8

Introduction to Greedoids

ANDERS BJÖRNER and GÜNTER M. ZIEGLER

8.1. Introduction

Greedoids were invented around 1980 by B. Korte and L. Lovász. Originally, the main motivation for proposing this generalization of the matroid concept came from combinatorial optimization. Korte and Lovász had observed that the optimality of a 'greedy' algorithm could in several instances be traced back to an underlying combinatorial structure that was not a matroid – but (as they named it) a 'greedoid'. In subsequent research greedoids have been shown to be interesting also from various non-algorithmic points of view.

The basic distinction between greedoids and matroids is that greedoids are modeled on the *algorithmic construction* of certain sets, which means that the *ordering of elements* in a set plays an important role. Viewing such ordered sets as words, and the collection of words as a formal language, we arrive at the general definition of a greedoid as a finite language that is closed under the operation of taking initial substrings and satisfies a matroid-type exchange axiom. It is a pleasant feature that greedoids can also be characterized in terms of set systems (the unordered version), but the language formulation (the ordered version) seems more fundamental.

Consider, for instance, the algorithmic construction of a spanning tree in a connected graph. Two simple strategies are: (1) pick one edge at a time, making sure that the current edge does not form a circuit with those already chosen; (2) pick one edge at a time, starting at some given node, so that the current edge connects a visited node with an unvisited node. These well known strategies are used respectively in Kruskal's and in Prim's minimal spanning tree algorithms. In both cases, the collection of feasible sequences of edges, i.e. sequences that are generated by the allowed strategy, forms a greedoid. However, in the first case, but not in the second, any permutation of a feasible sequence of edges is also feasible, so that ordering is irrelevant.

This is so because the first greedoid, but not the second, is a matroid. The optimality of Prim's algorithm, which is not explained by matroid theory, is indeed covered by greedoid theory.

In this chapter we shall give an introduction to greedoids. Our aim is to explain the basic ideas and to give a few glimpses of more specialized topics. In spite of its youth the subject is already large enough to make a complete account impossible in the available space. Due to the space limitation we have frequently chosen to omit detailed proofs, particularly when good proofs exist in the literature. Also, to unburden the main text, all references to original papers and additional comments are gathered in the 'Notes and Comments' at the end of the chapter.

Here is an outline of the contents. Section 8.2 discusses the axiomatics of greedoids and explains the equivalence of the ordered and unordered versions of the concept. Many of the basic definitions in the area are given here. In particular, the important class of interval greedoids is defined.

Many examples of greedoids are described in section 8.3. One of the interesting features of the greedoid concept is that it admits such a variety of combinatorial examples in addition to matroids: branchings in graphs, order ideals in posets, convex hull closures in Euclidean and other spaces, Gaussian elimination sequences, retract sequences, and many more.

In section 8.4 various structural properties of greedoids as combinatorial systems are discussed. Just like matroids, greedoids have cryptomorphic descriptions in terms of a rank function and a closure operator. Deletion, contraction, and some other operations on greedoids are defined, as well as a suitable notion of connectivity.

Connections with combinatorial optimization are presented in section 8.5. For a certain kind of objective function, the greedy algorithm is optimal over a greedoid. In fact, greedoids can be characterized in terms of this algorithmic property. Examples of greedoid optimization include, e.g., Dijkstra's shortest path algorithm. Linear objective functions pose special problems, which are briefly discussed.

Section 8.6 discusses a certain polynomial that is associated with every greedoid. It is a greedoid version of the Tutte polynomial of matroid theory. The polynomial has applications of an algorithmic and of a probabilistic nature. For instance, it is possible to express in terms of this polynomial the probability that rank will not decrease if elements are independently deleted with probability p. Finally, there is a brief discussion of what aspects of matroid duality can be said to exist for general greedoids.

Antimatroids form a special class of interval greedoids with considerable additional structure. They are discussed in section 8.7 as dual objects to convex geometries. Among the interval greedoids, matroids and antimatroids are from several points of view opposite classes. Each is connected with a closure operator, which for matroids abstracts *linear span* and for antimatroids abstracts *convex hull* in Euclidean spaces.

In section 8.8 the connections between greedoids and posets (particularly lattices) are discussed in some detail. Each greedoid has a poset of flats, which in general is not a lattice. For interval greedoids the poset of flats is a semimodular lattice, and every finite semimodular lattice arises in this way.

The following additional topics are briefly discussed in section 8.9: (1) the characterization of certain classes of greedoids by excluded minors; (2) the maximum number of feasible pivots needed to move from one basis to any other basis in a greedoid; (3) examples of greedoid languages that allow repetition of letters within feasible words.

8.2. Definitions and Basic Facts

8.2.A. Ordered and Unordered Versions

There are two equivalent definitions of greedoids, one as set systems and the other as languages. We will start by defining and discussing greedoids as set systems. The equivalence of the two approaches will be heavily used later by freely choosing, depending on context, whatever formulation seems more convenient or natural.

In the following, we will work over a finite ground set E. The set of all subsets of E will be denoted by 2^E, and a *set system* over E is a non-empty family $\mathcal{F} \subseteq 2^E$.

8.2.1. Definition. A *greedoid* is a pair $(E, \mathcal{F})$, where $\mathcal{F} \subseteq 2^E$ is a set system satisfying the following conditions.

(G1) For every non-empty $X \in \mathcal{F}$ there is an $x \in X$ such that $X - x \in \mathcal{F}$.
(G2) For $X, Y \in \mathcal{F}$ such that $|X| > |Y|$, there is an $x \in X - Y$ such that $Y \cup x \in \mathcal{F}$.

The axiom (G2) is the usual matroid exchange axiom. In fact, every matroid is a greedoid, and a greedoid is a matroid exactly if it is *hereditary*, that is, if the axiom

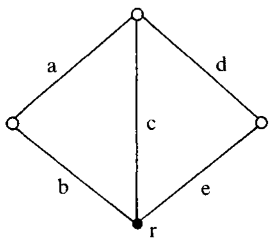
(M1) If $X \in \mathcal{F}$ and $Y \subseteq X$, then $Y \in \mathcal{F}$.

is satisfied. (M1) is a strengthening of (G1); (M1) and (G2) together define a matroid.

Many examples of greedoids that are not matroids will be given in the next section. To i-llustrate the definition, let us now look at one of these.

Let $\Gamma = (V, E, r)$ be a rooted graph, and let $\mathcal{F}$ be the family of subtrees in Γ that contain the root node r. We think of these subtrees as edge sets, so $\mathcal{F} \subseteq 2^E$. Now, if $X \neq \emptyset$ is such a tree, then it must have at least one leaf

Figure 8.1.



other than r, and if x is the edge adjacent to such a leaf then also $X - x \in \mathcal{F}$. Also, the cardinality $|X|$ of a subtree X equals the number of vertices other than r that are reached by X. Consequently, if $|X| > |Y|$ there must be some node $v \in V - r$ that is reached by X but not by Y. Follow the unique path in X from r to v and let x be the first edge of that path with a vertex not in Y. Then clearly $Y \cup x$ is also a subtree in $\mathcal{F}$. We have verified axioms (G1) and (G2), so $(E, \mathcal{F})$ is a greedoid. The greedoids that arise in this way (called 'undirected branching greedoids') will be further discussed in section 8.3.C. For the particular greedoid given by the rooted graph (V, E, r) in Figure 8.1 we observe e.g. that $\{b\}, \{a, c\}, \{b, c, d\} \in \mathcal{F}$ and $\{a\}, \{a, d\}, \{a, b, c\} \notin \mathcal{F}$.

