC'$.*

Proof. By symmetry we can assume that $J \notin \mathcal{I}_1$. Let N_1 and N_2 be the sets of arcs in C belonging to $D_{M_1}(I)$ and $D_{M_2}(I)$ respectively. As $J \notin \mathcal{I}_1$, there exists, by Theorem 39.13, a matching N_1' in $D_{M_1}(I)$ with union VC and with $N_1' \neq N_1$. Consider the directed graph $D = (VC, A)$ formed by the arcs in N_1, N_1' (taking arcs in $N_1 \cap N_1'$ parallel), and by the arcs in N_2 taking each of them twice (parallel). Then each vertex in VC is entered and left by exactly two arcs of D. Moreover, since $N_1' \neq N_1$, D contains a directed circuit C_1 with $VC_1 \subset VC$ (as N_1' contains a chord of C). As D is Eulerian, we can extend this to a decomposition of A into directed circuits $C_1, \ldots, C_k$. Then

(41.22) $$\chi^{VC_1} + \cdots + \chi^{VC_k} = 2 \cdot \chi^{VC}.$$

Since $VC_1 \neq VC$ we know that $VC_j = VC$ for at most one j. If, say $VC_k = VC$, then (41.22) implies that either $l(VC_j) < 0$ for some $j < k$ or $l(VC_j) \leq l(VC)$ for all $j < k$, implying the proposition.

Suppose next that $VC_j \neq VC$ for all j. If $l(VC_j) < 0$ for some $j \leq k$ we are done. So assume $l(VC_j) \geq 0$ for each $j \leq k$. We can assume that C_1 and C_2 traverse t. Then

(41.23) $$l(VC_1) + l(VC_2) \leq l(VC_1) + \cdots + l(VC_k) = 2l(VC).$$

Hence $l(VC_1) \leq l(VC)$ or $l(VC_2) \leq l(VC)$, and again we are done. ∎

This implies (Krogdahl [1976], Fujishige [1977a]):

Theorem 41.5. *Let $I \in \mathcal{I}_1 \cap \mathcal{I}_2$. Then I is extreme if and only if $D_{M_1,M_2}(I)$ has no directed circuit of negative length.*

Proof. To see necessity, suppose that $D_{M_1,M_2}(I)$ has a directed circuit C of negative length. Choose C with $|VC|$ minimal. Consider $J := I \triangle VC$. Since $w(J) = w(I) - l(C) > w(I)$, while $|J| = |I|$, we know that $J \notin \mathcal{I}_1 \cap \mathcal{I}_2$. Hence by Lemma 41.5$\alpha$, $D_{M_1,M_2}(I)$ has a negative-length directed circuit covering fewer than $|VC|$ vertices, contradicting our assumption.

To see sufficiency, consider a $J \in \mathcal{I}_1 \cap \mathcal{I}_2$ with $|J| = |I|$. By Corollary 39.12a, both $D_{M_1}(I)$ and $D_{M_2}(I)$ have a perfect matching on $I \triangle J$. These two matchings together form a vertex-disjoint union of a number of directed circuits $C_1, \ldots, C_t$. Then

$$(41.24) \qquad w(I) - w(J) = \sum_{j=1}^{t} l(VC_j) \geq 0,$$

implying $w(J) \leq w(I)$. So I is extreme. ∎

This theorem implies that we can find a shortest path P, in Case 1 of the algorithm, in strongly polynomial time (with the Bellman-Ford method). It also gives:

Theorem 41.6. *If Case 1 applies, I' is an extreme common independent set.*

Proof. We first show that $I' \in \mathcal{I}_1 \cap \mathcal{I}_2$. To this end, let t be a new element, and extend (for each $i = 1, 2$), M_i to a matroid $M_i' = (S + t, \mathcal{I}_i')$, where for each $T \subseteq S + t$:

$$(41.25) \qquad T \in \mathcal{I}_i' \text{ if and only if } T - t \in \mathcal{I}_i.$$

Note that $D_{M_1',M_2'}(I + t)$ arises from $D_{M_1,M_2}(I)$ by extending it with a new vertex t and adding arcs from t to each vertex in X_1, and from each vertex in X_2 to t.

Let P be the path found in the algorithm. Define

$$(41.26) \qquad w(t) := l(t) := -l(P).$$

As P is a shortest $X_1 - X_2$ path, this makes that $D_{M_1',M_2'}(I + t)$ has no negative-length directed circuit. Hence, by Theorem 41.5, $I + t$ is an extreme common independent set of M_1' and M_2'.

Let P run from $z_1 \in X_1$ to $z_2 \in X_2$. Extend P by the arcs (t, z_1) and (z_2, t) to a directed circuit C. So $J = (I + t) \triangle VC$. As P has a minimum number of arcs among all shortest $X_1 - X_2$ paths, and as $D_{M_1',M_2'}(I+t)$ has no negative-length directed circuits, by Lemma 41.5α we know that $J \in \mathcal{I}_1 \cap \mathcal{I}_2$.

Moreover, J is extreme, since $I + t$ is extreme and $w(J) = w(I + t)$. ∎

So the weighted common independent set augmenting algorithm is correct. It obviously has strongly polynomially bounded running time. Therefore:

Theorem 41.7. *A maximum-weight common independent set in two matroids can be found in strongly polynomial time.*

Proof. Starting with the extreme common independent set $I_0 := \emptyset$ we can find iteratively extreme common independent sets $I_0, I_1, \ldots, I_k$, where $|I_i| = i$ for $i = 0, \ldots, k$ and where I_k is a maximum-size common independent set. Taking one among $I_0, \ldots, I_k$ of maximum weight, we have a maximum-weight common independent set. ∎

The above algorithm gives a maximum-weight common independent set of size k, for each k. In particular, a maximum-weight common base can be found with the algorithm. Similarly for minimum-weight:

Theorem 41.8. *A minimum-weight common base in two matroids can be found in strongly polynomial time.*

Proof. The last extreme common independent set in the above algorithm is a maximum-weight common base. By flipping the signs of the weights, this can be turned into a minimum-weight common base algorithm. ∎

Notes. Frank [1981a] gave an $O(\tau n^3)$-time implementation of this algorithm, where τ is the time needed to test for any $I \in \mathcal{I}_i$ and any $s \in S$ whether or not $I \cup \{s\} \in \mathcal{I}_i$, and if not, to find a circuit of M_i contained in $I \cup \{s\}$.

