5

Fractional Arboricity and Matroid Methods

The material in this chapter is motivated by two notions of the density of a graph. The *arboricity* and the *maximum average degree* of a graph G measure the concentration of edges in the "thickest" part of the graph.

5.1 Arboricity and maximum average degree

Suppose we wish to decompose the edges of a graph G into acyclic subsets, i.e., if $G = (V, E)$ we want to find $E_1, E_2, \ldots, E_k \subseteq E$ so that (1) each of the subgraphs (V, E_i) is acyclic and (2) $E = E_1 \cup E_2 \cup \cdots \cup E_k$. The smallest size of such a decomposition is called the *arboricity* (or *edge-arboricity*) of G and is denoted $\Upsilon(G)$. If G is connected, the arboricity is also the minimum number of spanning trees of G that include all edges of G.

One can think of arboricity as being a variant of the edge chromatic number. We are asked to paint the edges of G with as few colors as possible. In the case of edge chromatic number, we do not want to have two edges of the same color incident with a common vertex. In the case of arboricity, we do not want to have a monochromatic cycle.

There is an obvious lower bound on $\Upsilon(G)$. Since G has $\varepsilon(G)$ edges and each spanning acyclic subgraph has at most $\nu(G) - 1$ edges we have $\Upsilon(G) \geq \varepsilon(G)/(\nu(G) - 1)$. Moreover, since Υ is an integer, we have $\Upsilon(G) \geq \left\lceil \frac{\varepsilon(G)}{\nu(G)-1} \right\rceil$.

This bound is not very accurate if the graph is highly "unbalanced"; for example, consider the graph G consisting of a K_9 with a very long tail attached—say 100 additional vertices. We have $\nu(G) = 109$, $\varepsilon(G) = 136$, and therefore $\Upsilon(G) \geq \left\lceil \frac{136}{108} \right\rceil = 2$. The actual value of $\Upsilon(G)$ is larger since we clearly cannot cover the edges of K_9 with two trees; indeed, the arboricity of a graph is at least as large as the arboricity of any of its subgraphs. Thus we have

$$\Upsilon(G) \geq \max \left\lceil \frac{\varepsilon(H)}{\nu(H) - 1} \right\rceil$$

where the maximum is over all subgraphs of H with at least 2 vertices. Indeed, this improved lower bound gives the correct value.

Theorem 5.1.1

$$\Upsilon(G) = \max \left\lceil \frac{\varepsilon(H)}{\nu(H) - 1} \right\rceil$$

where the maximum is over all subgraphs of H with at least 2 vertices.

The proof of this theorem of Nash-Williams [137, 138] is presented in §5.4 below.

Notice that the arboricity of a graph can be expressed as a hypergraph covering problem. Given a graph $G = (V, E)$ we define a hypergraph $\mathcal{H} = (E, \mathcal{X})$ where $\mathcal{X}$ is the set of all acyclic subsets of edges of G. Then $\Upsilon(G) = k(\mathcal{H})$.

The *fractional arboricity of* G, denoted $\Upsilon_f(G)$, is simply $k_f(\mathcal{H})$. In other words, to compute $\Upsilon_f(G)$ we assign weights to the various acyclic subsets of edges so that every edge is contained in sets of total weight at least 1 and we seek to minimize the total of all the weights.

The arboricity of G gives us some information about the "density" of G; graphs with higher arboricity are more tightly "packed" with edges. Another measure of the density of a graph is its *average degree*: let

$$\bar{d}(G) = \frac{\sum d(v)}{\nu(G)} = \frac{2\varepsilon(G)}{\nu(G)}.$$

Average degree gives us an overall measure of a graph's density, but does not indicate the density in the densest portion of the graph. Reconsider the graph G consisting of a K_9 with a long tail. The average degree in such a graph is fairly low: $\bar{d}(G) = 2 \times 136/109 \approx 2.5$. However, the dense part of this graph—the K_9—has average degree 8. The value 8, in this case, is a better measure of the densest portion of the graph.

We define the *maximum average degree* of G, denoted $\text{mad}(G)$, to be the maximum of $\bar{d}(H)$ over all subgraphs H of G.

The invariant $\text{mad}(G)$ arises naturally in the theory of random graphs; see, for example, [23, 141].

At first glance, $\text{mad}(G)$ does not seem to fit the paradigm of our other fractional graph-theoretic invariants. However, in §5.5, we see $\text{mad}(G)$ in the same light as $\Upsilon_f(G)$.

5.2 Matroid theoretic tools

The way to understand arboricity and maximum average degree is through matroid theory.

Basic definitions

A *matroid* is a pair $\mathcal{M} = (S, \mathcal{I})$ where S is a finite set and $\mathcal{I}$ is a subset of 2^S that satisfies the following three conditions:

- (nontriviality) $\emptyset \in \mathcal{I}$,
- (heredity) if $X \in \mathcal{I}$ and $Y \subseteq X$, then $Y \in \mathcal{I}$, and
- (augmentation) if $X, Y \in \mathcal{I}$ and $|X| > |Y|$, then there is an $x \in X - Y$ so that $Y \cup \{x\} \in \mathcal{I}$.

The set S is called the *ground set* of the matroid and the sets in $\mathcal{I}$ are called the *independent sets* of $\mathcal{M}$. Note that the term *independent* is motivated by the concept of linear independence and is not related to graph-theoretic independence (what some graph theorists also call stability).

The following two basic examples provide the motivation for this abstraction.

Example 5.2.1 *Columns of a matrix.* Let A be a matrix and let S be the set of its columns. We put a collection I of columns of A in $\mathcal{I}$ exactly when those columns are linearly independent. The resulting pair $\mathcal{M}(A) = (S, \mathcal{I})$ is a matroid. See exercise 3 on page 90.

Example 5.2.2 *Acyclic sets in a graph.* Let $G = (V, E)$ be a graph. The *cycle matroid* of G, denoted $\mathcal{M}(G)$, has ground set E and a subset $I \subseteq E$ is independent in $\mathcal{M}$ provided (V, I) has no cycles. Proving that $\mathcal{M}(G)$ forms a matroid is relegated to exercise 4 on page 90. A matroid is called *graphic* if it is (isomorphic to) the cycle matroid of a graph.

Bases

The maximal independent sets of $\mathcal{M}$ are called the *bases* of $\mathcal{M}$ and the set of all bases of $\mathcal{M}$ is denoted by $\mathcal{B}$ (or $\mathcal{B}(\mathcal{M})$). If G is a connected graph, then the bases of $\mathcal{M}(G)$ are exactly the (edge sets of) spanning trees of G. If A is a matrix, the bases of $\mathcal{M}(A)$ are those subsets of columns of A that form bases, in the sense of linear algebra, of the column space of A.

Theorem 5.2.3 *The set of bases $\mathcal{B}$ of a matroid $\mathcal{M}$ satisfies the following:*

- $\mathcal{B} \neq \emptyset$,
- *if $X, Y \in \mathcal{B}$, then $|X| = |Y|$, and*
- *if $X, Y \in \mathcal{B}$ and $x \in X$, then there is a $y \in Y$, so that $(X - \{x\}) \cup \{y\} \in \mathcal{B}$.*

Moreover, if $\mathcal{B}$ is any collection of subsets of some set S and $\mathcal{B}$ satisfies the above properties, then $\mathcal{B}$ forms the set of bases of a matroid on S.

Note: The fact that any two spanning trees of a graph have the same number of edges, and the fact that any two bases of a vector space have the same cardinality, are instances of this theorem.

Proof. Since $\mathcal{I} \neq \emptyset$, we know that there are maximal independent sets, hence $\mathcal{B} \neq \emptyset$.

Suppose $X, Y \in \mathcal{B}$ with $|X| > |Y|$. Since X and Y are independent, there is an $x \in X - Y$ so that $Y \cup \{x\} \in \mathcal{I}$, contradicting the maximality of Y. Thus any two bases have the same size.

To show the third property, we note that $X - \{x\} \in \mathcal{I}$ and has one less element than Y, so there is a $y \in Y - (X - \{x\})$ so that $X' = (X - \{x\}) \cup \{y\}$ is independent. Were X' not a basis, it would be contained in a strictly larger basis X'', but then $|X''| > |X'| = |X|$, which contradicts the second property.

Now suppose $\mathcal{B}$ satisfies the three stated properties. Define

$$\mathcal{I} = \{X \subseteq S : X \subseteq B \text{ for some } B \in \mathcal{B}\}.$$

Note that since $\mathcal{B} \neq \emptyset$, there is some $B \in \mathcal{B}$ and since $\emptyset \subseteq B$, we have $\emptyset \in \mathcal{I}$.

Clearly if $X \subseteq Y$ and $Y \in \mathcal{I}$, then $X \in \mathcal{I}$.