The axiom (G1) states that $\mathcal{F}$ is an *accessible* set system. It implies – because E is finite and $\mathcal{F}$ non-empty – that $\mathcal{F}$ contains the empty set. In fact, by (G1) every $X \in \mathcal{F}$ can be dismantled by successively removing elements to get a sequence $\varnothing = X_0 \subset X_1 \subset \ldots \subset X_k = X$, where every X_i is a set in $\mathcal{F}$ of cardinality i, $0 \leqq i \leqq k$. But the same also follows from (G2): if we assume $\varnothing \in \mathcal{F}$, then repeated application of (G2) implies the existence of a sequence $\varnothing = X_0 \subset X_1 \subset \ldots \subset X_k = X$, where $X_i \in \mathcal{F}$ and $|X_i| = i$ for $1 \leqq i \leqq k$. Hence, in Definition 8.2.1, (G1) could be replaced by the weaker axiom.

(G1′) $\varnothing \in \mathcal{F}$.

Just as for matroids, it is again sufficient (using (G1)) to require the exchange property of (G2) only for $|X| = |Y| + 1$:

(G2′) For $X, Y \in \mathcal{F}$, $|X| = |Y| + 1$, there is an $x \in X - Y$ such that $Y \cup x \in \mathcal{F}$.

The axioms (G1) and (G2′) together define greedoids as well as (G1′) and (G2). However, (G1′) and (G2′) together clearly do not suffice.

The following terminology will be used. For greedoids the sets in $\mathcal{F}$ are called *feasible* (rather than '*independent*'). As usual, the matroid exchange axiom (G2) implies that the (inclusion-wise) maximal feasible sets, the *bases*, have the same size r; $r = r(\mathcal{F})$ is called the *rank* of the greedoid $(E, \mathcal{F})$. For an arbitrary subset A of the ground set E we define its *rank* by $r(A) = \max\{|X|: X \subseteq A, X \in \mathcal{F}\}$. Thus A is feasible if and only if $r(A) = |A|$, and it is a basis if and only if $r(A) = |A| = r(\mathscr{F})$. The characteristic properties of the greedoid rank function will be discussed in section 8.4.A.

A *basis of a subset* $A \subseteq E$ is a maximal feasible subset of A. Equivalently, this is an $X \in \mathscr{F}$ such that $X \subseteq A$ and $r(X) = r(A)$, because the exchange axiom (G2) implies that every maximal feasible subset of A has size $r(A)$. In fact, for any set system $(E, \mathscr{F})$ the property

(B) For any subset $A \subseteq E$ all maximal feasible subsets of A have the same cardinality

is implied by the exchange axiom (G2). On the other hand, (B) together with (G1) does not imply (G2), as shown by $E = \{1, 2, 3\}$ and $\mathscr{F} = 2^E - \{\{1, 3\}, \{2, 3\}\}$. See also Exercise 8.1.

A *coloop* in a greedoid $(E, \mathscr{F})$ is an element $x \in E$ that is contained in every basis, and a *loop* is an element that is contained in no basis. If x is a loop then $r(\{x\}) = 0$, but not conversely. Another difference from the matroid case is that $r(\{x\}) = 0$ is possible for a coloop x. These facts are particularly easy to visualize for branching greedoids, cf. section 8.3.C. We will sometimes write just $\cup \mathscr{F}$ for the union of all feasible sets. Clearly, x is a loop if and only if $x \in E - \cup \mathscr{F}$. (Note that in matroid theory a coloop is often called an 'isthmus').

We will now describe the equivalent 'ordered' version of greedoids, in terms of exchange languages. For the finite ground set E, let E^* denote the free monoid of all *words* over the alphabet E. We use Greek letters $\alpha, \beta, \gamma, \ldots$ for words in E^* and Latin letters $x, y, z, \ldots$ for 'letters', i.e. elements of E. The concatenation of α and β (the string α followed by the string β) will be denoted by $\alpha\beta$. For any word $\alpha \in E^*$, $|\alpha|$ denotes the *length* of α, i.e. the number of (not necessarily distinct) letters in α. The *support* $\tilde{\alpha}$ of α is the set of letters in α. A word α is called *simple* if it does not contain any letter more than once, i.e. if $|\alpha| = |\tilde{\alpha}|$.

A *language* $\mathscr{L}$ over E is a non-empty set $\mathscr{L} \subseteq E^*$ of words over the alphabet E; it is called *simple* if every word in $\mathscr{L}$ is simple. Every simple language over a finite set E is again finite. Let E_s^* denote the (finite) set of simple words in E^*. By the *support* $\tilde{\mathscr{L}}$ of the language $\mathscr{L}$ we mean the set system $\tilde{\mathscr{L}} = \{\tilde{\alpha}: \alpha \in \mathscr{L}\}$.

8.2.2. Definition. A *greedoid language* over a finite ground set E is a pair $(E, \mathscr{L})$, where $\mathscr{L}$ is a simple language $\mathscr{L} \subseteq E_s^*$ satisfying the following conditions.

(L1) If $\alpha = \beta\gamma$ and $\alpha \in \mathscr{L}$, then $\beta \in \mathscr{L}$, i.e. every beginning section of a word in $\mathscr{L}$ is again in $\mathscr{L}$;

(L2) If $\alpha, \beta \in \mathscr{L}$ and $|\alpha| > |\beta|$, then α contains a letter x such that $\beta x \in \mathscr{L}$.

Here (L1) states that $\mathscr{L}$ is a *(left) hereditary* language; (L2) is an exchange axiom. Again, it would be sufficient to require that (L2) holds for $|\alpha| = |\beta| + 1$.

The words in $\mathscr{L}$ are called *feasible*. The maximal words in $\mathscr{L}$ (that is, the words that do not have extensions in $\mathscr{L}$) are called *basic words*. We call a language *pure* if all its maximal words have the same length. In particular, as a consequence of exchange axiom (L2), greedoid languages are pure. The common length of all basic words is called the *rank* of the greedoid $(E, \mathscr{L})$.

Let us illustrate Definition 8.2.2 by again considering a rooted graph $\Gamma = (V, E, r)$, as in the discussion following Definition 8.2.1. A string $x_1 x_2 \dots x_k$ of distinct edges $x_i \in E$ will be considered feasible if the subgraph $\{x_1, x_2, \dots, x_{i-1}\}$ connects the root node r to one endpoint of x_i but not the other, for $1 \leq i \leq k$. It is instructive to check that the language $\mathscr{L}$ of such feasible strings is a greedoid language. For instance, in Figure 8.1 we find that b, ca, $cbd \in \mathscr{L}$, but a, ac, $bdc \notin \mathscr{L}$.