Clearly, a maximum-weight common independent set need not be a common base, even if common bases exist and all weights are positive: Let $S = \{1, 2, 3\}$ and let M_i be the matroid on S with unique circuit $S \setminus \{i\}$ (for $i = 1, 2$). Define $w(1) := w(2) := 1$ and $w(3) := 3$. Then $\{3\}$ is the unique maximum-weight common independent set, while $\{1, 2\}$ is the unique common base.

41.3a. Speeding up the weighted matroid intersection algorithm

The algorithm described in Section 41.3 is strongly polynomial-time, since we can find a shortest path P in strongly polynomial time, as in each iteration the graph $D_{M_1,M_2}(I)$ has no negative-length directed circuit. Hence we can apply the Bellman-Ford method. To bound the running time, suppose that we can construct, for any $I \in \mathcal{I}_1 \cap \mathcal{I}_2$ the graph $D_{M_1,M_2}(I)$ in time T. Then any iteration can be done in time $O(T + n^3)$, where $n := |S|$.

We can improve this to $O(T + n \log n)$ as follows (Frank [1981a], Brezovec, Cornuéjols, and Glover [1986]). The idea is that, in each iteration, with the extreme common independent set I, we give a 'certificate' of extremity, by specifying a potential for the length function; that is, a function $p \in \mathbb{Q}^S$ satisfying

(41.27) $$l(v) \geq p(v) - p(u)$$

for each arc (u, v) of $D_{M_1,M_2}(I)$. By Theorem 41.5, such a potential certifies extremity of I. We call such a p a potential *for* I.

Having the potential, we can apply Dijkstra's method instead of the Bellman-Ford method, as with the potential we can transform the length function (if defined on arcs) to a nonnegative length function.

It is convenient to associate the following functions $w_1, w_2 : S \to \mathbb{R}$ to $p, w : S \to \mathbb{R}$:

(41.28) $$w_1(v) = p(v) \text{ and } w_2(v) = w(v) - p(v) \text{ if } v \in I,$$
$$w_1(v) = w(v) + p(v) \text{ and } w_2(v) = -p(v) \text{ if } v \in S \setminus I.$$

So $w = w_1 + w_2$. Then:

Theorem 41.9. *Let $I \in \mathcal{I}_1 \cap \mathcal{I}_2$ and let $p, w, w_1, w_2 : S \to \mathbb{R}$ satisfy* (41.28). *Then p is a potential for $D_{M_1,M_2}(I)$ if and only if for $i = 1, 2$ one has*

(41.29) *I maximizes $w_i(X)$ over all $J \in \mathcal{I}_i$ satisfying $|J| = |I|$.*

Proof. The theorem follows easily with Corollary 39.12b. Indeed, there is an arc (u, v) leaving I if and only if $I - u + v \in \mathcal{I}_1$. Then

(41.30) $$w_1(v) \leq w_1(u) \iff l(v) \geq p(v) - p(u),$$

since $l(v) = -w(v) = -w_2(v) - w_1(v)$ and $-w_2(v) - w_1(u) = p(v) - p(u)$.

Similarly, there is an arc (u, v) entering I if and only if $I - v + u \in \mathcal{I}_2$. Then

(41.31) $$w_2(v) \geq w_2(u) \iff l(v) \geq p(v) - p(u),$$

since $l(v) = w(v) = w_2(v) + w_1(v)$ and $w_2(u) + w_1(v) = p(v) - p(u)$. ∎

We trivially have a potential for $I := \emptyset$. Consider next an arbitrary iteration, with as input a common independent set I and a potential p for I. Construct $D_{M_1,M_2}(I)$ and l as before. Let P be an $X_1 - X_2$ path with $l(P)$ minimum, and, under this condition, with $|VP|$ minimum. (Using the potential described above, we can find P with Dijkstra's algorithm.) Let $I' := I \triangle VP$.

We now reset the potential p such that for any $v \in S$ with v reachable from X_1, $p(v)$ is equal to the distance from X_1 to v (= the minimum of $l(VQ)$ over all $X_1 - v$ paths Q in $D_{M_1,M_2}(I)$).

Let w_1 and w_2 satisfy (41.28) with respect to I, (the new) p, and w. Then:

Theorem 41.10. *w_1, w_2 satisfy* (41.29) *with respect to I'.*

Proof. Extend M_1 and M_2 to matroids $M'_1 = (S + t, \mathcal{I}'_1)$ and $M'_2 = (S + t, \mathcal{I}'_2)$ as in (41.25). Let P run from $z_1 \in X_1$ to $z_2 \in X_2$. Define $w(t) := l(t) := -l(P)$, $p(t) := 0$, $w_1(t) := 0$, and $w_2(t) := w(t)$. Now it suffices to show:

(41.32) (i) $w_i(I + t) = w_i(I')$ for $i = 1, 2$;
(ii) w_1, w_2 satisfy (41.29) with respect to M'_1, M'_2, and $I + t$.

Let C be the directed circuit obtained by extending P by the arcs (t, z_1) and (z_2, t). Now, since $I' = (I+t) \triangle VC$, to show (41.32), it suffices to show, for each arc (u, v):

(41.33) if (u, v) leaves $I + t$, then $w_1(v) \leq w_1(u)$, with equality if (u, v) is on C;
if (u, v) enters $I + t$, then $w_2(u) \leq w_2(v)$, with equality if (u, v) is on C.

Note that for each arc (u, v) of $D_{M'_1,M'_2}(I + t)$ one has $p(v) \leq p(u) + l(v)$, with equality if (u, v) is on C. Hence, if (u, v) leaves $I + t$, then:

(41.34) $w_1(v) = p(v) + w(v) = p(v) - l(v) \leq p(u) = w_1(u),$

with equality if (u, v) is on C.

Similarly, if (u, v) enters $I + t$, then:

(41.35) $w_2(v) = w(v) - p(v) = l(v) - p(v) \geq -p(u) = w_2(u),$

with equality if (u, v) is on C. This proves (41.33). ∎

Using (41.28) and Theorem 41.9, we can obtain from w_1, w_2 a potential for I'. This implies:

Corollary 41.10a. *A maximum-weight common independent set can be found in time $O(k(T+n\log n))$, where $n := |S|$, k is the maximum size of a common independent set, and T is the time needed to find $D_{M_1,M_2}(I)$ for any common independent set I.*

Proof. Each iteration can be done in time $O(T + n \log n)$, since constructing the graph $D_{M_1,M_2}(I)$ takes T time, implying that there are $O(T)$ arcs. Hence, by Corollary 7.7a, a shortest $X_1 - X_2$ path P can be found in $O(T + n \log n)$ time. Hence I', and a potential for I' can be found in time $O(T + n \log n)$.