Finally, we show that $\mathcal{I}$ has the augmentation property. Suppose $X, Y \in \mathcal{I}$ and $|X| < |Y|$. Choose $A, B \in \mathcal{B}$ so that $X \subseteq A$ and $Y \subseteq B$. We know that A and B have the same cardinality, which we call r. We may also assume that we have chosen A and B so that $|A \cap B|$ is as large as possible.

Write

$$\begin{aligned}
X &= \{x_1, x_2, \ldots, x_j\}, \\
A &= \{x_1, x_2, \ldots, x_j, a_1, a_2, \ldots, a_{r-j}\}, \\
Y &= \{y_1, y_2, \ldots, y_j, \ldots, y_{j+k}\}, \text{ and} \\
B &= \{y_1, y_2, \ldots, y_j, \ldots, y_{j+k}, b_1, \ldots, b_{r-j-k}\}.
\end{aligned}$$

If some $a_i \in Y$ then $X \cup \{a_i\} \in \mathcal{I}$ and we're done. So we consider the case that no $a_i \in Y$. Without loss of generality, we may also assume $a_1 \notin B$ (since there are more a's than b's). Consider $A - \{a_1\}$. By hypothesis, we know there is a $z \in B$ so that $A' = (A - \{a_1\}) \cup \{z\} \in \mathcal{B}$. Since all members of $\mathcal{B}$ have the same cardinality, we know that $z \notin A$ and therefore $z \notin X$. If we are lucky, $z \in Y$ (and thus $z \in Y - X$) so then $X \cup \{z\} \subseteq A'$ so $X \cup \{z\} \in \mathcal{I}$ and we're done.

Otherwise ($z \notin Y$), we can replace A by A' and observe that A' and B have more elements in common than do A and B. This contradicts the maximality of $|A \cap B|$. □

Rank

We have seen that any two bases of a matroid have the same cardinality. Just as the maximum number of linearly independent columns in a matrix is called the rank of the matrix, so too do we define the *rank* of a matroid $\mathcal{M}$, denoted $\rho(\mathcal{M})$, to be the maximum size of an independent set in $\mathcal{M}$.

We can extend the notion of rank further. Let X be any subset of the ground set of a matroid $\mathcal{M}$. One readily checks that any two maximal independent subsets of X are necessarily of the same size. We therefore define the *rank* of X, denoted $\rho(X)$, to be the maximum cardinality of an independent subset of X.

When A is a matrix and X is a subset of its columns, then the rank of X in the matroid $\mathcal{M}(A)$ is exactly the rank (in the usual linear algebra sense) of the matrix composed of those columns.

The rank function for a graphic matroid can be described as follows. Let $G = (V, E)$ be a graph and let $\mathcal{M}(G)$ be its cycle matroid. Let $F \subseteq E$. The rank $\rho(F)$ is the maximum size of an acyclic subset of F. If the graph (V, F) has c components (including isolated vertices, if any), then $\rho(F) = \nu(G) - c$.

Properties of the rank function of a general matroid are collected in the following result.

Theorem 5.2.4 *Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid and let ρ be its rank function. Then:*

- $\rho(\emptyset) = 0$,
- *if $X \subseteq Y \subseteq S$, then $\rho(X) \le \rho(Y)$,*
- *if $X \subseteq S$, then $0 \le \rho(X) \le |X|$,*
- *if $X \subseteq S$ and $y \in S$, then $\rho(X) \le \rho(X \cup \{y\}) \le \rho(X) + 1$,*
- *if $X \subseteq S$ and $x, y \in S$ satisfy $\rho(X \cup \{x\}) = \rho(X \cup \{y\}) = \rho(X)$, then $\rho(X \cup \{x, y\}) = \rho(X)$, and*
- *if $X, Y \subseteq S$, then $\rho(X \cup Y) + \rho(X \cap Y) \le \rho(X) + \rho(Y)$.*

Proof. The verification of most of these properties is routine; here we just prove the last property, which is known as the *submodular inequality*.

Pick $X, Y \subseteq S$. Let I be a maximal independent subset of $X \cap Y$. Using the augmentation property, we can construct a $J \supseteq I$ that is maximal independent in $X \cup Y$. Furthermore, let $K = J - Y$ and $L = J - X$. See Figure 5.1. Note that $K \cup I$ is an independent subset of X and

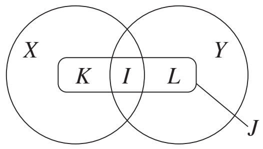

Figure 5.1. Understanding why the rank function of a matroid satisfies the submodular inequality.

$L \cup I$ is an independent subset of Y. Thus we may compute

$$\begin{aligned}\rho(X) + \rho(Y) &\geq |K \cup I| + |L \cup I| \\ &= 2|I| + |K| + |L| \\ &= |I| + |J| \\ &= \rho(X \cap Y) + \rho(X \cup Y).\end{aligned}$$

□

Circuits

Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid. The subsets of S not in $\mathcal{I}$ are called *dependent*. The minimal dependent subsets of $\mathcal{M}$ are called the *circuits* of $\mathcal{M}$. If $\mathcal{M}$ is the cycle matroid of a graph G, the circuits of $\mathcal{M}$ correspond exactly to the (edges sets of) cycles in G. The set of circuits of $\mathcal{M}$ is denoted $\mathcal{C}$ (or $\mathcal{C}(\mathcal{M})$).

If a singleton $\{x\}$ is dependent, it forms a circuit called a *loop* of the matroid. Note that a loop of a multigraph G corresponds exactly to a loop of $\mathcal{M}(G)$. If A is a matrix, a loop of $\mathcal{M}(A)$ corresponds to a column of all zeros.

If neither x nor y is a loop but $\{x, y\}$ is dependent (and therefore a circuit), then the elements x and y are called *parallel*. Edges $x, y \in E(G)$ are parallel edges (i.e., have the same endpoints) if and only they are parallel elements of $\mathcal{M}(G)$. Nonzero columns x, y of a matrix A are parallel elements of $\mathcal{M}(A)$ just when each is a scalar multiple of the other.

Theorem 5.2.5 *If $\mathcal{M}$ is a matroid and $\mathcal{C}$ is the set of circuits of $\mathcal{M}$ then the following conditions hold.*

- *If $X, Y \in \mathcal{C}$ and $X \subseteq Y$, then $X = Y$, and*
- *if $X, Y \in \mathcal{C}$, $X \neq Y$, and $a \in X \cap Y$, then there exists a $Z \in \mathcal{C}$ with $Z \subseteq (X \cup Y) - \{a\}$.*

Proof. The first property is trivial. For the second, suppose X, Y are distinct circuits with $a \in X \cap Y$. Then $X \cap Y$ is independent, and so

$$\begin{aligned}\rho(X) &= |X| - 1, \\ \rho(Y) &= |Y| - 1, \text{ and} \\ \rho(X \cap Y) &= |X \cap Y|.\end{aligned}$$

The submodularity inequality then gives

$$\rho(X \cup Y) \leq \rho(X) + \rho(Y) - \rho(X \cap Y)$$

$$= |X| - 1 + |Y| - 1 - |X \cap Y|$$
$$= |X \cup Y| - 2.$$

But then

$$\rho((X \cup Y) - \{a\}) \leq \rho(X \cup Y) < |(X \cup Y) - \{a\}|.$$

Thus $X \cup Y - \{a\}$ is dependent, and so contains a circuit. □

We know that if we add an edge between two vertices of a tree we create a unique cycle. A similar result holds for matroids.

Theorem 5.2.6 *Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid, let I be independent, and let $x \in S - I$. If $I \cup \{x\} \notin \mathcal{I}$, then there is a unique $C \in \mathcal{C}$ with $C \subseteq I \cup \{x\}$.*

Proof. Since $I \cup \{x\}$ is dependent, it contains a minimal dependent subset $C \in \mathcal{C}$. To show uniqueness, suppose $C_1, C_2 \in \mathcal{C}$ and $C_1, C_2 \subseteq I \cup \{x\}$, but $C_1 \neq C_2$. Note that $x \in C_1 \cap C_2$ (otherwise one of C_1 or C_2 would be contained in I, contradicting the independence of I) and therefore, by Theorem 5.2.5 on the facing page, there is a $C \in \mathcal{C}$ with $C \subseteq (C_1 \cup C_2) - \{x\} \subseteq I$, a contradiction. □

5.3 Matroid partitioning

Let us now revisit the arboricity of a graph in the language of matroid theory. Given a graph $G = (V, E)$, we wish to partition[1] E into the smallest number of sets $E_1, E_2, \ldots, E_k$ so that each of the graphs (V, E_i) is acyclic. In matroid language, we seek to partition S, the ground set of $\mathcal{M} = (S, \mathcal{I})$, into $S_1, S_2, \ldots, S_k$ so that each $S_i \in \mathcal{I}$.