Definitions 8.2.1 and 8.2.2 are tied together by:

8.2.3. Proposition. *Greedoids and greedoid languages are equivalent in the following sense.*

(i) *If $(E, \mathscr{L})$ is a greedoid language, then the support $\tilde{\mathscr{L}}$ is a greedoid.*
(ii) *If $(E, \mathscr{F})$ is a greedoid, then*

$$\mathscr{L}(\mathscr{F}) = \{x_1, x_2 \dots x_k \in E_s^*: \{x_1, x_2, \dots, x_i\} \in \mathscr{F} \text{ for } 1 \leq i \leq k\}$$

is a greedoid language.
(iii) *Furthermore, $\mathscr{L}(\tilde{\mathscr{L}}) = \mathscr{L}$ and $\tilde{\mathscr{L}}(\mathscr{F}) = \mathscr{F}$, so these constructions give a one-to-one correspondence between greedoids and greedoid languages.*

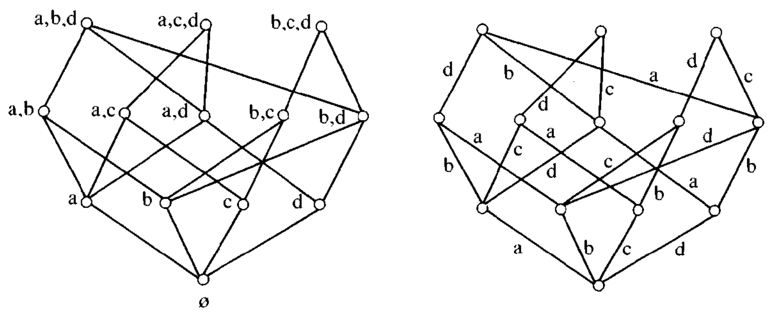
The verification of parts (i) and (ii) is straightforward and very easy. For part (iii), the only point that requires a small argument is the inclusion $\mathscr{L}(\tilde{\mathscr{L}}) \subseteq \mathscr{L}$, which follows by induction on the length of words from the exchange axiom (L2) together with the simplicity of the language $\mathscr{L}$.

In view of this equivalence between greedoids (as set systems) and greedoid languages, the two concepts will from now on be used interchangeably. For a greedoid G we will freely write $G = (E, \mathscr{F}) = (E, \mathscr{L})$, and if the ground set is clear from the context G will often be denoted by just $\mathscr{F}$ or $\mathscr{L}$. (If the ground set is not given by context, one can always take $E = \cup \mathscr{F}$ or $E = \cup \tilde{\mathscr{L}}$ to recover it, except for loops.)

It is often convenient to think of a greedoid $(E, \mathscr{F})$ as a poset $(\mathscr{F}, \subseteq)$, with the partial order given by inclusion. This poset has a least element $\emptyset$, and every unrefinable chain from $\emptyset$ to a maximal element B (i.e. a basis) has the same length $r(\mathscr{F}) = |B|$. More generally, if $A \subseteq C$, where $A, C \in \mathscr{F}$, then every unrefinable chain $A = A_0 \subset\!\cdot\, A_1 \subset\!\cdot\, \dots \subset\!\cdot\, A_k = C$ in $\mathscr{F}$ has the same length $k = |C - A|$, since A can be repeatedly augmented from C one element at a time.

For instance, let $E = \{a, b, c, d\}$ and consider the greedoid $\mathscr{F} = 2^E -$

Figure 8.2.



(a) (b)

$\{\{a, b, c, d\}, \{a, b, c\}, \{c, d\}\}$. (A systematic reason why this *is* a greedoid is given in Exercise 8.4.) The poset $\mathscr{F}$ is depicted in Figure 8.2a where for simplicity set brackets are omitted.

To see clearly the connection with the language version $(E, \mathscr{L})$ of $\mathscr{F}$, reformulate the poset $\mathscr{F}$ of Figure 8.2a into an abstract edge-labeled poset as in Figure 8.2b. Here each covering edge $X \subset\!\cdot\, Y$ is labeled by the single element of $Y - X$. Then the words in $\mathscr{L} = \mathscr{L}(\mathscr{F})$ can be read off as the sequences of labels along unrefinable chains starting at the bottom. For instance, the basic words beginning with '*b*' are: *bad*, *bcd*, *bda*, and *bdc*. Also, the feasible set corresponding to an element in this abstract labeled poset can be reconstructed as the set of labels on any unrefinable chain from the bottom to that element.

We remark that non-isomorphic greedoids can have isomorphic unlabeled posets of feasible sets (non-trivial examples are mentioned at the end of section 8.4.E; see also section 8.8.D). Hence, the edge labels as in Figure 8.2(b) are essential for such a poset representation of a greedoid.

8.2.B. Interval Greedoids and Antimatroids

The following 'interval property' characterizes a very large class of greedoids that covers many of the main examples (see section 8.3). Greedoids with this property, usually called *interval greedoids*, behave better than general greedoids in many respects. In some types of study the interval property has to be assumed to obtain meaningful results.

8.2.4. Definition. A greedoid $(E, \mathscr{F})$ has the *interval property* if $A \subseteq B \subseteq C$, $A, B, C \in \mathscr{F}$, $x \in E - C$, $A \cup x \in \mathscr{F}$, and $C \cup x \in \mathscr{F}$ imply that $B \cup x \in \mathscr{F}$. Equivalently, in terms of greedoid language $(E, \mathscr{L})$ this means that $\alpha x, \alpha\beta\gamma x \in \mathscr{L}$ implies $\alpha\beta x \in \mathscr{L}$.

We observe that the greedoid in Figure 8.2 does not have the interval property, since e.g. $\varnothing$, $\{c\}$, and $\{b, c\}$ are feasible, and the first and third can be augmented by d but not the second. Clearly, every greedoid of rank less than three has the interval property.

The following exchange property characterization of interval greedoids is often useful.

8.2.5. Proposition. *A hereditary language $(E, \mathscr{L})$ is an interval greedoid if and only if it satisfies the following strong exchange property:*

(L2′) *If $\alpha, \beta \in \mathscr{L}$ and $|\alpha| > |\beta|$, then α contains a subword α' of length $|\alpha'| = |\alpha| - |\beta|$ such that $\beta\alpha' \in \mathscr{L}$.*

Here a *subword* of $\alpha = x_1 x_2 \ldots x_n$ is a not necessarily consecutive substring of α, i.e. a word of the form $\alpha' = x_{i_1} x_{i_2} \ldots x_{i_k}$ with $1 \leqq i_1 < i_2 < \ldots < i_k \leqq n$. Obviously, the axiom (L2′) implies the regular exchange property (L2).

Proof. We will here prove that (L2′) implies the interval property. The proof of the converse will be postponed until section 8.8.C.

Suppose that $\alpha x, \alpha\beta\gamma x \in \mathscr{L}$. Strong exchange gives, since $\mathscr{L}$ is simple, that $\alpha x \beta\gamma \in \mathscr{L}$. Since $\mathscr{L}$ is left hereditary, $\alpha\beta, \alpha x\beta \in \mathscr{L}$, and a second application of strong exchange yields $\alpha\beta x \in \mathscr{L}$. □

A relatively special but very important class of interval greedoids is the class of antimatroids. Some of their special properties will be discussed in section 8.7.

8.2.6. Definition. A greedoid $(E, \mathscr{F})$ is called an *antimatroid* if it satisfies the following *interval property without upper bounds*: if $A \subseteq B$, $A, B \in \mathscr{F}$, $x \in E - B$, and $A \cup x \in \mathscr{F}$, then $B \cup x \in \mathscr{F}$. Equivalently, in terms of greedoid language, if $\alpha x, \alpha\beta \in \mathscr{L}$ and $x \notin \tilde{\beta}$, then $\alpha\beta x \in \mathscr{L}$.