Since there are k iterations, we have the time bound given. ∎

In applications where the matroids are specifically given, one can often derive a better time bound, by obtaining $D_{M_1,M_2}(I')$ not from scratch, but by adapting $D_{M_1,M_2}(I)$. See also Brezovec, Cornuéjols, and Glover [1986] and Gabow and Xu [1989,1996].

41.4. Intersection of the independent set polytopes

It turns out that the intersection of the independent set polytopes of two matroids gives exactly the convex hull of the common independent sets, as was shown by Edmonds [1970b][27].

We first prove a very useful theorem, due to Edmonds [1970b], which we often will apply in this part. (A more general statement and interpretation in terms of network matrices will be given in Section 13.4.)

A family $\mathcal{C}$ of sets is called *laminar* if

(41.36) $Y \subseteq Z$ or $Z \subseteq Y$ or $Y \cap Z = \emptyset$

for all $Y, Z \in \mathcal{C}$.

Theorem 41.11. *Let $\mathcal{C}$ be the union of two laminar families of subsets of a set X. Let A be the $\mathcal{C} \times X$ incidence matrix of $\mathcal{C}$. Then A is totally unimodular.*

[27] Lawler [1976b] wrote that this result was announced by Edmonds 'at least as long ago as 1964'.

Proof. Let A be a counterexample with $|\mathcal{C}| + |X|$ minimal, and (secondly) with a minimal number of 1's. Then A is nonsingular and has determinant $\neq \pm 1$. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be laminar families, with union $\mathcal{C}$.

If each $\mathcal{C}_i$ consists of pairwise disjoint sets, then A is the incidence matrix of a bipartite graph, added with some unit base vectors. Hence A is totally unimodular, a contradiction.

If say $\mathcal{C}_1$ does not consist of pairwise disjoint sets, $\mathcal{C}_1$ contains a smallest nonempty set Y that is contained in some other set Z in $\mathcal{C}_1$. Choose Z smallest. Replacing Z by $Z \setminus Y$, maintains laminarity of $\mathcal{C}_1$. As this does not change the determinant of the corresponding matrix (as it amounts to subtracting row indexed Y from row indexed Z), we would have a counterexample with a smaller number of 1's, a contradiction. ∎

Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be matroids, with rank functions r_1 and r_2. By Corollary 40.2a, the intersection $P_{\text{independent set}}(M_1) \cap P_{\text{independent set}}(M_2)$ of the independent set polytopes associated with the matroids $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ is determined by:

(41.37) (i) $x_s \geq 0$ for $s \in S$,
(ii) $x(U) \leq r_i(U)$ for $i = 1, 2$ and $U \subseteq S$.

Trivially, this intersection contains the convex hull of the incidence vectors of common independent sets of M_1 and M_2. We shall see that these two polytopes are equal.

Basis is the following result of Edmonds [1970b], whose proof we follow (it constitutes the base of a fundamental technique developed further in several other results).

Theorem 41.12. *System* (41.37) *is box-totally dual integral.*

Proof. Choose $w \in \mathbb{Z}^S$. Consider the linear programming problem dual to maximizing $w^{\mathsf{T}} x$ over the constraints (41.37)(ii):

(41.38)
$$\begin{aligned} \text{minimize} \quad & \sum_{U \subseteq S} (y_1(U) r_1(U) + y_2(U) r_2(U)) \\ \text{where} \quad & y_1, y_2 \in \mathbb{R}_+^{\mathcal{P}(S)}, \\ & \sum_{U \subseteq S} (y_1(U) + y_2(U)) \chi^U = w. \end{aligned}$$

Let y_1, y_2 attain this minimum, such that

(41.39)
$$\sum_{U \subseteq S} (y_1(U) + y_2(U)) |U| |S \setminus U|$$

is minimized. Define

(41.40) $\mathcal{F}_i := \{U \subseteq S \mid y_i(U) > 0\}$,

for $i = 1, 2$. We show that for $i = 1, 2$, the collection $\mathcal{F}_i$ is a chain; that is, (41.41) if $T, U \in \mathcal{F}_i$, then $T \subseteq U$ or $U \subseteq T$.

Suppose not. Choose $\alpha := \min\{y_i(T), y_i(U)\}$, and decrease $y_i(T)$ and $y_i(U)$ by α, and increase $y_i(T \cap U)$ and $y_i(T \cup U)$ by α. Since

(41.42) $\chi^T + \chi^U = \chi^{T\cap U} + \chi^{T\cup U}$,

y_1, y_2 remains a feasible solution of (41.38); and since

(41.43) $r_i(T) + r_i(U) \geq r_i(T \cap U) + r_i(T \cup U)$,

it remains optimum. However, sum (41.39) decreases (by Theorem 2.1), contradicting the minimality assumption. So $\mathcal{F}_1$ and $\mathcal{F}_2$ are chains.

As the constraints in (41.37)(ii) corresponding to $\mathcal{F}_1$ and $\mathcal{F}_2$ form a totally unimodular matrix (by Theorem 41.11), by Theorem 5.35 system (41.37)(ii) is box-TDI, and hence (41.37) is box-TDI. ∎

(The fact that the $\mathcal{F}_i$ can be taken to be chains also follows directly from the proof method of Theorem 40.2.)

This implies a characterization of the *common independent set polytope*

(41.44) $P_{\text{common independent set}}(M_1, M_2)$

of two matroids $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$, being the convex hull of the incidence vectors of the common independent sets of M_1 and M_2:

Corollary 41.12a. $P_{\text{common independent set}}(M_1, M_2)$ *is determined by* (41.37).

Proof. Directly from Theorem 41.12, since it implies that the vertices of the polytope determined by (41.37) are integer, and hence are the incidence vectors of common independent sets. ∎

Another way of stating this is:

Corollary 41.12b.

(41.45) $P_{\text{common independent set}}(M_1, M_2)$
$= P_{\text{independent set}}(M_1) \cap P_{\text{independent set}}(M_2)$.