Since a matroid $\mathcal{M} = (S, \mathcal{I})$ is a hypergraph, it makes sense to speak of the covering number of a matroid and to write $k(\mathcal{M}(G))$. Note that $\Upsilon(G) = k(\mathcal{M}(G))$.

It is perhaps useful to use the language of "coloring" to discuss matroid partitioning. We seek to color the elements of the ground set S so that each color class (the set of elements of a given color) is independent. Indeed, we call a coloring of the elements *proper* in just this case.

Our goal is to find formulas and efficient (polynomial-time) algorithms for computing k and k_f of a matroid $\mathcal{M}$. For convenience we present the formulas here:

$$k(\mathcal{M}) = \max_{Y \subseteq S} \left\lceil \frac{|Y|}{\rho(Y)} \right\rceil \qquad \text{and} \qquad k_f(\mathcal{M}) = \max_{Y \subseteq S} \left(\frac{|Y|}{\rho(Y)} \right).$$

(See Corollary 5.3.3 on page 81 and Theorem 5.4.1 on page 83.)

We begin by developing an algorithm, known as the Matroid Partitioning Algorithm, that computes whether or not a given matroid $\mathcal{M}$ can be partitioned into a given number k of independent sets. In this algorithm, we begin with all the elements of the ground set S uncolored. As the algorithm progresses, colors are assigned to elements of S and, at various times, the color assigned to an element can change. We first prove a lemma assuring ourselves that the recoloring step of the algorithm maintains the independence of the color classes. We then formulate the complete algorithm (on page 80) by iterating the lemma. We conclude with proofs of the formulas for $k(\mathcal{M})$ and $k_f(\mathcal{M})$.

[1] We may either *partition* or *cover* E by acyclic subsets of edges—there is no difference since a subset of an acyclic set of edges is also acyclic. See exercise 7 on page 13.

Suppose S_i (with $1 \le i \le k$) represents the set of elements that are currently color i and let x be any element. We can (re)assign color i to x (and still have a proper coloring) exactly when $S_i \cup \{x\} \in \mathcal{I}$. It is useful to have a notation for this. Let us write

$$x \leftarrow \langle i \rangle \quad \text{to mean} \quad x \notin S_i \text{ and } S_i \cup \{x\} \in \mathcal{I},$$

which we read as "x may get color i".

At times, the algorithm transfers a color from one element to another. Let x and y be elements in S with y colored and x either uncolored or colored differently from y. Let us write

$$x \leftarrow y \quad \text{to mean} \quad y \in S_i,\ x \notin S_i,\ \text{and } (S_i - y) \cup \{x\} \in \mathcal{I} \text{ for some } i.$$

We read this as "y may relinquish its color to x" or, more tersely, "x may get y's color".

Thus, given a partial proper coloring $S_1, \ldots, S_k$ of $\mathcal{M}$, the $\leftarrow$ relation defines a digraph on the set $S \cup \{\langle 1\rangle, \langle 2\rangle, \ldots, \langle k\rangle\}$.

The central idea in the Matroid Partitioning Algorithm is the following.

Lemma 5.3.1 *Suppose $\mathcal{M} = (S, \mathcal{I})$ is a matroid and $S_1, S_2, \ldots, S_k$ forms a partial proper coloring of $\mathcal{M}$. Suppose there is a directed path of the form*

$$x_0 \leftarrow x_1 \leftarrow x_2 \leftarrow \cdots \leftarrow x_n \leftarrow \langle a \rangle$$

that is minimal in the sense that $x_i \not\leftarrow x_j$ for $j > i+1$ and $x_i \not\leftarrow \langle a \rangle$ for $0 \le i < n$. Furthermore suppose we recolor the elements of S as follows: The new color of x_i is the old color of x_{i+1} (for $0 \le i < n$) and the new color of x_n is a. Then the recoloring described results in a proper partial coloring of $\mathcal{M}$.

The notation $x_i \leftarrow x_{i+1}$ means that x_i can acquire a new color from x_{i+1}. The assertion here is that it is permissible to make all these changes simultaneously. Because the $\leftarrow$ relation changes globally every time the partial coloring is amended, the lemma is not trivial.

Proof. We proceed by induction on n, the length of the path.

The basis case, $n = 0$, is trivial.

For $n \ge 1$, suppose

$$x_0 \leftarrow x_1 \leftarrow x_2 \leftarrow \cdots \leftarrow x_n \leftarrow \langle a \rangle$$

is a minimal path as described in the statement of the lemma and suppose the lemma has been proved for all smaller values of n. Suppose the current color of x_n is b.

Recolor the matroid so that x_n now has color a. That is, let $S_1', S_2', \ldots, S_k'$ be the new partial coloring with

$$S_i' = S_i \quad (i \ne a, b)$$
$$S_a' = S_a \cup \{x_n\}$$
$$S_b' = S_b - \{x_n\}$$

We know that all S_i''s are independent because $x_n \leftarrow \langle a \rangle$ was true for the old coloring.

We claim that in this new partial coloring

$$x_0 \leftarrow x_1 \leftarrow x_2 \leftarrow \cdots \leftarrow x_{n-1} \leftarrow \langle b \rangle$$

is a valid minimal path. Note that once we have verified this claim, this proof is complete by induction.

To this end, we need to verify four claims.

1. *In the new coloring $x_{n-1} \leftarrow \langle b \rangle$.*

   This is correct because in the old coloring we had $x_{n-1} \leftarrow x_n$ and $x_n \in S_b$. This means that $S_b - \{x_n\} \cup \{x_{n-1}\}$ is independent. Since x_n is not in S'_b, we may recolor x_{n-1} to color b as required.

2. *In the new coloring $x_i \leftarrow x_{i+1}$ for all $0 \le i < n-1$.*

   If x_{i+1} is any color *other* than a, then we clearly have $x_i \leftarrow x_{i+1}$ in the new coloring because $x_i \leftarrow x_{i+1}$ was true in the old coloring. Thus we may assume the color of x_{i+1} is a.

   To show $x_i \leftarrow x_{i+1}$ (in the new coloring) we must show that the set $S''_a = S'_a - \{x_{i+1}\} \cup \{x_i\} = S_a - \{x_{i+1}\} \cup \{x_i, x_n\}$ is independent.

   Suppose S''_a were dependent. Since S''_a is the union of an independent set ($S'_a - \{x_{i+1}\}$) and a single element (x_i), by Theorem 5.2.6, S''_a contains a unique cycle C. Note that $x_i \in C$. Also, $x_n \in C$ since S''_a is the union of the independent set $S_a - \{x_{i+1}\} \cup \{x_i\}$ (independent because $x_i \leftarrow x_{i+1}$ in the old partial coloring) and the single element x_n.

   Now $x_i \not\leftarrow \langle a \rangle$ (by minimality) so $S_a \cup \{x_i\}$ is dependent and contains a unique circuit C'. Since $x_i \leftarrow x_{i+1}$ in the old coloring, C' must contain both x_i and x_{i+1}. Since $x_n \in S_b$, we know that $x_n \notin C'$. Thus C and C' are distinct circuits both containing x_i. By Theorem 5.2.5, there is a circuit $C'' \subseteq (C \cup C') - \{x_i\}$.

   Summarizing, we have

   $$\begin{aligned} C &\subseteq S''_a = S_a \cup \{x_i, x_n\} - \{x_{i+1}\}, \\ C' &\subseteq S_a \cup \{x_i\}, \text{ and} \\ C'' &\subseteq (C \cup C') - \{x_i\} \subseteq S_a \cup \{x_n\} \in \mathcal{I}, \end{aligned}$$

   which gives a contradiction.

3. *In the new coloring $x_i \not\leftarrow \langle b \rangle$ (for $i < n-1$).*

   If $x_i \leftarrow \langle b \rangle$ were true in the new coloring, then $x_i \leftarrow x_n$ would have been true in the old coloring—a contradiction to minimality.

4. *In the new coloring $x_i \not\leftarrow x_j$ (for $0 \le i < j+1 \le n$).*

   Suppose that, in the new coloring, we had $x_i \leftarrow x_j$ with $0 \le i < j+1 \le n$. If the color of x_j is not b (say it's c), then we have $S'_c \cup \{x_i\} - \{x_j\} \supseteq S_c \cup \{x_i\} - \{x_j\} \notin \mathcal{I}$ (with strict containment only if $c = a$), and therefore $S'_c \cup \{x_i\} - \{x_j\}$ is dependent, a contradiction. We may therefore restrict to the case that x_j has color b.

   Since $x_i \leftarrow x_j$ in the new coloring, the set $I = S'_b - \{x_j\} \cup \{x_i\}$ is independent. Since $x_i \not\leftarrow x_j$ in the old coloring, the set $D = S_b - \{x_j\} \cup \{x_i\}$ is dependent. Now $D = I \cup \{x_n\}$ so D contains a unique circuit C. Notice that C necessarily contains x_n and x_i.