In many cases, the easiest way to recognize an antimatroid is via the following characterization.

8.2.7. Proposition. *Let $\mathscr{F} \subseteq 2^E$ be a set system. Then the following conditions are equivalent.*

(i) *$(E, \mathscr{F})$ is an antimatroid.*
(ii) *$\mathscr{F}$ is accessible and closed under union.*
(iii) *$\varnothing \in \mathscr{F}$, and $\mathscr{F}$ satisfies the following exchange axiom.*
   (A) *For $X, Y \in \mathscr{F}$ such that $X \nsubseteq Y$, there is an $x \in X - Y$ such that $Y \cup x \in \mathscr{F}$.*

Proof. (ii) $\Rightarrow$ (iii). Suppose that $\mathscr{F}$ is accessible (i.e. satisfies axiom (G1)) and is closed under union (i.e. $A, B \in \mathscr{F}$ implies $A \cup B \in \mathscr{F}$). Let $X, Y \in \mathscr{F}$ such that $X \nsubseteq Y$. Accessibility means that we can find a sequence $\emptyset = X_0 \subset X_1 \subset \ldots \subset X_k = X$ such that $X_i \in \mathscr{F}$ and $|X_i| = i$, for $0 \leqq i \leqq k$. Let i be the least integer for which $X_i \nsubseteq Y$. Then $Y \cup X_i = Y \cup x \in \mathscr{F}$, where $x \in X_i - Y \subseteq X - Y$.

(iii)$\Rightarrow$(i). Axiom (A) implies axiom (G2), and axiom (G1′) is assumed, so $\mathscr{F}$ is a greedoid. Suppose that $A, B, A \cup x \in \mathscr{F}$, $A \subset B$, and $A \cup x \nsubseteq B$. By axiom (A), the set B can be augmented from the set $(A \cup x) - B = \{x\}$, so $B \cup x \in \mathscr{F}$. This proves the interval property without upper bounds.

(i)$\Rightarrow$(ii). We leave this step, which is similar to the other two, as an exercise for the reader. □

Notice that, since $\mathscr{F}$ is closed under union, every subset in an antimatroid has a unique basis.

The following result expresses some of the ways in which interval greedoids and antimatroids are related. The proof is a simple exercise with the interval property, with and without upper bounds.

8.2.8. Proposition. *Let $(E, \mathscr{F})$ be a greedoid. Then*

(i) *$(E, \mathscr{F})$ is an antimatroid if and only if it is an interval greedoid and has a unique basis;*
(ii) *$(E, \mathscr{F})$ is an interval greedoid if and only if the restriction to each feasible set $X \in \mathscr{F}$, meaning $\{Y \in \mathscr{F}\colon Y \subseteq X\}$, is an antimatroid.*

A greedoid $(E, \mathscr{F})$ is said to be *full* if $E \in \mathscr{F}$. It follows from the preceding result that if an antimatroid has no loops then it is full. In any case, an antimatroid has one and only one basis, namely $\cup \mathscr{F}$. We remark in this connection that the wealth of examples of antimatroids, all with only one basis, shows that greedoids cannot in general be reconstructed from or axiomatically characterized in terms of their set of bases. On the other hand, a greedoid is of course completely determined by its basic words.

For a general set system $\mathscr{F} \subseteq 2^E$ such that $\emptyset \in \mathscr{F}$, define its *accessible kernel* (with some mild abuse of set notation) by

$$A(\mathscr{F}) = \{\{x_1, \ldots, x_k\} \in \mathscr{F}\colon \{x_1, \ldots, x_i\} \in \mathscr{F} \text{ for all } 1 \leqq i \leqq k\},$$

or – recursively – by: $X \in A(\mathscr{F})$ iff $X \in \mathscr{F}$ and $X = \emptyset$ or there is an $x \in X$ such that $X - x \in A(\mathscr{F})$. The *hereditary closure* of a set system $\mathscr{F}$ is defined as

$$H(\mathscr{F}) = \{Y \subseteq E\colon Y \subseteq X \text{ for some } X \in \mathscr{F}\}.$$

Thus, for every greedoid $\mathscr{F}$, (G1) states that $A(\mathscr{F}) = \mathscr{F}$, and $\mathscr{F}$ is a matroid precisely when $H(\mathscr{F}) = \mathscr{F}$. But we note that in general for a greedoid $\mathscr{F}$, $H(\mathscr{F})$ need not be the collection of feasible sets of a greedoid.

8.3. Examples

In this section we shall survey some major classes of greedoids. Many other classes are known, and the reader will find new examples constructed in nearly every paper on the subject.

8.3.A. Matroids

As was remarked before, the independent sets of a matroid form the feasible sets of a greedoid. Much of the terminology for greedoids is adapted from matroid theory, so that there is no translation problem. In particular, the rank function and bases of a matroid and its associated greedoid coincide. Matroids are clearly interval greedoids. In fact, they can be characterized as greedoids satisfying the '*interval property without lower bounds*': If $B \subseteq C$, B, $C \in \mathscr{F}$ and $x \in E - C$, then $C \cup x \in \mathscr{F}$ implies $B \cup x \in \mathscr{F}$. This is equivalent to the statement that $B \subseteq C$, $C \in \mathscr{F}$ implies $B \in \mathscr{F}$, i.e. that $\mathscr{F}$ is hereditary.

Matroids give rise to greedoids in more than one way. For example, the following construction produces '*twisted matroids*'. Let $M = (E, \mathscr{I})$ be a matroid; choose an independent set $A \in \mathscr{I}$. We define a simple language $\mathscr{L}_{M,A}$ by $\mathscr{L}_{M,A} = \{a_1 \dots a_k \in E_s^*\colon A\Delta\{a_1, \dots, a_i\} \in \mathscr{I} \text{ for all } 1 \leqq i \leqq k\}$. Here, $A\Delta B$ denotes the symmetric difference $(A - B) \cup (B - A)$. This language is clearly left hereditary and the exchange axiom (L2) can be checked. For $A = \varnothing$ we get the standard matroid greedoid $\mathscr{L} = \mathscr{L}_{M,\varnothing}$. However, $\mathscr{L}_{M,A}$ depends heavily on A, and is in general not an interval greedoid. The feasible sets of the twisted matroid $\mathscr{L}_{M,A}$ are the sets whose symmetric difference with A is independent. The basic words describe the ways to move from A to a basis of $E - A$ through a sequence of intermediate independent sets.

Some greedoids are related to matroids in the following way: the greedoid $G = (E, \mathscr{F})$ is a *slimming* of the matroid $M = (E, \mathscr{I})$ if G and M have the same set of bases (which implies that $\mathscr{F} \subseteq \mathscr{I}$). For instance, the twisted matroid $(E, \mathscr{L}_{M,A})$ defined above is a slimming of the direct sum of the free matroid on A and the restriction $M - A$. We will encounter another example in section 8.3.C.

8.3.B. Antimatroids

Antimatroids are in several ways 'opposite' to matroids. For example, while matroids are precisely the greedoids satisfying the interval property without lower bounds, antimatroids are precisely those characterized by the interval property without upper bounds. Also, whereas the matroid closure operator is characterized by the MacLane exchange axiom, the closure operator of antimatroids can be characterized by the opposite 'anti-exchange' axiom (see section 8.7.A).