Proof. From Corollary 41.12a, using the fact that (41.37) is the union of the constraints for the independent set polytopes of M_1 and M_2, by Corollary 40.2b. ∎

The total dual integrality of (41.37) gives the following extension of the matroid intersection theorem:

Corollary 41.12c. *Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be matroids, with rank functions r_1 and r_2, respectively, and let $w \in \mathbb{Z}_+^S$. Then the maximum value of $w(I)$ over $I \in \mathcal{I}_1 \cap \mathcal{I}_2$ is equal to the minimum value of*

(41.46) $r_1(U_1) + \cdots + r_1(U_k) + r_2(T_1) + \cdots + r_2(T_l)$,

where $U_1 \subseteq \cdots \subseteq U_k \subseteq S$ *and* $T_1 \subseteq \cdots \subseteq T_l \subseteq S$ *such that each element* s *of* S *occurs in precisely* $w(s)$ *sets among* $U_1, \ldots, U_k, T_1, \ldots, T_l$.

Proof. Directly from Theorem 41.12 and its proof. ∎

(Edmonds [1979] gave an algorithmic proof of this result.)

These corollaries cannot be extended to the intersection of the independent set polytopes of three matroids. Let $S = \{1, 2, 3\}$, and for $i = 1, 2, 3$, let M_i be the matroid on S with $S \setminus \{i\}$ as unique circuit. Then $P_{\text{independent set}}(M_1) \cap P_{\text{independent set}}(M_2) \cap P_{\text{independent set}}(M_3)$ contains the all-$\frac{1}{2}$ vector, while each integer vector in this intersection contains at most one 1. So the intersection is *not* the convex hull of the common independent sets.

Similar results hold for the common base polytope. For matroids M_1 and M_2, let the *common base polytope* $P_{\text{common base}}(M_1, M_2)$ be the convex hull of the incidence vectors of common bases of M_1 and M_2. Then:

Corollary 41.12d. $P_{\text{common base}}(M_1, M_2) = P_{\text{base}}(M_1) \cap P_{\text{base}}(M_2)$.

Proof. Directly from the foregoing. ∎

So the common base polytope is determined by:

(41.47)
$$\begin{array}{ll} x_s \geq 0 & \text{for } s \in S, \\ x(U) \leq r_i(U) & \text{for } i = 1, 2 \text{ and } U \subseteq S, \\ x(S) = r_i(S) & \text{for } i = 1, 2. \end{array}$$

Corollary 41.12e. *System* (41.47) *is box-TDI.*

Proof. From Theorem 41.12, with Theorem 5.25. ∎

Moreover, similar results hold for the common spanning set polytope. For matroids M_1 and M_2, let the *common spanning set polytope*, in notation $P_{\text{common spanning set}}(M_1, M_2)$, be the convex hull of the incidence vectors of common spanning sets of M_1 and M_2. Then:

Corollary 41.12f.

(41.48)
$$\begin{aligned} &P_{\text{common spanning set}}(M_1, M_2) \\ &= P_{\text{spanning set}}(M_1) \cap P_{\text{spanning set}}(M_2). \end{aligned}$$

Proof. This can be reduced to Corollary 41.12b on the common independent set polytope, by duality: x belongs to the spanning set polytope of M_i if and only if $\mathbf{1} - x$ belongs to the independent set polytope of M_i^*.

Similarly, x belongs to the common spanning set polytope of M_1 and M_2 if and only if $\mathbf{1} - x$ belongs to the common independent set polytope of M_1^* and M_2^*. ∎

So the common spanning set polytope is determined by:

(41.49) $$\begin{array}{ll} 0 \leq x_s \leq 1 & \text{for } s \in S, \\ x(U) \leq r_i(S) - r_i(S \setminus U) & \text{for } i = 1, 2 \text{ and } U \subseteq S. \end{array}$$

Corollary 41.12g. *System* (41.49) *is box-TDI.*

Proof. Again, this can be derived from Theorem 41.12, by replacing x by $\mathbf{1} - x$. ∎

Another consequence of Theorem 41.12 is:

Corollary 41.12h. *Let* $M_1 = (S, \mathcal{I}_1)$ *and* $M_2 = (S, \mathcal{I}_2)$ *be matroids and let* $x \in \mathbb{R}_+^S$. *Then*

(41.50) $$\begin{array}{l} \max\{z(S) \mid z \leq x, z \in P_{\text{common independent set}}(M_1, M_2)\} \\ = \min\{r(U) + x(S \setminus U) \mid U \subseteq S\}, \end{array}$$

where $r(U)$ *denotes the maximum size of a common independent set contained in* U.

Proof. This follows from the box-total dual integrality of (41.37), using the fact that $r(U_1 \cup U_2) \leq r_1(U_1) + r_2(U_2)$ for disjoint U_1, U_2. ∎

Cunningham [1984] showed that, if matroids $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ are given by independence testing oracles, one can find in strongly polynomial time for any $x \in \mathbb{Q}^S$, optimum solutions of (41.50). This will follow from the results in Section 47.4.

The result of Cunningham [1984] also implies:

Theorem 41.13. *Given matroids* $M_1 = (S, \mathcal{I}_1)$ *and* $M_2 = (S, \mathcal{I}_2)$ *by independence testing oracles, and given* $x \in \mathbb{Q}^S$*, one can test in strongly polynomial time if* x *belongs to the common independent set polytope, and if so, decompose* x *as a convex combination of incidence vectors of common independent sets.*

Proof. Let r_i be the rank function of M_i ($i = 1, 2$) and let $r(U) := \min\{r_1(U), r_2(U)\}$ for $i = 1, 2$. Let P be the common independent set polytope. Corollaries 40.4a and 41.12b imply that one can test in strongly polynomial time if x belongs to P.

So we can assume that x belongs to P. We decompose x as a convex combination of incidence vectors of common independent sets. Iteratively resetting x, we keep a collection $\mathcal{U}$ of subsets of S with $x(U) = r(U)$ for each $U \in \mathcal{U}$. Initially, $\mathcal{U} := \emptyset$. We describe the iteration.

Define

$$(41.51)\qquad F := \{y \in P \mid \forall s \in S : x_s = 0 \Rightarrow y_s = 0; \forall U \in \mathcal{U} : y(U) = r(U)\}.$$

So F is a face of P containing x.