   Now $x_i \not\leftarrow x_n$ in the old coloring, so $S_b - \{x_n\} \cup \{x_i\}$ must be dependent and contain a circuit C'. Note that C' must contain x_i but not x_n, and therefore $C' \neq C$ and both contain x_i. Thus there is a circuit $C'' \subseteq C \cup C' - \{x_i\}$. However $C \cup C' - \{x_i\} \subseteq S_b$, a contradiction.

This completes the proof. □

Lemma 5.3.1 gives us a method for extending a partial coloring of a matroid. Given a partial coloring $S_1, \ldots, S_k$ of $\mathcal{M}$, we construct the $\leftarrow$ digraph on $S \cup \{\langle 1\rangle, \ldots, \langle k\rangle\}$ and search for a minimal directed path to an unlabeled element of S from one of $\langle 1\rangle$, $\langle 2\rangle$, ..., or $\langle k\rangle$.

If this is successful at each stage, we ultimately construct a partition of $\mathcal{M}$ into k independent sets. However, at some point we might not be able to find such a path. In such a case, we would like to know that no partition of $\mathcal{M}$ into k independent sets is possible.

There is a simple test we can apply to see if $\mathcal{M}$ is *not* partitionable into k independent sets. Recall that $\rho(\mathcal{M})$, the rank of $\mathcal{M}$, is the maximum size of an independent set. If $\mathcal{M}$ had a partition $S = S_1 \cup S_2 \cup \cdots \cup S_k$ we would have $|S| \le k\rho(\mathcal{M})$.

More generally, if $Y \subseteq S$, then $Y = (S_1 \cap Y) \cup (S_2 \cap Y) \cup \cdots \cup (S_k \cap Y)$ and therefore $|Y| \le k\rho(Y)$. Thus if there is a $Y \subseteq S$ for which $|Y| > k\rho(Y)$ there can be no partition of $\mathcal{M}$ into k independent sets.

Theorem 5.3.2 *Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid and let k be a positive integer. Then $\mathcal{M}$ has a partition into k independent sets if and only if $|Y| \le k\rho(Y)$ for all $Y \subseteq S$.*

The proof of this theorem is closely tied to the following algorithm.

Matroid Partitioning Algorithm

Input: A matroid $\mathcal{M} = (S, \mathcal{I})$ and a positive integer k.
Output: Either a partition $S = S_1 \cup \cdots \cup S_k$ with each $S_i \in \mathcal{I}$ or a $Y \subseteq S$ with $|Y| > k\rho(Y)$.

1. Let $S_1, S_2, \ldots, S_k$ all equal $\emptyset$.
2. If $S = S_1 \cup S_2 \cup \cdots \cup S_k$, output this partition and stop.
3. Otherwise (there are uncolored vertices), form the $\leftarrow$ directed graph based on the partial coloring $S_1, \ldots, S_k$.
4. If there is dipath to an uncolored vertex x_i from a color class $\langle a\rangle$, recolor along a minimal such dipath, and go to step 2.
5. Otherwise (there is no such path), let Y be the set of all vertices that can reach to uncolored vertices in the $\leftarrow$ digraph (i.e., $y \in Y$ if there is a $u \in S$ (with u uncolored) and a dipath[2] $u \leftarrow \cdots \leftarrow y$). Output the set Y.

Proof (of Theorem 5.3.2 and the correctness of the Matroid Partitioning Algorithm).

It is enough to prove that if the algorithm reaches step 5, then $|Y| > k\rho(Y)$. Let U denote the (nonempty) set of elements uncolored in the algorithm and let $V = Y - U$ (perhaps $V = \emptyset$). Since $U \ne \emptyset$, $|Y| > |V|$.

We claim that $\rho(Y) = |Y \cap S_i|$ for each i. Since $Y \cap S_i \in \mathcal{I}$ we know that $\rho(Y) \ge |Y \cap S_i|$. Suppose $\rho(Y) > |Y \cap S_i|$. This implies there is an $x \in Y - S_i$ with $(Y \cap S_i) \cup \{x\} \in \mathcal{I}$. If $S_i \cup \{x\} \in \mathcal{I}$ then we would have $x \leftarrow \langle i\rangle$, contradicting the fact that the algorithm passed step 4. Thus $S_i \cup \{x\}$ contains a unique circuit C. Now C is not contained in Y (since $(Y \cap S_i) \cup \{x\}$ is independent) so

[2] We allow dipaths of length zero. In other words, Y contains all uncolored vertices.

there is an element $z \in C - Y$ and, since $z \neq x$, we have $z \in S_i - Y$. Therefore, $(S_i - \{z\}) \cup \{x\} \in \mathcal{I}$, so $x \leftarrow z$. However, this is a contradiction because $x \in Y$ but $z \notin Y$.

Finally, we compute:

$$\begin{aligned} k\rho(Y) &= \sum_{i=1}^{k} \rho(Y) \\ &= \sum_{i=1}^{k} |Y \cap S_i| \\ &= \left| \bigcup_{i=1}^{k} Y \cap S_i \right| \\ &= |V| < |Y| \end{aligned}$$

as required. □

If $\mathcal{M}$ has no loops (every singleton subset is independent), then there is a least k so that $S = S_1 \cup \cdots \cup S_k$ with each $S_i \in \mathcal{I}$; indeed, this is just the covering number $k(\mathcal{M})$ of $\mathcal{M}$.

Corollary 5.3.3 *If $\mathcal{M} = (S, \mathcal{I})$ is a matroid, then*

$$k(\mathcal{M}) = \max_{Y \subseteq S} \left\lceil \frac{|Y|}{\rho(Y)} \right\rceil.$$

Proof. Immediate from Theorem 5.3.2. □

A few words are in order about the worst-case run time of the Matroid Partitioning Algorithm. The "price" of a matroid-theoretic algorithm is often assessed by the number of times we "ask" the matroid if a given set is independent.

Suppose the ground set of a matroid $\mathcal{M}$ has n elements. We run through steps 2, 3, and 4 at most n times (we add one element to the partial coloring on each pass). To create the digraph in step 3, we check every pair of elements x, y to see if $x \leftarrow y$. Each $x \leftarrow y$ and $x \leftarrow \langle a \rangle$ determination can be made by determining if a certain set is independent. We also compare each element x to each color class $\langle a \rangle$ to check if $x \leftarrow \langle a \rangle$. These determinations take $O(n^2)$ tests of independence.

Thus, overall, we make $O(n^3)$ independence queries in the worst case.

To determine $k(\mathcal{M})$ we can run the Matroid Partitioning Algorithm for $k = 1, 2, 3, \ldots$ until we succeed in finding a partition. Even if this is done naively, we perform $O(n^4)$ independence queries. Thus $k(\mathcal{M})$ can be computed in polynomial time.

5.4 Arboricity again

Let G be a graph. Recall that $\Upsilon(G)$ is the smallest size of a partition of $E(G)$ into acyclic subsets. In other words, $\Upsilon(G) = k(\mathcal{M}(G))$. The Matroid Partitioning Algorithm gives us a method to compute $k(\mathcal{M}(G))$ with a number of queries of independence that is a polynomial in the size of the matroid. Since we can check if a subset of $E(G)$ is acyclic in polynomial time, the overall time complexity of Υ is polynomial.

We may use Theorem 5.3.2 on the facing page to obtain a proof of Nash-Williams's Theorem 5.1.1 on page 72.

Proof (of Theorem 5.1.1). Let $G = (V, E)$ be a graph. We want to show

$$\Upsilon(G) = \max_H \left\lceil \frac{\varepsilon(H)}{\nu(H) - 1} \right\rceil \qquad (*)$$

where the maximum is over all subgraphs H (with at least 2 vertices). We claim that we may restrict our attention to connected H. Suppose the maximum on the right-hand side of $(*)$ is achieved for a graph H with more than one component. If one of those components is an isolated vertex v, then $\nu(H - v) < \nu(H)$, but $\varepsilon(H - v) = \varepsilon(H)$, a contradiction. Thus each component of H has at least two vertices. If the components of H are $H_1, H_2, \ldots, H_c$, then we have

$$\begin{aligned} \frac{\varepsilon(H)}{\nu(H) - 1} &= \frac{\sum_i \varepsilon(H_i)}{(\sum_i \nu(H_i)) - 1} \\ &\le \frac{\sum_i \varepsilon(H_i)}{\sum_i (\nu(H_i) - 1)} \\ &\le \max_i \left\{ \frac{\varepsilon(H_i)}{\nu(H_i) - 1} \right\}. \end{aligned}$$

(See exercise 10 on page 91.)