We shall now describe several classes of antimatroids occurring 'in nature'. They are all easy to identify using the following criterion (Proposition 8.2.7): a set system $\mathscr{F} \subseteq 2^E$ is an antimatroid exactly if it is accessible and closed under union.

(1) Let $P = (E, \leqq)$ be a finite partially ordered set and $\mathscr{F}$ the set of ideals of E (a subset $A \subseteq E$ is an *ideal* if $x \leqq y \in A$ implies $x \in A$). Then $(E, \mathscr{F})$ is an antimatroid, the *poset greedoid* of P. In this case $\mathscr{F}$ is closed both under union and intersection and hence $(\mathscr{F}, \subseteq)$ forms a distributive lattice. Conversely, by a theorem of G. Birkhoff, every finite distributive lattice occurs this way. The basic words of the poset greedoid are the linear extensions of P.

(2) Let $\Gamma = (V, E, r)$ be a finite rooted graph, and let $V' = V - r$ be the set of vertices distinct from the root r. Then the *vertex search greedoid* of Γ is $(V', \mathscr{F})$, where $\mathscr{F}$ is given by $\mathscr{F} = \{X \subseteq V' \colon X \cup r$ is the vertex set of a connected subgraph of $G\}$. If Γ is connected, the basic words of this antimatroid correspond to the orderings in which nodes are visited by the standard search procedures starting at r.

(3) In the case of a rooted digraph $\Delta = (V, E, r)$, we again let V' be the set of vertices distinct from the root r, and $\mathscr{F} = \{X \subseteq V' \colon X \cup r$ is the vertex set of a tree in Δ that is directed away from $r\}$. Then $(E, \mathscr{F})$ is the *vertex search greedoid* of the digraph Δ.

(4) Let E be the vertex set of a tree and $\mathscr{F}$ the collection of complements of subtrees. Again, $(E, \mathscr{F})$ is an antimatroid, the *vertex pruning greedoid* of the tree. The same construction can be repeated for the edge set of a tree, to get the *edge pruning greedoid* of the tree, also an antimatroid.

(5) Both the vertex pruning and the edge pruning greedoids of trees are special cases of the *simplicial vertex pruning greedoids* of graphs. A vertex of a graph (V, E) is *simplicial* if all its neighbors are pairwise adjacent. Successive removal of simplicial vertices gives a hereditary language $(V, \mathscr{L})$ that is easily seen to be an antimatroid.

For $\mathscr{L}$ to be non-trivial a sufficient supply of simplicial vertices in (V, E) and its subgraphs is needed. This is guaranteed if the graph is *chordal*, meaning that no induced subgraph on k vertices is a k-cycle, for $k \geqslant 4$. Chordal graphs are characterized by the property that every induced subgraph has a simplicial vertex. It follows that the simplicial vertex pruning greedoid of a graph is full if and only if the graph is chordal.

(6) Our final example is crucial for the geometric interpretation of antimatroids. Let E be a finite subset of $\mathbb{R}^n$, and for $A \subseteq E$ define $\bar{A}$ to be the convex hull of A intersected with E. We call $A \subseteq E$ *convex* if $A = \bar{A}$, and define $\mathscr{F}$ to be the family $\mathscr{F} = \{X \subseteq E \mid E - X$ is convex$\}$. Then $(E, \mathscr{F})$ is an antimatroid, the *convex pruning greedoid* on E. This

example in fact generalizes in a straightforward way to oriented matroids with an appropriate notion of convexity.

Antimatroids have a lot of additional structure, which makes them quite special among greedoids. We will study antimatroids in greater detail in section 8.7, and proceed here to describe more general classes of greedoids.

8.3.C. Branching Greedoids

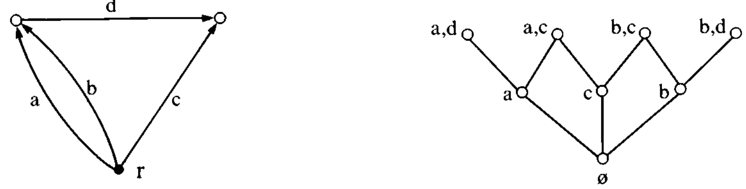
Let $\Delta = (V, E, r)$ be a finite rooted directed graph. Let $\mathscr{F}$ be the collection of edge sets of trees in Δ that contain the root and are directed away from it (such trees are called *branchings* or *arborescences*). Then $(E, \mathscr{F})$ is a greedoid, the *directed branching greedoid* (or *line search greedoid*) on Δ. Every non-empty tree in $\mathscr{F}$ has a leaf, which can be removed to get another tree in $\mathscr{F}$. This verifies axiom (G1). To check (G2), one observes that for two trees X and Y in Δ, $|X| > |Y|$ implies that X reaches a vertex v that Y does not reach ($|X|$ is the number of vertices of $V - r$ reached by X). Now the first arc along the path in X from r to v that reaches a vertex not reached by Y can be added to Y.

Figure 8.3 illustrates a particular rooted digraph Δ and the associated branching greedoid $\mathscr{F}$. This greedoid is of rank 2 with bases $\{a, d\}$, $\{a, c\}$, $\{b, c\}$, and $\{b, d\}$. The language is $\mathscr{L} = \{\emptyset, a, b, c, ac, ad, bc, bd, ca, cb\}$.

The rooted digraph in Figure 8.8a on p. 315 gives a branching greedoid of rank 6. That greedoid has two loops and two coloops, which shows that these greedoid concepts do not in this case have their standard graph-theoretic meaning.

It is clear, as was also observed with the example of Figure 8.2, that an analogous construction works for every finite rooted *undirected* graph $\Gamma = (V, E, r)$. The construction then yields the *undirected branching greedoid* $(E, \mathscr{F})$, where $\mathscr{F}$ is the set of trees in Γ that contain the root. We note that – ignoring the root – the graph Γ also gives rise to the graphic matroid. $(E, \mathscr{I})$. If Γ is connected, then the bases of the branching greedoid and of the graphic matroid are the same, namely the spanning trees of Γ. Hence,

Figure 8.3.



(a) (b)

$(E, \mathscr{F})$ is a slimming of $(E, \mathscr{I})$, or equivalently, the hereditary closure of $\mathscr{F}$ is the graphic matroid.

The common algorithmic search procedures on a rooted graph (directed or not) visit the nodes one at a time so that the currently visited node (originally just r) is at each stage reached along some edge from a previously visited node. It is clear that the basic words of the associated branching greedoid record the sequences of *edges* generated by such search procedures. Similarly, the associated vertex search greedoid (defined in section 8.3.B) records the possible orders in which the *nodes* are reached.

Both directed and undirected branching greedoids are interval greedoids. If an edge x is a legal continuation at any given stage, that means that x leads from a visited node v to an unvisited node u. But then clearly x will remain a legal choice at a later stage if and only if u remains unvisited. This verifies the interval property.