Find a common independent set I with $\chi^I \in F$. This can be done by finding a common independent set $I \subseteq \text{supp}(x)$ maximizing $w^\mathsf{T} x$, where $w := \sum_{U \in \mathcal{U}} \chi^U$. (Here $\text{supp}(x)$ is the support of x; so $\text{supp}(x) = \{s \in S \mid x_s > 0\}$.)

If $x = \chi^I$ we stop. Otherwise, define $u := x - \chi^I$. Let λ be the largest rational such that

$$(41.52)\qquad \chi^I + \lambda u$$

belongs to P.

We describe an inner iteration to find λ. We consider vectors z along the halfline $L = \{\chi^I + \lambda u \mid \lambda \geq 0\}$. First we let λ be the largest rational with $\chi^I + \lambda u \geq \mathbf{0}$, and set $z := \chi^I + \lambda u$.

We iteratively reset z. We check if z belongs to the common independent set polytope, and if not, we find a $U \subseteq S$ minimizing $r(U) - z(U)$ (with Corollary 40.4c). Let z' be the (unique) vector on L achieving $x(U) \leq r(U)$ with equality; that is, satisfying $z'(U) = r(U)$.

Consider any inequality $x(U') \leq r(U')$ violated by z'. Then

$$(41.53)\qquad r(U') - |U' \cap I| < r(U) - |U \cap I|.$$

This can be seen by considering the function

$$(41.54)\qquad d(y) := (r(U) - y(U)) - (r(U') - y(U')).$$

We have $d(z) \leq 0$ (since U minimizes $r(U) - z(U)$) and $d(z') > 0$ (since $z'(U) = r(U)$ and $z'(U') > r(U')$). Hence, as d is linear, $d(\chi^I) > 0$; that is, we have (41.53). This implies that resetting $z := z'$, there are at most $r(S)$ inner iterations.

Let x' be the final z found. If we apply no inner iteration, then $x'_s = 0$ for some $s \in I \subseteq \text{supp}(x)$ (since we chose λ largest with $\chi^I + \lambda u \geq \mathbf{0}$). If we do at least one inner iteration, we find a U such that x' satisfies $x'(U) = r(U)$ while $|U \cap I| < r(U)$ (since x' is the unique vector on L satisfying $x'(U) = r(U)$ and since $x' \neq \chi^I$).

In the latter case, set $\mathcal{U}' := \mathcal{U} \cup \{U\}$; otherwise set $\mathcal{U}' := \mathcal{U}$. Then resetting x to x' and $\mathcal{U}$ to $\mathcal{U}'$, the dimension of F decreases (as χ^I does not belong to the new F). So the number of iterations is at most $|S|$. This shows that the method is strongly polynomial-time. ∎

41.4a. Facets of the common independent set polytope

Since the common independent set polytope of two matroids is the intersection of their independent set polytopes, each facet-inducing inequality for the intersection is facet-inducing for (at least) one of the independent set polytopes, but not necessarily conversely. Giles [1975] characterized which inequalities are facet-inducing for the common independent set polytope. If this polytope is full-dimensional, then each inequality $x_s \geq 0$ is facet-inducing. As for the other inequalities, Giles proved:

Theorem 41.14. *Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be loopless matroids, with rank functions r_1 and r_2. For $U \subseteq S$, define $r(U) := \min\{r_1(U), r_2(U)\}$. Then, for $U \subseteq S$, the inequality*

$$(41.55) \qquad x(U) \leq r(U)$$

is facet-inducing for $P_{\text{common independent set}}(M_1, M_2)$ if and only if there is no partition of U into nonempty proper subsets U_1, U_2 with

$$(41.56) \qquad r(U) \geq r(U_1) + r(U_2)$$

and there is no proper superset U' of U with $r(U') \leq r(U)$.

Proof. By symmetry, we can assume that $r(U) = r_1(U)$.

Necessity is easy: Assume that $x(U) \leq r_1(U)$ is facet-inducing. If (41.56) would hold, then each common independent set I with $|I \cap U| = r_1(U)$ satisfies $|I \cap U_1| = r(U_1)$ (since $|I \cap U_1| = |I \cap U| - |I \cap U_2| \geq r(U) - r(U_2) \geq r(U_1)$). Hence each x in the facet determined by $x(U) \leq r_1(U)$ satisfies $x(U_1) = r(U_1)$, a contradiction. Similarly, if $r(U') \leq r_1(U)$ for some proper superset U' of U, then each common independent set I with $|I \cap U| = r_1(U)$ satisfies $|I \cap U'| = r(U')$, implying that each x in the facet determined by $x(U) \leq r_1(U)$ satisfies $x(U') = r(U')$, again a contradiction.

To see sufficiency, suppose that (41.55) satisfies the conditions, but is not facet-inducing for the common independent set polytope. This implies that the inequality $x(U) \leq r_1(U)$ is implied by other inequalities in (41.37). So there exist $\lambda_i : \mathcal{P}(S) \to \mathbb{Q}_+$ (for $i = 1, 2$) such that

$$(41.57) \qquad \sum_{T \in \mathcal{P}(S)} (\lambda_1(T) + \lambda_2(T))\chi^T \geq \chi^U \text{ and}$$
$$\sum_{T \in \mathcal{P}(S)} (\lambda_1(T) r_1(T) + \lambda_2(T) r_2(T)) \leq r_1(U),$$

and such that $\lambda_i(U) = 0$ for $i = 1, 2$. Let D be the least common denominator of the values of the λ_i. Choose the λ_i such that D is as small as possible and (secondly) such that

$$(41.58) \qquad D \cdot \sum_{T \subseteq S} (\lambda_1(T) + \lambda_2(T))|T|(|S \setminus T| + 1)$$

is as small as possible. For $i = 1, 2$, define

$$(41.59) \qquad \mathcal{F}_i := \{T \subseteq S \mid \lambda_i(T) > 0\}.$$

We claim that for $i = 1, 2$:

(41.60) $\quad \mathcal{F}_i$ is a chain.