Now by Corollary 5.3.3 on the preceding page we know that

$$\Upsilon(G) = \max_{Y \subseteq E(G)} \left\lceil \frac{|Y|}{\rho(Y)} \right\rceil.$$

We claim that the maximum here is achieved at a set $Y \subseteq E(G)$ such that (V, Y), the subgraph of G induced on the set Y, has just one nontrivial[3] component. To see this, suppose Y can be partitioned into $Y = Y_1 \cup Y_2 \cup \cdots \cup Y_c$ where each (V, Y_i) is a nontrivial component of (V, Y). Then

$$\begin{aligned} \frac{|Y|}{\rho(Y)} &= \frac{|Y_1| + \cdots + |Y_c|}{\rho(Y_1) + \cdots + \rho(Y_c)} \\ &\le \max_i \left\{ \frac{|Y_i|}{\rho(Y_i)} \right\}. \end{aligned}$$

(Again, see exercise 10.) If $H = (V, Y)$ has a single, nontrivial component on h vertices, then $\rho(V, Y) = h - 1$. Thus we have

$$\Upsilon(G) = \max_H \left\lceil \frac{\varepsilon(H)}{\nu(H) - 1} \right\rceil,$$

where the maximum is over all connected subgraphs H of G with at least 2 vertices. □

We now turn to a discussion of the fractional arboricity of G. We define $\Upsilon_f(G)$ to be $k_f(\mathcal{M}(G))$, the fractional covering number of the cycle matroid of G. Equivalently, we write $\Upsilon_t(G)$ to denote the t-fold arboricity of G: the minimum number of acyclic subsets of edges that include all edges of G at least t times. Alternatively, $\Upsilon_t(G) = \Upsilon(t \cdot G)$ where $t \cdot G$ is the multigraph formed from G by replacing each edge of G with t parallel edges. Finally, $\Upsilon_f(G) = \lim_{t \to \infty} \Upsilon_t(G)/t$.

In a similar vein, the t-fold covering number $k_t(\mathcal{M})$ of a matroid $\mathcal{M}$ is the size of a smallest collection of independent sets of $\mathcal{M}$ that includes every element of $\mathcal{M}$ at least t times. Alternatively, let $t \cdot \mathcal{M}$ denote the matroid formed by replacing each element of the ground set S with t parallel elements. (We leave it to the reader to make this precise; see exercise 11 on page 91.) Then $k_t(\mathcal{M}) = k(t \cdot \mathcal{M})$ and $k_f(\mathcal{M}) = \lim k_t(\mathcal{M})/t = \lim k(t \cdot \mathcal{M})/t$.

The following result is an analogue of Corollary 5.3.3 on the preceding page.

[3] A *trivial* component is an isolated vertex.

Theorem 5.4.1 *Let $\mathcal{M}$ be a matroid. Then*

$$k_f(\mathcal{M}) = \max_{X \subseteq S} \frac{|X|}{\rho(X)}.$$

Proof. We know that $k_t(\mathcal{M}) = k(t \cdot \mathcal{M})$ so by Corollary 5.3.3 on page 81 we have

$$k_t(\mathcal{M}) = \max_{Y \subseteq t \cdot S} \left\lceil \frac{|Y|}{\rho(Y)} \right\rceil$$

where $t \cdot S$ is the ground (multi)set of $t \cdot \mathcal{M}$. Consider a $Y \subseteq t \cdot S$ that achieves the maximum. Notice that if $y \in Y$ then all elements of $t \cdot S$ parallel to y are also in Y; otherwise we could add them to Y without increasing Y's rank. Thus we may restrict the scope of the maximization to those Y of the form $t \cdot X$ where $X \subseteq S$. We have

$$k_t(\mathcal{M}) = \max_{X \subseteq S} \left\lceil \frac{t|X|}{\rho(X)} \right\rceil = \max_{X \subseteq S} \left\{ \frac{t|X|}{\rho(X)} \right\} + O(1) = t \max_{X \subseteq S} \left\{ \frac{|X|}{\rho(X)} \right\} + O(1)$$

and therefore

$$\frac{k_t(\mathcal{M})}{t} = \max_{X \subseteq S} \left\{ \frac{|X|}{\rho(X)} \right\} + O(1/t)$$

so as $t \to \infty$ we finally have

$$k_f(\mathcal{M}) = \max_{X \subseteq S} \left\{ \frac{|X|}{\rho(X)} \right\}.$$ □

We know (see Corollary 1.3.2 on page 4) that there is a finite positive integer t for which $k_f(\mathcal{M}) = k_t(\mathcal{M})/t$. We claim that there is such a t with $t \leq \rho(\mathcal{M}) \leq |S|$.

Corollary 5.4.2 *Let $\mathcal{M}$ be a matroid. There exists a positive integer t with $t \leq \rho(\mathcal{M})$ so that $k_f(\mathcal{M}) = k_t(\mathcal{M})/t$.*

Proof. We know that $k_f(\mathcal{M}) = \max_Y |Y|/\rho(Y)$. Let Y be a subset of S that achieves this maximum and let $t = \rho(Y)$. It follows that

$$k_t(\mathcal{M}) = \max_X \left\lceil \frac{t|X|}{\rho(X)} \right\rceil$$

and this maximum is certainly achieved by Y, so, since we chose $t = \rho(Y)$, we also have

$$k_t(\mathcal{M}) = \left\lceil \frac{t|Y|}{\rho(Y)} \right\rceil = |Y| = \frac{t|Y|}{\rho(Y)}$$

and therefore

$$\frac{k_t(\mathcal{M})}{t} = \frac{|Y|}{\rho(Y)} = k_f(\mathcal{M}).$$ □

It follows that there is a polynomial-time algorithm to compute $k_f(\mathcal{M})$: we simply compute $k(t \cdot \mathcal{M})/t$ for $1 \leq t \leq \rho(\mathcal{M})$ and take the smallest answer.

Let us apply what we have learned about $k_f(\mathcal{M})$ to give an analogue of Theorem 5.1.1 on page 72 for fractional arboricity.

Corollary 5.4.3 *Let G be a graph. Then*

$$\Upsilon_f(G) = \max_H \left\{ \frac{\varepsilon(H)}{\nu(H) - 1} \right\}$$

where the maximum is over all subgraphs H of G with at least 2 vertices. □

Thus the fractional arboricity of a graph can never be much less than its arboricity.

5.5 Maximum average degree again

Recall that $\text{mad}(G)$ denotes the maximum average degree of G, i.e., $\text{mad}(G) = \max_H \bar{d}(H)$ where $\bar{d}(H) = 2\varepsilon(H)/\nu(H)$ and the maximization is over all subgraphs H of G.

Our goal is to show that $\text{mad}(G)$ is actually an instance of the fractional covering problem for a matroid that is quite similar to $\mathcal{M}(G)$.

First note that if G is a tree then $\text{mad}(G) = \bar{d}(G) = 2(1 - \frac{1}{n})$ where $n = \nu(G)$. If G is not connected, one checks (e.g., using exercise 10 on page 91) that $\text{mad}(G)$ equals the maximum average degree of one of its components. Thus in the sequel we assume that G is connected, but not a tree.

The first step is to define a new matroid on such graphs. Let $G = (V, E)$ be a connected graph that is not a tree. Let us call a subset Y of the edges *nearly acyclic* if (V, Y) has at most one cycle. Let $\mathcal{M}_1(G)$ be the pair $(E, \mathcal{I}_1)$ where $\mathcal{I}_1$ consists of all nearly acyclic subsets of edges. We claim that this forms a matroid. (In exercise 13 on page 91 this matroid is seen to be a special case of the *elongation* of a matroid.) The proof uses the following simple lemma.

Lemma 5.5.1 *Let G be a connected graph. Then G has exactly one cycle if and only if $\nu(G) = \varepsilon(G)$.*

Proof. Exercise 12 on page 91. □

Theorem 5.5.2 *Let $G = (V, E)$ be a connected graph. Then $\mathcal{M}_1(G)$ is a matroid.*

Proof. If G is a tree then $\mathcal{M}(G) = \mathcal{M}_1(G)$ and the result is trivial, so we may assume that G is not a tree.

Let the set of maximal members of $\mathcal{I}_1$ be denoted $\mathcal{B}_1$. In other words, $B \in \mathcal{B}_1$ just when (V, B) is a spanning, connected, unicyclic subgraph of G. We use Theorem 5.2.3 on page 74 to show that $\mathcal{M}_1 = \mathcal{M}_1(G)$ is a matroid.

Note first that since $\emptyset \in \mathcal{I}_1$ we have that $\mathcal{B}_1 \neq \emptyset$.

Next, by Lemma 5.5.1, note that any two members of $\mathcal{B}_1$ have the same size $\nu(G)$.

Finally, choose $B, B' \in \mathcal{B}_1$ and $x \in B$. We must prove there is a $y \in B'$ so that $B - \{x\} \cup \{y\} \in \mathcal{B}_1$. If $x \in B'$ this is trivial, so we may suppose $x \notin B'$.