For many purposes branching greedoids can serve as 'canonical examples' of greedoids. Being easy to represent graphically they play a role similar to the role graphic matroids play in matroid theory. However, branching greedoids are relatively well behaved and do not exhibit all the pathologies that can occur. For example, the intervals in the poset $(\mathscr{F}, \subseteq)$ are distributive lattices – given any two branchings $X \subseteq Y$ in Δ, the interval $[X, Y]$ of $\mathscr{F}$ corresponds to the order ideals of $Y \backslash X$, ordered by 'precedence along the paths in Y emanating from the root'. Greedoids with the property that all the intervals in $\mathscr{F}$ are distributive are called *local poset greedoids*. This name comes from the fact that $\mathscr{F}$ is a local poset greedoid if and only if the restriction of $\mathscr{F}$ to any feasible set is a poset greedoid ('restriction' will be defined in section 8.4.D as a straightforward generalization of the matroid operation). From this it is easy to see that all local poset greedoids are interval greedoids.

The following class of greedoids is closely related to the branching greedoids. Let (V, E, r) be a rooted undirected graph, and let $\mathscr{A}$ be the collection of edge sets of all connected subgraphs covering r. Then $(E, \mathscr{A})$ is a greedoid, in fact an antimatroid. An analogous construction associates an antimatroid with every rooted directed graph.

The construction of these greedoids is part of a much more general procedure: if $(E, \mathscr{F})$ is a greedoid and $\mathscr{A}$ is the collection of unions of feasible sets from $\mathscr{F}$, then $(E, \mathscr{A})$ is an antimatroid. The elements of $\mathscr{A}$ are called the *partial alphabets* of $(E, \mathscr{F})$. See Exercise 8.5.

8.3.D. Polymatroid Greedoids

A pair (E, f), consisting of a finite ground set and a function $f\colon 2^E \to \mathbb{N}$, is called a *polymatroid* if for all $X, Y \subseteq E$:

(PM1) $f(\emptyset)=0$;
(PM2) $X\subseteq Y$ implies $f(X)\leqq f(Y)$;
(PM3) $f(X\cap Y)+f(X\cup Y)\leqq f(X)+f(Y)$.

Polymatroids are generalizations of matroids: f is the rank function of a matroid if in addition $f(X)\leqq |X|$ for all $X\subseteq E$.

Polymatroids give rise to greedoids in the following way. Suppose that (E,f) is a polymatroid, and let

$$\mathscr{L}=\{x_1x_2\ldots x_k: f(\{x_1,\ldots,x_i\})=i \text{ for } 1\leqq i\leqq k\}.$$

Then $(E, \mathscr{L})$ is a greedoid, called a *polymatroid greedoid*. Such greedoids are local poset greedoids (as defined in section 8.3.C), and hence they have the interval property.

We give three examples of polymatroid greedoids that have been discussed before:

(1) If (E,f) is a matroid, then the polymatroid greedoid is the greedoid usually associated with this matroid (cf. section 8.3.A). In fact, the above construction will reconstruct any greedoid $\mathscr{L}$ from its rank function f.
(2) Let $\Gamma=(V, E, r)$ be a rooted undirected graph. For $X\subseteq E$, let $f(X)$ be the number of vertices in $V\backslash r$ covered by X. Then it is easy to check that (E,f) is a polymatroid. The associated polymatroid greedoid is the undirected branching greedoid of Γ, since if $\{x_1,\ldots,x_{i-1}\}$ is a tree in Γ containing the root r, then the same is true for $\{x_1,\ldots,x_i\}$ if and only if x_i covers exactly one additional vertex.
   In contrast, directed branching greedoids are not in general polymatroid greedoids.
(3) Poset greedoids are polymatroid greedoids. The corresponding rank function measures the size $f(X)$ of the ideal in P generated by a subset X of $P=(E, \leqq)$.

8.3.E. Faigle Geometries

For this class of greedoids the ground set is assumed to be partially ordered in a way that is suitably compatible with the greedoid structure. Both matroids and poset greedoids belong to this class.

A *Faigle geometry* is a triple $(E, \mathscr{L}, \leqq)$ where $(E, \mathscr{L})$ is a greedoid language and $(E, \leqq)$ a poset such that:

(F1) for $x_1x_2\ldots x_k\in\mathscr{L}$, $x_i\leqq x_j$ implies $i\leqslant j$ (that is, the ordering of every word in $\mathscr{L}$ is compatible with the partial order on E);
(F2) if A, B are ideals in $(E, \leqq)$ with $A\subseteq B$, then every $p\in A$ that occurs in every maximum length word in $B^*\cap\mathscr{L}$ also occurs in every maximum length word in $A^*\cap\mathscr{L}$.

We note that when the poset $(E, \leqq)$ is an antichain (meaning that $x \leqq y$ implies $x = y$), the axiom (F1) is vacuously fulfilled, and (F2) implies that $(E, \mathscr{L})$ is a matroid. In general, if $(E, \mathscr{L}, \leqq)$ is a Faigle geometry then $(E, \mathscr{L})$ is an interval greedoid. However, not every interval greedoid admits the structure of a Faigle geometry (e.g. some branching greedoids do not).

There is a second rank function f on a Faigle geometry $(E, \mathscr{L}, \leqq)$, which is in general different from the greedoid rank function r of $(E, \mathscr{L})$. This *ideal rank* function f: $2^E \rightarrow \mathbb{N}$ is defined by

$$f(X) = r(I(X)), \text{ for } X \subseteq E,$$

where $I(X)$ denotes the ideal generated by X, i.e. $I(X) = \{y \in E\colon y \leqq x$ for some $x \in X\}$.

The ideal rank f of a Faigle geometry is a polymatroid rank function, i.e. it satisfies axioms (PM1)–(PM3) of section 8.3.D. The corresponding polymatroid greedoid is also a Faigle geometry $(E, \mathscr{L}', \leqq)$ over the same poset and with the same ideal rank function, and $\mathscr{L} \subseteq \mathscr{L}'$. However, in general $\mathscr{L} \neq \mathscr{L}'$, which shows that a Faigle geometry is not uniquely determined by its ideal rank function.

8.3.F. Retract Greedoids

A *retract* of a poset $(E, \leqq)$ is a subposet $Q \subseteq E$ such that there is an order-preserving map r: $E \rightarrow Q$ with $r(x) = x$ for all $x \in Q$. In this case r is called a *retraction* of E to Q. We are going to consider retracts such that $|Q| = |E| - 1$, that is, Q corresponds to the deletion of a single element x from E. There are two cases; either x and $r(x)$ are not comparable, or one of x and $r(x)$ covers the other. In the second case Q is the poset obtained from E by deleting a meet or join irreducible element x (i.e. an element with a unique cover or a unique cocover). We will call this a *monotone* retract.

This situation gives rise to the following two greedoids: the *retract greedoid* $(E, \mathscr{L})$ given by

$$\mathscr{L} = \{x_1 x_2 \ldots x_k\colon \text{for } 1 \leqq i \leqq k,\ E - \{x_1, \ldots, x_i\} \text{ is a retract of } E - \{x_1, \ldots, x_{i-1}\}\}$$

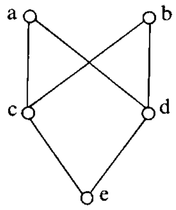
and the *dismantling greedoid* $(E, \mathscr{L}')$ defined by

$$\mathscr{L}' = \{x_1 x_2 \ldots x_k\colon \text{for } 1 \leqq i \leqq k,\ E - \{x_1, \ldots, x_i\} \text{ is a monotone retract of } E - \{x_1, \ldots, x_{i-1}\}\}.$$

For instance, consider the poset shown in Figure 8.4. Here, $b \in \mathscr{L}$, $ab \notin \mathscr{L}$, and $acb \in \mathscr{L}$. Also, $cd \in \mathscr{L}'$, $ced \notin \mathscr{L}'$, and $cead \in \mathscr{L}'$.