Suppose to the contrary that $T_1, T_2 \in \mathcal{F}_i$ satisfy $T_1 \not\subseteq T_2 \not\subseteq T_1$. Then decreasing $\lambda_i(T_1)$ and $\lambda_i(T_2)$ by $1/D$ and increasing $\lambda_i(T_1 \cap T_2)$ and $\lambda_i(T_1 \cup T_2)$ by $1/D$ maintains (41.57) but decreases (41.58). This would be a contradiction, except if $T_1 \cap T_2$ or $T_1 \cup T_2$ equals U. If one of these sets equals U and $D \geq 2$, we can reset $\lambda_i(U) := 0$, and multiply all values of λ_1 and λ_2 by $D/(D-1)$. This again maintains (41.57) but decreases the least common divisor of the denominators. So the contradiction would remain, except if $D = 1$. Then (41.57) implies $r_i(T_1) + r_i(T_2) \le r_1(U)$. Now if $T_1 \cap T_2 = U$, then $U \subset T_1$ and

$$r(T_1) \le r_i(T_1) \le r_i(T_1) + r_i(T_2) \le r_1(U), \tag{41.61}$$

contradicting the condition. If $T_1 \cup T_2 = U$, then

$$r(T_1) + r(U \setminus T_1) \le r_i(T_1) + r_i(U \setminus T_1) \le r_i(T_1) + r_i(T_2) \le r_1(U), \tag{41.62}$$

again contradicting the condition.

This proves (41.60). As each $\mathcal{F}_i$ is a chain, the incidence matrix of $\mathcal{F}_1 \cup \mathcal{F}_2$ is totally unimodular (by Theorem 41.11). Therefore, there are integer-valued λ_i satisfying (41.57), with $\lambda_i(T) = 0$ for $T \notin \mathcal{F}_i$. Then we can assume that $|\mathcal{F}_i| \le 1$ for $i = 1, 2$, since if $T, T' \in \mathcal{F}_i$ and $T \subset T'$, we can decrease $\lambda_i(T)$ by 1 without violating (41.57). If $U' \in \mathcal{F}_i$ with $U' \supset U$, then $r(U') \le r_i(U') \le r(U)$, contradicting the condition. So each $\mathcal{F}_i$ contains a set $U_i \not\supseteq U$, implying $r(U_1) + r(U \setminus U_1) \le r(U_1) + r(U_2) \le r_1(U_1) + r_2(U_2) \le r(U)$, again contradicting the condition. ∎

This theorem can be seen to imply a variant of it, in which, instead of $r(U) := \min\{r_1(U), r_2(U)\}$, we define

$$r(U) := \max\{|I| \mid I \in \mathcal{I}_1 \cap \mathcal{I}_2\} = \min_{T \subseteq U}(r_1(T) + r_2(U \setminus T)). \tag{41.63}$$

Fonlupt and Zemirline [1983] characterized the dimension of the common base polytope of two matroids.

41.4b. Up and down hull of the common base polytope

We saw in Corollary 41.12d a characterization of the common base polytope $P_{\text{common base}}(M_1, M_2)$ of two matroids $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$. The up hull of this polytope:

$$P^{\uparrow}_{\text{common base}}(M_1, M_2) := P_{\text{common base}}(M_1, M_2) + \mathbb{R}^S_+ \tag{41.64}$$

was characterized by Cunningham [1977] and McDiarmid [1978] as follows (proving a conjecture of Fulkerson [1971a]).

Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be matroids having a common base. Then $P^{\uparrow}_{\text{common base}}(M_1, M_2)$ is determined by:

$$x(U) \ge r(S) - r(S \setminus U) \text{ for } U \subseteq S, \tag{41.65}$$

where $r(Z) :=$ the maximum size of a common independent set contained in Z. (A weaker version of this was proved by Edmonds and Giles [1977].)

For a proof we refer to Section 46.7a, where it is also shown that (41.65) is TDI (Gröflin and Hoffman [1981]). (Frank and Tardos [1984a] derived this, with a direct algorithmic construction, from the total dual integrality of (41.47).)

Note that by the matroid intersection theorem, the inequalities (41.65) are equivalent to:

$$x(U) \ge k - r_1(A) - r_2(B) \text{ for each partition } U, A, B \text{ of } S, \tag{41.66}$$

where r_1 and r_2 are the rank functions of M_1 and M_2 respectively, and where k is the size of a common base. This implies that if we add $x \leq \mathbf{1}$ to (41.66) we obtain the convex hull of the subsets of S that contain a common base.

Similarly, the down hull of the common base polytope:

(41.67) $$P^{\downarrow}_{\text{common base}}(M_1, M_2) := P_{\text{common base}}(M_1, M_2) - \mathbb{R}^S_+,$$

is determined by

(41.68) $$x(U) \leq r_1(S \setminus A) + r_2(S \setminus B) - k \text{ for each partition } U, A, B \text{ of } S.$$

This can be derived from the description of the up hull of the common base polytope, since

(41.69) $$P^{\downarrow}_{\text{common base}}(M_1, M_2) = \mathbf{1} - P^{\uparrow}_{\text{common base}}(M_1^*, M_2^*)$$

(where $\mathbf{1}$ stands for the all-one vector in $\mathbb{R}^S$).

This implies that the convex hull of the incidence vectors of the subsets of common bases is determined by $x \geq \mathbf{0}$ and (41.68).

Cunningham [1984] gave a strongly polynomial-time algorithm to test if a vector belongs to $P^{\uparrow}_{\text{common base}}(M_1, M_2)$, or to $P^{\downarrow}_{\text{common base}}(M_1, M_2)$, using only independence testing oracles for M_1 and M_2.

41.5. Further results and notes

41.5a. Menger's theorem for matroids

Tutte [1965b] showed a special case of the matroid intersection theorem, namely when both M_1 and M_2 are minors of one matroid. Specialized to graphic matroids, it gives the vertex-disjoint, undirected version of Menger's theorem.

Let $M = (E, \mathcal{I})$ be a matroid, with rank function r, and let U and W be disjoint subsets of E. Then the maximum size of a common independent set in $M/U \setminus W$ and $M/W \setminus U$ is equal to the minimum value of

(41.70) $$r(X) - r(U) + r(E \setminus X) - r(W)$$

taken over sets X with $U \subseteq X \subseteq E \setminus W$. This is the special case of the matroid intersection theorem for the matroids $M/U \setminus W$ and $M/W \setminus U$, since for $Y \subseteq E \setminus (U \cup W)$ one has

(41.71) $$r_{M/U \setminus W}(Y) = r(Y \cup U) - r(U),$$

and similarly for $M/W \setminus U$.

To see that this implies the vertex-disjoint, undirected version of Menger's theorem, let $G = (V, E)$ be a graph and let S and T be disjoint nonempty subsets of V. We show that the above theorem implies that the maximum number of disjoint $S - T$ paths in G is equal to the minimum number of vertices intersecting each $S - T$ path.