Now (V, B) has a unique cycle C. Either $x \in C$ or $x \notin C$.

If $x \in C$, then $T = (V, B - \{x\})$ is a spanning tree of G. Now $|B'| = |B|$ and $B' \neq B$, so we select y to be any member of $B' - B$. Observe that $T + y$ is therefore a spanning unicyclic subgraph of G, i.e., $B - \{x\} \cup \{y\} \in \mathcal{B}_1$ as required.

Otherwise $x \notin C$ and so $(V, B - \{x\})$ consists of two connected components: one containing C and one that is acyclic. Now let T' be a spanning tree of (V, B') (we form T' by deleting any edge in the unique cycle of (V, B')). There must be some edge $y \in B'$ that links the two components of $(V, B - \{x\})$. Thus $(V, B - \{x\} \cup \{y\})$ is a spanning unicyclic subgraph of G, i.e., $B - \{x\} \cup \{y\} \in \mathcal{B}_1$ as required. □

Now that we have verified that $\mathcal{M}_1(G)$ is a matroid, we can use Theorem 5.3.2 on page 80 to show that finding $\text{mad}(G)$ is (essentially) an instance of the fractional covering problem for a matroid.

Theorem 5.5.3 *Let G be a connected graph that is not a tree. Then* $\text{mad}(G) = 2k_f(\mathcal{M}_1(G))$.

Proof. Let $G = (V, E)$ be as in the theorem. Let H be a subgraph of G for which $\bar{d}(H) = \text{mad}(G)$.

Now consider an optimal fractional covering of $\mathcal{M}_1(G)$, i.e., we have $J_1, J_2, \ldots, J_k \in \mathcal{I}_1$ and weights $w_1, w_2, \ldots, w_k \in [0, 1]$ so that for any edge e of G we have

$$\sum_{i:e \in J_i} w_i \geq 1.$$

Since the fractional covering is optimal, $\sum w_i = k_f(\mathcal{M}_1(G))$.

Now we calculate:

$$\begin{aligned}
\varepsilon(H) &\leq \sum_{e \in E(H)} \sum_{i:e \in J_i} w_i \\
&= w_1|J_1 \cap E(H)| + \cdots + w_k|J_k \cap E(H)| \\
&\leq w_1\nu(H) + \cdots + w_k\nu(H) \\
&= k_f(\mathcal{M}_1(G))\nu(H)
\end{aligned}$$

and therefore

$$\operatorname{mad}(G) = \bar{d}(H) = \frac{2\varepsilon(H)}{\nu(H)} \leq 2k_f(\mathcal{M}_1(G)).$$

We now prove the opposite inequality. By Corollary 5.3.3 on page 81

$$k_f(\mathcal{M}_1(G)) = \max_{Y \subseteq E} \frac{|Y|}{\rho_1(Y)} \tag{$*$}$$

where ρ_1 is the rank function for $\mathcal{M}_1(G)$. As in the proof of Theorem 5.1.1, it is enough to consider in $(*)$ just those Y so that (V, Y) has only one nontrivial connected component.

We know, by hypothesis, that G has at least one cycle. If C is the set of edges in that cycle, note that $|C|/\rho_1(C) = 1$. If T is an acyclic set of edges, then $|T|/\rho_1(T) = 1$, so in computing $(*)$ we may further restrict our attention to those Y for which (V, Y) has one nontrivial component that is *not* a tree; let H be the nontrivial component associated with an optimal Y. Note that $\rho_1(Y) = \nu(H)$, so we have

$$k_f(\mathcal{M}_1(G)) = \frac{|Y|}{\rho_1(Y)} = \frac{\varepsilon(H)}{\nu(H)} = \frac{1}{2}\bar{d}(H) \leq \frac{1}{2}\operatorname{mad}(G)$$

and we are done. □

Thus the Matroid Partitioning Algorithm can be used to compute, in polynomial time, the maximum average degree of a graph. Note that a "greedy" heuristic to compute $\operatorname{mad}(G)$ does not work (exercise 2 on page 90).

5.6 Duality, duality, duality, and edge toughness

We have expended considerable effort studying the covering number of a matroid $\mathcal{M}$. However, as we discussed in §1.5, there are three other related invariants closely allied to the covering number: the packing number, the transversal number, and the matching number. We consider each of these in turn.

We show that of the four invariants from Chapter 1 (covering, packing, transversal, and matching) only $k(\mathcal{M})$ is interesting.

We then introduce yet another notion of duality—matroid duality—and we consider the covering and fractional covering number of this dual. Finally, we show how the fractional covering number of the (matroid) dual of $\mathcal{M}(G)$ is closely connected to the edge toughness of G.

Hypergraph duality and mathematical programming duality

We have examined k and k_f of a matroid in extensive detail. Here we consider p, τ, and μ of a matroid (and their fractional counterparts) and find little of interest.

Let us begin with $p(\mathcal{M})$, the packing number of a matroid. Here we seek the maximum number of elements of S no two of which are together in an independent set. Let P be a maximal subset of S with no two elements of P forming an independent set. Certainly all loops of $\mathcal{M}$ are in P. Further, if $x, y \in P$ are not loops, then we must have x and y parallel. We therefore have the following.

Theorem 5.6.1 *Let $\mathcal{M}$ be a matroid. Let ℓ denote the number of loops in $\mathcal{M}$ and let m denote the maximum number of pairwise parallel elements of $\mathcal{M}$. Then $p(\mathcal{M}) = \ell + m$.* □

We know that $p(\mathcal{M}) \le k(\mathcal{M})$, but Theorem 5.6.1 does not give an interesting lower bound for $k(\mathcal{M})$. If $\mathcal{M}$ has a loop, then $k(\mathcal{M}) = \infty$. If $\mathcal{M}$ is loopless, then $k(\mathcal{M})$ is finite but certainly we have $k(\mathcal{M}) \ge m$, the maximum number of pairwise parallel elements, since parallel elements must have different colors.

Next consider the transversal number of $\mathcal{M} = (S, \mathcal{I})$, the minimum number of elements of the ground set S that intersect every member of $\mathcal{I}$. Since $\emptyset \in \mathcal{I}$, no transversal can exist, and therefore $\tau(\mathcal{M})$ is undefined (or ∞).

Finally, consider the matching number of $\mathcal{M}$. We want to compute the maximum number of pairwise disjoint independent sets of $\mathcal{M}$. It is clear that the best we can do is to take all the singleton sets in $\mathcal{I}$ (the non-loops) and $\emptyset$.

(See exercise 17 on page 91 for another approach to matching and packing.)

Theorem 5.6.2 *Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid. Let $n = |S|$ and suppose $\mathcal{M}$ has ℓ loops. Then $\mu(\mathcal{M}) = n - \ell + 1$.* □

Now the fractional packing and fractional covering numbers of a matroid are the same, and these are covered by Theorem 5.4.1 on page 83. The common value of the fractional transversal and fractional matching number is ∞.

Matroid duality

In addition to mathematical programming duality and hypergraph duality, matroids admit yet another form of duality. Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid and let $\mathcal{B}$ be the bases of $\mathcal{M}$. The *dual* of the matroid $\mathcal{M}$, denoted[4] $\tilde{\mathcal{M}} = (S, \tilde{\mathcal{I}})$, has the same ground set as S and a subset X of $\mathcal{S}$ is independent in $\tilde{\mathcal{M}}$ (i.e., is in $\tilde{\mathcal{I}}$) if and only if $X \subseteq S - B$ for some $B \in \mathcal{B}$. Indeed, it is easier to describe $\tilde{\mathcal{M}}$ in terms of its bases: The bases of $\tilde{\mathcal{M}}$ are the complements (relative to S) of the bases of $\mathcal{M}$. In symbols: $B \in \mathcal{B} \iff S - B \in \tilde{\mathcal{B}}$. We call $x \in S$ a *coloop* of $\mathcal{M}$ if $\{x\}$ is a dependent set in $\tilde{\mathcal{M}}$. In other words, a coloop of a matroid is an element contained in every basis.

The first order of business is to verify that $\tilde{\mathcal{M}}$ is a matroid.

Theorem 5.6.3 *If $\mathcal{M}$ is a matroid, then so is its dual, $\tilde{\mathcal{M}}$.*

[4]The customary notation for a matroid dual is $\mathcal{M}^*$. We use the $*$ superscript to denote hypergraph duality, so we adopt this unconventional notation.

Proof. We check that $\tilde{\mathcal{B}}$ satisfies the conditions of Theorem 5.2.3 on page 74. It is immediate that $\tilde{\mathcal{B}}$ is nonempty and any two members of $\mathcal{B}$ have the same cardinality.