This example shows that the greedoids $(E, \mathscr{L})$ and $(E, \mathscr{L}')$ in general fail to have the interval property. Also, they may have many loops. Observe that by definition $\mathscr{L}' \subseteq \mathscr{L}$.

Figure 8.4.



There is a straightforward generalization of retract sequences and dismantling sequences from posets (i.e. comparability graphs) to finite digraphs, which leads to a generalization of the corresponding greedoids. In this case, one works with digraphs with a loop at every vertex and defines retracts as above, using graph maps instead of order preserving maps. For monotone retracts, the additional requirement is that every vertex is mapped to an adjacent vertex.

The following construction gives a very general framework for retract greedoids. Let Φ be a set of mappings of a finite set E into itself, which is closed under composition and contains the identity (in other words, Φ is a submonoid of E^E). Call a subset $X \subseteq E$ a *retract* if $X = \phi(E)$ for some idempotent element $\phi \in \Phi$. Now, define a left hereditary language by

$$\mathscr{L} = \{x_1 \, x_2 \ldots x_k \colon \text{for } 1 \leqq i \leqq k,\ E - \{x_1, \ldots, x_k\} \text{ is a retract}\}.$$

Then $(E, \mathscr{L})$ is greedoid. Taking Φ to be the monoid of order preserving self-maps of a poset $(E, \leqq)$ we get the special retract greedoids that were originally defined.

A general formulation of dismantling greedoids along similar lines is also possible.

8.3.G. Transposition Greedoids

A greedoid $(E, \mathscr{F})$ is said to have the *transposition property* if it satisfies the axiom

(TP) If A, $A \cup x$, $A \cup y \in \mathscr{F}$ and $A \cup x \cup y \notin \mathscr{F}$, then $A \cup x \cup B \in \mathscr{F}$ implies $A \cup y \cup B \in \mathscr{F}$, for all $B \subseteq E - (A \cup x \cup y)$.

If an accessible set system has the transposition property then it is a greedoid, but the converse does not hold. Hence, the axioms (G1) and (TP) form an axiom system for a proper subclass that we call *transposition greedoids*.

The best known way to prove that a retract greedoid actually *is* a greedoid is to verify (G1) and (TP). In fact, both retract greedoids and dismantling greedoids are transposition greedoids. Examples of greedoids that lack the transposition property can be found among the twisted matroids.

Let us now verify (TP) for an arbitrary interval greedoid $(E, \mathscr{F})$. In the given situation $A \cup y$ can be augmented from the larger set $A \cup x \cup B$ to a set $A \cup y \cup B' \in \mathscr{F}$, where $B' \subseteq B \cup x$ and $|B'| = |B|$. Because of the interval property, $x \notin B'$, i.e. $B' = B$. So, all interval greedoids have the transposition property.

For later use we record the following trivial strengthening of the preceding paragraph: property (TP) with the last words replaced by 'for all $B \subseteq E - A$', is satisfied by every interval greedoid. Note that this stronger formulation is not possible for transposition greedoids in general.

8.3.H. Gaussian Greedoids

The Gaussian algorithm for solving systems of equations gives rise to greedoids in the following way. Let $M = (m_{ij})$ be an $m \times n$ matrix over an arbitrary field. Perform Gaussian elimination working downward row by row from the top and keep track of the column indices of the pivot elements. Each possible such procedure gives rise to a sequence of column indices (for row 1, row 2, and so on), and these sequences are the basic words of the *Gaussian elimination greedoid* $(E, \mathscr{F})$ of M, $E = \{1, 2, ..., n\}$. Equivalently, this greedoid can be defined directly by

$$\mathscr{F} = \{A \subseteq E\text{: the submatrix } M_{\{1, 2, ..., |A|\}, A} \text{ is non-singular}\}.$$

Gaussian elimination greedoids are not in general transposition greedoids, as may be checked for the matrix

$$N = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

whose associated greedoid $(\{1, 2, 3, 4\}, \mathscr{F})$ has $\{1\}, \{2\}, \{1, 3, 4\} \in \mathscr{F}$, but $\{1, 2\}, \{2, 3, 4\} \notin \mathscr{F}$.

We shall now discuss a special case of Gaussian elimination greedoids and then generalize their construction. Suppose that $\Gamma = (V, U, E)$ is a bipartite graph, $E \subseteq V \times U$, and fix an ordering $u_1, u_2, ..., u_n$ of the elements of the color class U. Now, let

$$\mathscr{F} = \{A \subseteq V\text{: } A \text{ can be matched to } \{u_1, u_2, ..., u_{|A|}\} \text{ in } \Gamma\}.$$

Then $(V, \mathscr{F})$ is a greedoid, called a *medieval marriage greedoid.*

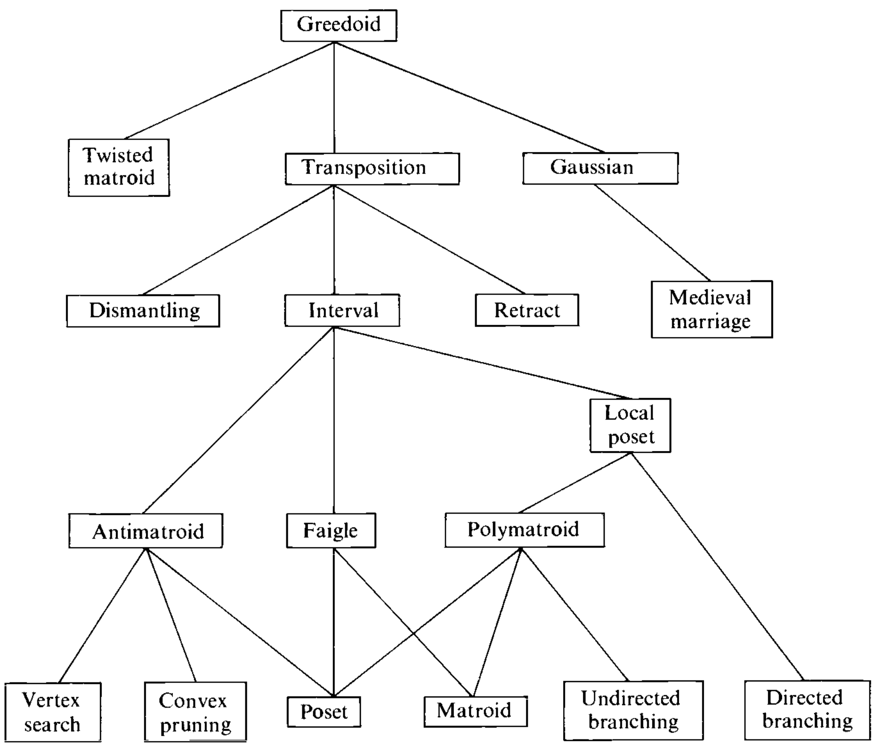
Every medieval marriage greedoid is a Gaussian elimination greedoid over a suitable field. To see this, take the $U \times V$ incidence matrix and replace the elements that are unity, if necessary, by algebraically independent field elements. The matrix N given above is the incidence matrix of a bipartite graph, which shows that in general medieval marriage greedoids also lack the transposition property.