To this end, we can assume that G is connected, and that E contains subsets U and W such that (S, U) and (T, W) are trees. (Adding appropriate edges does not modify the result to be proved.)

Let $M := M(G)$ be the cycle matroid of G. Define $R := V \setminus (S \cup T)$. Then

(41.72) the maximum number of disjoint $S-T$ paths is at least the maximum size of a common independent set I of $M/U\setminus W$ and $M/W\setminus U$, minus $|R|$.

(In fact, there is equality.)

To prove (41.72), let I be a maximum-size common independent in $M/U\setminus W$ and $M/W\setminus U$. So I is a forest. Consider any component K of I. Since I is independent in M/U, K intersects S in at most one vertex. Similarly, K intersects T in at most one vertex. Let p be the number of components K intersecting both S and T. By deleting p edges we obtain a forest I' such that no component of I' intersects both S and T. So $|I'|\leq|R|$ (since I' remains a forest after contracting (in the graphical sense) $S\cup T$ to one vertex). Hence $p=|I|-|I'|\geq|I|-|R|$. So we have (41.72).

On the other hand,

(41.73) the minimum size of a set of vertices intersecting each $S-T$ path is at most the minimum value of (41.70), minus $|R|$.

(Again, we have in fact equality.)

To prove (41.73), let X attain the minimum value of (41.70). So $U\subseteq X\subseteq E\setminus W$. Let K be the component of (V,X) containing S and let L be the component of $(V,E\setminus X)$ containing T. We choose X with $|K\cup L|$ maximized.

Then $K\cup L=V$. For suppose not. Then, as G is connected, there is an edge e of G leaving $K\cup L$. By symmetry, we can assume that $e\in X$. Let K' be the component of (V,X) containing e. So $K'\neq K$ and $E[K']\cap U=\emptyset$. Resetting X by $X\setminus E[K']$, $r(X)$ decreases by $|K'|-1$, while $r(E\setminus X)$ increases by at most $|K'|-1$. So the new X again attains the minimum in (41.70), while $K\cup L$ increases. This contradicts our maximality assumption.

So $K\cup L=V$. Hence $K\cap L$ intersects each $S-T$ path (since $S\subseteq K$ and $T\subseteq L$, and there is no edge connecting $K\setminus L$ and $L\setminus K$). Moreover

(41.74)
$$\begin{aligned}|K\cap L| &= |K|+|L|-|V|\leq(r(X)+1)+(r(E\setminus X)+1)-|V|\\ &= r(X)+r(E\setminus X)-|V|+2=r(X)+r(E\setminus X)-r(U)-r(W)-|R|.\end{aligned}$$

So we have (41.73).

Since the maximum number of disjoint $S-T$ paths is trivially not more than the minimum number of vertices intersecting all $S-T$ paths, we thus obtain Menger's theorem (and also equality in (41.72) and (41.73)).

(Tomizawa [1976a] gave an algorithm for Menger's theorem for matroids.)

41.5b. Exchange properties

Kundu and Lawler [1973] showed the following extension of the exchange property of bipartite graphs given in Theorem 16.8. Let $M_1=(S,\mathcal{I}_1)$ and $M_2=(S,\mathcal{I}_2)$ be matroids, with span functions span_1 and span_2. Then

(41.75) For any $I_1,I_2\in\mathcal{I}_1\cap\mathcal{I}_2$ there exists an $I\in\mathcal{I}_1\cap\mathcal{I}_2$ with $I_1\subseteq\text{span}_1(I)$ and $I_2\subseteq\text{span}_2(I)$.

(Theorem 16.8 is equivalent to the case where the M_i are partition matroids.)

To prove (41.75), choose $I\in\mathcal{I}_1\cap\mathcal{I}_2$ with $I_1\subseteq\text{span}_1(I)$ and $|I\cap I_2|$ maximized. Suppose that $I_2\not\subseteq\text{span}_2(I)$. Choose $s\in I_2\setminus\text{span}_2(I)$ with $I\cup\{s\}\in\mathcal{I}_2$. By the maximality of $|I\cap I_2|$ we know that $I\cup\{s\}\notin\mathcal{I}_1$. So M_1 has a circuit C

contained in $I \cup \{s\}$. Since $I_2 \in \mathcal{I}_1$ we know that $C \not\subseteq I_2$. Choose $t \in C \setminus I_2$. Then for $I' := I - t + s$ we have $I' \in \mathcal{I}_1 \cap \mathcal{I}_2$, while $\text{span}_1(I') = \text{span}_1(I)$. Since $|I' \cap I_2| > |I \cap I_2|$ this contradicts the maximality assumption.

A second exchange property was shown by Davies [1976]:

(41.76) Two matroids M_1 and M_2 have bases B_1 and B_2 (respectively) with $|B_1 \cap B_2| = k$ if and only if M_1 has bases X_1 and Y_1 and M_2 has bases X_2 and Y_2 with $|X_1 \cap X_2| \leq k$ and $|Y_1 \cap Y_2| \geq k$.

To see this, we may assume that $X_2 = Y_2$, since if $|X_1 \cap Y_2| \leq k$ we can reset $X_2 := Y_2$, and if $|X_1 \cap Y_2| > k$ we can reset $Y_1 := X_1$ and exchange indices.

By (39.33)(ii), there exists a series of bases $Z_0, \ldots, Z_t$ of M_1 such that $Z_0 = X_1$, $Z_t = Y_1$, and $|Z_{i-1} \triangle Z_i| = 2$ for $i = 1, \ldots, t$. Hence

(41.77) $\big||Z_{i-1} \cap X_2| - |Z_i \cap X_2|\big| \leq 1$

for $i = 1, \ldots, t$. Since $|Z_0 \cap X_2| \leq k$ and $|Z_t \cap X_2| \geq k$, we know $|Z_i \cap X_2| = k$ for some i. This proves (41.76).

41.5c. Jump systems

A framework that includes both matroid intersection and maximum-size matching was introduced by Bouchet and Cunningham [1995]. For $x, y \in \mathbb{Z}^n$, let $[x, y]$ be the set of vectors $z \in \mathbb{Z}^n$ with $\|x - y\|_1 = \|x - z\|_1 + \|z - y\|_1$. So $[x, y]$ consists of all integer vectors z in the box $x \wedge y \leq z \leq x \vee y$.