Suppose $\tilde{B}_1, \tilde{B}_2 \in \tilde{\mathcal{B}}$ (so that their complements B_1, B_2 are bases of $\mathcal{M}$) and let $x \in \tilde{B}_1$. We seek a $y \in \tilde{B}_2$ so that $\tilde{B}_1 \cup \{y\} - \{x\} \in \tilde{\mathcal{B}}$. If $x \in \tilde{B}_2$ we may take $y = x$, so suppose $x \notin \tilde{B}_2$. We now know that $x \in B_2 - B_1$. Now $B_1 \cup \{x\}$ must be dependent, and contain a unique circuit C. Note that C is not entirely contained in B_2, so there must be a $y \in C - B_2$. Since $y \notin B_2$, we have $y \in \tilde{B}_2$. Finally $B_1 \cup \{x\} - \{y\}$ must be independent (since we have removed y from the unique cycle C of $B_1 \cup \{x\}$) and therefore a basis (since it is of full rank). Therefore its complement, $S - (B_1 \cup \{x\} - \{y\}) = \tilde{B}_1 \cup \{y\} - \{x\}$ is in $\tilde{\mathcal{B}}$ as desired. □

Let $G = (V, E)$ be a graph. The dual of the cycle matroid $\mathcal{M}(G)$ is denoted $\tilde{\mathcal{M}}(G)$ and is called the *cocycle matroid* of G. Let us describe the independent sets of $\tilde{\mathcal{M}}(G)$. Call a subset F of E *disconnecting* if $G - F$ has more components than G. Let K be a maximal, nondisconnecting set of edges of G, and consider the graph $H = (V, E - K)$. Notice that H has the same number of components as G (since K is nondisconnecting), but every edge of H is a cut edge (since K is maximal). Thus H is a maximal spanning forest of G, i.e., $E - K$ is a basis of $\mathcal{M}$. We have shown that if K is maximal, nondisconnecting, then K is a basis of $\tilde{\mathcal{M}}$. One easily checks that the converse is true as well. Thus the independent sets of edges of $\tilde{\mathcal{M}}(G)$ are the subsets of the maximal, nondisconnecting subsets of edges of G. A less convoluted way of saying this is the following.

Proposition 5.6.4 *Let $G = (V, E)$ be a graph and let $\tilde{\mathcal{M}}(G)$ be its cocycle matroid. A set of edges $F \subseteq E$ is independent in $\tilde{\mathcal{M}}$ if and only if F is nondisconnecting in G.* □

It follows that the circuits of $\tilde{\mathcal{M}}(G)$ are the minimal disconnecting sets of edges of G; these are known as *bonds* or *cocycles* of G. A single edge of G is a loop of $\tilde{\mathcal{M}}(G)$ if and only if it is a cut edge. A pair of edges of G are parallel in $\tilde{\mathcal{M}}(G)$ if and only if they are not cut edges, but together they form a disconnecting set. Observe that in case G is planar, then a loop of G corresponds to a cut edge of G's planar dual, and a pair of edges forming a minimal disconnecting set in G corresponds to a pair of parallel edges in the dual. This is more than a curious observation. When $G = (V, E)$ is a planar graph, then $\tilde{\mathcal{M}}(G)$ has a particularly nice description.

Theorem 5.6.5 *Let G be a planar graph (embedded in the plane) and let $\tilde{G}$ be its dual. Then $\tilde{\mathcal{M}}(G) = \mathcal{M}(\tilde{G})$.* □

The equal sign in the conclusion makes sense since we can identify the edge set of an embedded planar graph with that of its dual. The proof of Theorem 5.6.5 is relegated to exercise 14 on page 91. A corollary of this result is that if G is connected and planar, then the complement of a spanning tree in G corresponds to a spanning tree in the dual of G.

Dual rank

Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid and let $\tilde{\mathcal{M}}$ be its dual. Denote by ρ and $\tilde{\rho}$ the rank functions of these matroids. Since bases of $\mathcal{M}$ and $\tilde{\mathcal{M}}$ are complements of one another, we clearly have $\rho(\mathcal{M}) + \tilde{\rho}(\tilde{\mathcal{M}}) = |S|$. The formula for $\tilde{\rho}(X)$ for any $X \subseteq S$ is presented next.

Theorem 5.6.6 *Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid and let $\tilde{\mathcal{M}}$ be its dual. Denote by $\rho, \tilde{\rho}$ the rank functions of $\mathcal{M}, \tilde{\mathcal{M}}$ respectively. Then for any $X \subseteq S$ we have*

$$\tilde{\rho}(X) = |X| - \rho(S) + \rho(S - X).$$

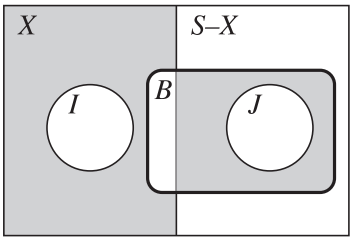

Figure 5.2. Deriving the formula for $\tilde{\rho}$.

Proof. Let $X \subseteq S$ and let $I \subseteq X$ be a maximal member of $\tilde{\mathcal{I}}$ contained in X. Let $J \subseteq S - X$ be a maximal member of $\mathcal{I}$ contained in the complement of X. Thus $\tilde{\rho}(X) = |I|$ and $\rho(S - X) = |J|$.

Since $I \in \tilde{\mathcal{I}}$, there exists a basis $B_0 \in \mathcal{B}$ for which $I \subseteq S - B_0$, or equivalently, $B_0 \subseteq S - I$. Observe, also, that $J \subseteq S - I$. Applying augmentation, we can add elements of J to B_0 until we build a basis $B \in \mathcal{B}$ for which $J \subseteq B \subseteq S - I$. See Figure 5.2. Consider the shaded regions in the figure. By the maximality of I, the shaded portion of X must be empty. Likewise, by the maximality of J, the shaded portion of $S - X$ is also empty. We may now calculate:

$$\begin{aligned}\tilde{\rho}(X) &= |I| \\ &= |X| - |X - I| \\ &= |X| - |B \cap X| \\ &= |X| - (|B| - |J|) \\ &= |X| - \rho(S) + \rho(S - X).\end{aligned}$$ □

Covering number of the dual of a matroid

If $\mathcal{M} = (S, \mathcal{I})$ is a matroid and $\tilde{\mathcal{M}} = (S, \tilde{\mathcal{I}})$ is its dual, then $k(\tilde{\mathcal{M}})$ is the minimum size of a covering of S by members of $\tilde{\mathcal{I}}$. Without loss of generality, we may assume the members of the cover are in $\tilde{\mathcal{B}}$. Thus we have

$$k(\tilde{\mathcal{M}}) \le k \iff S = \tilde{B}_1 \cup \cdots \cup \tilde{B}_k \text{ with } \tilde{B}_i \in \tilde{\mathcal{B}}.$$

Taking complements with respect to S we can rewrite this as

$$k(\tilde{\mathcal{M}}) \le k \iff \emptyset = B_1 \cap \cdots \cap B_k \text{ with } B_i \in \mathcal{B}.$$

Thus we have the following.

Theorem 5.6.7 *Let $\mathcal{M}$ be a matroid. Then $k(\tilde{\mathcal{M}})$ is the smallest number of bases of $\mathcal{M}$ whose intersection is empty. In case $\mathcal{M} = \mathcal{M}(G)$ for a connected graph G, then $k(\tilde{\mathcal{M}})$ is the minimum number of spanning trees of G whose intersection is empty.* □

We can fractionalize this result. We weight the bases of $\mathcal{M} = (S, \mathcal{I})$ so that the total weight of bases not containing each $x \in S$ is at least 1. The minimum total weight is $k_f(\tilde{\mathcal{M}})$. When G is a graph, there is a nice connection between $k_f(\tilde{\mathcal{M}}(G))$ and the *edge toughness* of G.

Edge toughness

Let G be a graph with at least one edge. The edge toughness of G is an invariant that measures how resilient the graph is against edge deletion. Formally, the *edge toughness* of G, denoted $\sigma'(G)$, is defined to be

$$\sigma'(G) = \min_F \frac{|F|}{c(G-F) - c(G)}$$

where the minimization is over all disconnecting subsets of edges of G. For example, if G is a connected graph with edge toughness σ', then we must delete at least $\lceil \sigma' j \rceil$ edges to break G into $j+1$ components. This invariant is also known as the *strength* of the graph.

Now the denominator $c(G-F)-c(G)$ is computable from the cycle matroid of G. If $F \subseteq E(G)$, then $\rho(F) = n - c(V,F)$ where $n = |V(G)|$ and $c(V,F)$ is, of course, the number of components of the graph (V,F). Thus

$$c(G-F) - c(G) = [n - c(G)] - [n - c(G-F)] = \rho(E) - \rho(E-F).$$

Thus, letting $X = E - F$, we can rewrite the definition of edge toughness as

$$\sigma'(G) = \min_X \frac{|E-X|}{\rho(E) - \rho(X)}$$

where the minimization is over all subsets of edges X for which (V,X) has more components than G.