Call a vector z a *step from x to y* if $z \in [x, y]$ and $\|z - x\|_1 = 1$. A *jump system* is a finite subset J of $\mathbb{Z}^n$ satisfying the following axiom:

(41.78) if $x, y \in J$ and z is a step from x to y, then $z \in J$ or J contains a step from z to y.

Trivially, for any jump system J and any $x, y \in \mathbb{Z}^n$, the intersection $J \cap [x, y]$ is again a jump system. Moreover, being a jump system is maintained under translations by an integer vector and by reflections in a coordinate hyperplane. Bouchet and Cunningham [1995] showed that the sum of jump systems is again a jump system (attributing the proof below to A. Sebő):

Theorem 41.15. *If J_1 and J_2 are jump systems in $\mathbb{Z}^n$, then $J_1 + J_2$ is a jump system.*

Proof. For $x, y \in J_1 + J_2$ we prove (41.78) by induction on the minimum value of

(41.79) $\|y' - x'\|_1 + \|y'' - x''\|_1$,

where $x', y' \in J_1$, $x'', y'' \in J_2$, $x' + x'' = x$, and $y' + y'' = y$.

Let z be a step from x to y. By reflection and permutation of coordinates, we can assume that $z = x + \chi^1$. So $x_1 < y_1$. Hence, by symmetry of J_1 and J_2, we can assume that $x'_1 < y'_1$. Next, by reflection, we can assume that $x' \leq y'$.

Now $x' + \chi^1$ is a step from x' to y'. If $x' + \chi^1 \in J_1$, then $z = x' + \chi^1 + x'' \in J_1 + J_2$, and we have (41.78). So we can assume that $x' + \chi^1 \not\in J_1$. Hence, by (41.78) applied to J_1, there is an $i \in \{1, \ldots, n\}$ with $\tilde{x}' := x' + \chi^1 + \chi^i \in J_1$ and $\tilde{x}' \leq y'$.

So $z + \chi^i = \tilde{x}' + x'' \in J_1 + J_2$. If $z + \chi^i \in [x, y]$, we have (41.78). If $z + \chi^i \not\in [x, y]$, then as $z \in [x, y]$, we have $z_i = y_i$. So z is a step from $z + \chi^i$ to y. Also,

$\|y' - \tilde{x}'\|_1 = \|y' - x'\|_1 - 2$. Hence, by our induction hypothesis applied to $z + \chi^i$ and y, we have (41.78). ∎

As Bouchet and Cunningham [1995] observed, this theorem implies that the following two constructions give jump systems $J \subseteq \mathbb{Z}^V$.

For any matroid $M = (S, \mathcal{I})$, the set $\{\chi^B \mid B \text{ base of } M\}$ is a jump system in $\mathbb{Z}^S$, as follows directly from the axioms (39.33). With Theorem 41.15, this implies that for matroids $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$, the set

(41.80) $$J := \{\chi^{B_1} - \chi^{B_2} \mid B_i \text{ base of } M_i \ (i = 1, 2)\}$$

is a jump system.

Let $G = (V, E)$ be an undirected graph and let

(41.81) $$J := \{\deg_F \mid F \subseteq E\} \subseteq \mathbb{Z}^V;$$

that is, J is the collection of degree sequences of spanning subgraphs of G. Again, J is a jump system. This follows from Theorem 41.15, since for each edge $e = uv$ the set $\{\mathbf{0}, \chi^{\{u,v\}}\}$ is trivially a jump system in $\mathbb{Z}^V$ and since J is the sum of these jump systems.

Bouchet and Cunningham [1995] showed that the following greedy approach finds, for any $w \in \mathbb{R}^n$, a vector $x \in J$ maximizing $w^{\mathsf{T}}x$. By reflecting, we can assume that $w \geq \mathbf{0}$. We can also assume that $w_1 \geq w_2 \geq \cdots \geq w_n$. Let $J_0 := J$, and for $i = 1, \ldots, n$, let J_i be the set of vectors x in J_{i-1} maximizing x_i over J_{i-1}. Trivially, J_n consists of one vector, y say. Then:

Theorem 41.16. *y maximizes $w^{\mathsf{T}}x$ over J.*

Proof. It suffices to show that the maximum value of $w^{\mathsf{T}}x$ over J_1 is the same as over J (since applying this to the jump systems $J_1, \ldots, J_n$ gives the theorem). Let the maximum over J be attained by x and over J_1 by y. Suppose $w^{\mathsf{T}}y < w^{\mathsf{T}}x$. So $x \notin J_1$, and hence $x_1 < y_1$. We choose x, y such that $y_1 - x_1$ is minimal. Let $z := x + \chi^1$. So z is a step from x to y.

Then $w^{\mathsf{T}}z = w^{\mathsf{T}}x + w_1 \geq w^{\mathsf{T}}x$. Hence $z \notin J$, since otherwise we can replace x by z, contradicting the minimality of $y_1 - x_1$. So, by (41.78), J contains a step u from z to y. So $u = z \pm \chi^i$ for some $i \in \{1, \ldots, n\}$. Then

(41.82) $$w^{\mathsf{T}}u = w^{\mathsf{T}}z \pm w_i \geq w^{\mathsf{T}}z - w_i = w^{\mathsf{T}}x + w_1 - w_i \geq w^{\mathsf{T}}x.$$

So we can replace x by u, again contradicting the minimality of $y_1 - x_1$ (as $u_1 > x_1$). ∎

Lovász [1997] gave a min-max relation for the minimum l_1-distance of an integer vector to a jump system of special type. It can be considered as a common generalization of the matroid intersection theorem (Theorem 41.1) and the Tutte-Berge formula (Theorem 24.1).

For a survey, see Cunningham [2002].

41.5d. Further notes

A special case of the weighted matroid intersection algorithm (where one matroid is a partition matroid) was studied by Brezovec, Cornuéjols, and Glover [1988].

Data structures for on-line updating of matroid intersection solutions were given by Frederickson and Srinivas [1984,1987], and a randomized parallel algorithm for linear matroid intersection by Narayanan, Saran, and Vazirani [1992,1994].

An extension of matroid intersection to 'supermatroid' intersection was given by Tardos [1990]. Fujishige [1977a] gave a primal approach to weighted matroid intersection, and Shigeno and Iwata [1995] a dual approximation approach. Camerini and Maffioli [1975,1978] studied 3-matroid intersection problems.