Notice that this new definition easily extends to matroids. If $\mathcal{M} = (S, \mathcal{I})$ is a matroid, define

$$\sigma'(\mathcal{M}) = \min_X \frac{|S-X|}{\rho(S) - \rho(X)}$$

where the minimization is over all $X \subseteq S$ for which $\rho(X) < \rho(S)$.

Note that if G has at least one edge, then $\sigma'(G) = 1$ if and only if G has a cut edge (exercise 15 on page 91). In a similar vein, $\Upsilon(G) = \infty$ if and only if G has a loop.

More generally, for a matroid $\mathcal{M}$ (with rank at least 1) we have $\sigma'(\mathcal{M}) = 1$ if and only if $\mathcal{M}$ has a coloop (exercise 16), and $\Upsilon(\mathcal{M}) = \infty$ if and only if $\mathcal{M}$ has a loop.

If $\mathcal{M}$ has neither loops nor coloops, then there is a nice connection between $\sigma'(\mathcal{M})$ and $k_f(\tilde{\mathcal{M}})$.

Theorem 5.6.8 *Let $\mathcal{M}$ be a matroid and suppose both $\mathcal{M}$ and $\tilde{\mathcal{M}}$ are loopless. Then*

$$\frac{1}{\sigma'(\mathcal{M})} + \frac{1}{k_f(\tilde{\mathcal{M}})} = 1.$$

Proof. Rewrite Theorem 5.6.6 on page 87 as $\rho(X) = |X| - \tilde{\rho}(S) + \tilde{\rho}(S-X)$ and use it to compute:

$$\begin{aligned}
1 - \frac{1}{\sigma'(\mathcal{M})} &= 1 - \max\left(\frac{\rho(S) - \rho(X)}{|S-X|}\right) \\
&= \min\left(1 - \frac{\rho(S) - \rho(X)}{|S-X|}\right) \\
&= \min\left(1 - \frac{[|S| - \tilde{\rho}(S)] - [|X| - \tilde{\rho}(S) + \tilde{\rho}(S-X)]}{|S-X|}\right) \\
&= \min\left(1 - \frac{|S| - |X| - \tilde{\rho}(S-X)}{|S-X|}\right)
\end{aligned}$$

$$= \min\left(\frac{\tilde{\rho}(S-X)}{|S-X|}\right)$$
$$= \frac{1}{k_f(\tilde{\mathcal{M}})}. \qquad \square$$

We can rewrite Theorem 5.6.8 in graph terms.

Corollary 5.6.9 *Let G be a graph. The edge toughness of G is given by*

$$\sigma'(G) = \frac{k_f(\tilde{\mathcal{M}}(G))}{k_f(\tilde{\mathcal{M}}(G)) - 1}.$$

If G is planar with planar dual $\tilde{G}$, then

$$\sigma'(G) = \frac{\Upsilon_f(\tilde{G})}{\Upsilon_f(\tilde{G}) - 1}. \qquad \square$$

In particular, the Matroid Partitioning Algorithm gives us a polynomial-time algorithm for computing the edge toughness of a graph.

5.7 Exercises

1. Prove that the arboricity of a planar graph is at most 3 and that this bound is the best possible.

2. Prove that $\mathrm{mad}(G)$ cannot be computed in a greedy fashion. Specifically, find a graph G for which the following algorithm fails.

   Input: a graph G.

   Output: (allegedly) $\mathrm{mad}(G)$.

   (a) Let $X \leftarrow \emptyset$.
   (b) Let $X \leftarrow X \cup \{\bar{d}(G)\}$.
   (c) Let v be a vertex of minimum degree in G.
   (d) Let $G \leftarrow G - v$.
   (e) If G is not empty, go to step (b).
   (f) Output the largest element of X.

3. Prove that for any matrix A, $\mathcal{M}(A)$ is indeed a matroid.

4. Prove that for any graph G, $\mathcal{M}(G)$ is indeed a matroid.

5. Create a good definition of isomorphic matroids.

6. Prove or disprove: For all graphs G, $\mathcal{M}(G)$ is isomorphic to $\mathcal{M}(A(G))$ where $A(G)$ is the adjacency matrix of G.

7. Prove or disprove: For every graph G, there is a matrix A so that $\mathcal{M}(G)$ is isomorphic to $\mathcal{M}(A)$.

8. Let $U_{n,k}$ denote the pair $(S, \mathcal{I})$ where S is an n-set and $\mathcal{I}$ is the set of all subsets of S with at most k elements.

   Prove that $U_{n,k}$ is a matroid. We call $U_{n,k}$ a *uniform* matroid.

   Compute $k(U_{n,k})$ and $k_f(U_{n,k})$.

9. Prove that there is no graph G for which $\mathcal{M}(G) \cong U_{4,2}$. (The preceding exercise defines $U_{n,k}$.)

10. Prove that if $a_i, b_i > 0$ (for $i = 1, \ldots, c$) then

$$\frac{a_1 + \cdots + a_c}{b_1 + \cdots + b_c} \leq \max_i \left\{ \frac{a_i}{b_i} \right\}.$$

    (We use this fact in the proof of Theorem 5.1.1.)

11. Write a careful definition for $t \cdot \mathcal{M}$: the matroid formed from $\mathcal{M}$ by replacing each element of the ground set of $\mathcal{M}$ with t parallel elements.

12. Prove Lemma 5.5.1 on page 84: *Let G be a connected graph. Then G has exactly one cycle if and only if $\nu(G) = \varepsilon(G)$.*

13. Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid with rank function ρ. Let t be a positive integer. Define

$$\mathcal{I}_t = \{X \subseteq S : \rho(I) \geq |I| - t\},$$

    i.e., all sets that can be formed from independent sets by the addition of at most t additional elements.

    Prove that $\mathcal{M}_t = (S, \mathcal{I}_t)$ is a matroid, called an *elongation* of $\mathcal{M}$.

    Observe that this is a generalization of Theorem 5.5.2 on page 84 since if $\mathcal{M} = \mathcal{M}(G)$, then $\mathcal{M}_1 = \mathcal{M}_1(G)$.

14. Prove Theorem 5.6.5 on page 87: *Let G be a planar graph (embedded in the plane) and let $\tilde{G}$ be its dual. Then $\tilde{\mathcal{M}}(G) = \mathcal{M}(\tilde{G})$.*

15. Let G be a graph with at least one edge. Prove that $\sigma'(G) \geq 1$ and that $\sigma'(G) = 1$ if and only if G has a cut edge.

16. (Generalizing the previous problem.) Let $\mathcal{M}$ be a matroid of rank at least 1. Prove that $\sigma'(\mathcal{M}) \geq 1$ and $\sigma'(\mathcal{M}) = 1$ if and only if $\mathcal{M}$ has a coloop.

17. Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid. Define a hypergraph $\mathcal{H} = (S, \mathcal{I} - \{\emptyset\})$, i.e., delete $\emptyset$ as a member of $\mathcal{I}$. Note that $k(\mathcal{M}) = k(\mathcal{H})$. State and prove simple formulas for $\mu(\mathcal{H})$, $p(\mathcal{H})$, and $\tau(\mathcal{H})$.

5.8 Notes

Matroids were invented by Hassler Whitney [189] in the 1930s. For background reading on matroids, see the monographs by Recski [152] or Welsh [186].

The results on arboricity (e.g., Theorem 5.1.1 on page 72) are due to Nash-Williams [137, 138]. See also Edmonds [49].

The Matroid Partitioning Algorithm is due to Edmonds [49], but he treats a more general situation. Given several matroids on the same ground set, e.g., $\mathcal{M}_i = (S, \mathcal{I}_i)$ for $1 \leq i \leq k$, Edmonds seeks a partition $S = I_1 \cup I_2 \cup \cdots \cup I_k$ where each $I_i \in \mathcal{I}_i$. We only needed the case in which all k matroids are the same.

The material on fractional arboricity is based on the work of Payan [143] and Catlin, Grossman, Hobbs, and Lai [33, 34]. See the latter papers for a treatment of edge toughness.

Random graph theorists (and others) are concerned with graphs G whose densest portion is G itself. In particular, a graph G is called *balanced* provided $\text{mad}(G) = \bar{d}(G)$. Maximum average degree and balanced graphs arise in the theory of evolution of graphs. Let G be a fixed graph and let n be a very large positive integer. Form a graph by gradually adding edges at random between n fixed vertices. As the density of the evolving random graph increases we eventually reach a stage where it is very likely that the random graph contains G as a subgraph. The point at which this happens depends only on $\text{mad}(G)$. See Bollobás [23] or Palmer [141]. For more information on balanced graphs, see the papers by Lai and Lai [112] or Ruciński and Vince [155].