A more general class of greedoids is obtained as follows. Suppose that $M_i = (E, \mathscr{I}_i)$, $i = 0, 1, \ldots, m$, is a sequence of matroids on the same ground set E such that (1) if $A \subseteq E$ is closed in M_{i-1} then it is closed in M_i, for $1 \leqslant i \leqslant m$, and (2) rank $M_i = i$, for $0 \leqslant i \leqslant m$. A greedoid $(E, \mathscr{F})$ of rank m is then defined by $\mathscr{F} = \{A \subseteq E\colon A$ is a basis of $M_{|A|}\}$. Greedoids of this kind are called *Gaussian*. Clearly Gaussian elimination greedoids are special cases of Gaussian greedoids (take for M_i the column matroid determined by the first i rows of the given matrix). Notice that in a Gaussian greedoid the matroids M_i can be uniquely recovered from $\mathscr{F}$, namely, M_i is the hereditary closure of the feasible sets of cardinality i. Also, every matroid M is a Gaussian greedoid (for M_i take the rank i truncation of M).

8.3.I. Relations Among Classes of Greedoids

We have discussed several cases where one class of greedoids is seen to be a generalization or a specialization of another class. For an overview, these containment relations between classes of greedoids are gathered in the form of a poset diagram in Figure 8.5. (The containment of the class of matroids

Figure 8.5.



in the classes of twisted matroids and Gaussian greedoids is not indicated in the diagram.)

8.4. Structural Properties

In this section we will discuss the greedoid rank function and the closure operator to which it gives rise. Just like matroids, greedoids have cryptomorphic definitions in terms of rank and closure. Also, various elementary constructions on greedoids will be defined.

8.4.A. Rank Function

Recall that the rank function of a greedoid $(E, \mathcal{F})$ is defined by $r(A) = \max\{|X| : X \subseteq A,\ X \in \mathcal{F}\}$, for $A \subseteq E$. Clearly, $\mathcal{F} = \{A \subseteq E\colon\ r(A) = |A|\}$, which means that the greedoid $\mathcal{F}$ is completely determined by its rank function. This proves the last sentence of the following result.

8.4.1. Theorem. *A function $r\colon 2^E \to \mathbb{N}$ is the rank function of a greedoid if and only if for all $A, B \subseteq E$ and $x, y \in E$:*

(R1) $r(A) \leqq |A|$;
(R2) $A \subseteq B$ *implies* $r(A) \leqq r(B)$;
(R3) $r(A) = r(A \cup x) = r(A \cup y)$ *implies* $r(A) = r(A \cup \{x, y\})$.

Furthermore, the greedoid with rank function r is then uniquely determined.

Let us check the necessity of these axioms. (R1) states that r is *subcardinal*. This is clear from the definition of r, and in particular implies $r(\emptyset) = 0$. (R2) states that r is *monotone*; this is equally clear by definition. (R3) essentially codes the exchange axiom (G2) and is easily proved from it.

A greedoid rank function on E is the rank function of a matroid if additionally it satisfies the *unit increase property*: $r(A \cup x) \leqq r(A) + 1$ for every $A \subseteq E$ and $x \in E$. Together with $r(\emptyset) = 0$ this implies (R1), and together with the other axioms it is sufficient to prove *submodularity* of the matroid rank function:

$$r(A \cap B) + r(A \cup B) \leqq r(A) + r(B), \text{ for all } A, B \subseteq E.$$

For an example of the failure of the unit increase property and of submodularity on a general greedoid, consider the undirected branching greedoid $(E, \mathcal{F})$ of a rooted graph Γ. Let $X \in \mathcal{F}$ be a large tree in Γ that contains exactly one edge x adjacent to the root. Then we have $r(X)$ large, but $r(\{x\}) = 1$ and $r(X - x) = 0$.

8.4.B. Closure Operator

Using the rank function, we define the *(rank) closure operator* $\sigma : 2^E \rightarrow 2^E$ of a greedoid $(E, \mathscr{F})$ by

$$\sigma(A) = \{x \in E: r(A \cup x) = r(A)\}.$$

Clearly, σ is *increasing*: $A \subseteq \sigma(A)$. Furthermore, for all $A \subseteq E$,

$$r(A) = r(\sigma(A)). \tag{8.1}$$

To see this, suppose $r(A) < r(\sigma(A))$ and let X and Y be bases of A and $\sigma(A)$ respectively. Then X can be augmented by some $y \in Y$ so that $X \cup y$ is feasible and of cardinality $r(A)+1$. But then $r(A \cup y) \geqq r(X \cup y) = r(A)+1$, which means that $y \notin \sigma(A)$, contradicting $y \in Y \subseteq \sigma(A)$.

As a consequence of the preceding we find that σ is *idempotent*:

$$\sigma\sigma(A) = \sigma(A), \tag{8.2}$$

for all $A \subseteq E$. Namely, if $x \in \sigma\sigma(A)$, then $r(\sigma(A)) = r(A) \leqq r(A \cup x) \leqq r(\sigma(A) \cup x) = r(\sigma(A))$, where the first equality is by (8.1) and the last by the definition of the closure of $\sigma(A)$. So, $x \in \sigma\sigma(A)$ implies $x \in \sigma(A)$.

A serious shortcoming of greedoid closure is that it is *not necessarily monotone*, i.e. $A \subseteq B$ does not in general imply $\sigma(A) \subseteq \sigma(B)$. For an easy counterexample, take the full greedoid with exactly one basic word xy. In this greedoid, which is both a poset greedoid and a branching greedoid, $\sigma(\emptyset) = \{y\}$, $\sigma(\{x\}) = \{x\}$. The *closed sets* (i.e. $A = \sigma(A)$) of this greedoid are $\{x\}$, $\{y\}$, and $\{x, y\}$.

The failure of monotonicity means that greedoid closure is not a closure operator in the usual sense (i.e. as defined in section 8.7.A). Many of the characteristic properties of ordinary closure are absent; for instance the intersection of two closed sets is not always closed, and the closed sets ordered by inclusion do not form a lattice. It can be shown that greedoid closure σ is monotone only in the matroid case.

In spite of what has just been said, it turns out that greedoid closures can be axiomatically characterized in a way that is reminiscent of matroid closure.

8.4.2. Theorem. *A function $\sigma : 2^E \rightarrow 2^E$ is the rank closure operator of a greedoid if and only if for all A, $B \subseteq E$ and x, $y \in E$ the following conditions hold.*

(RC1) $A \subseteq \sigma(A)$.

(RC2) $A \subseteq B \subseteq \sigma(A)$ *implies* $\sigma(B) = \sigma(A)$.

(RC3) *Suppose* $x \notin A$, $z \notin \sigma(A \cup x - z)$ *for all* $z \in A \cup x$. *Then* $x \in \sigma(A \cup y)$ *implies* $y \in \sigma(A \cup x)$.

Furthermore, the greedoid with closure σ is uniquely determined.

Suppose that $(E, \mathscr{F})$ is a greedoid with closure operator σ. Now, $\mathscr{F} = \{A \subseteq E: x \notin \sigma(A - x) \text{ for all } x \in A\}$, so the greedoid can be uniquely reconstructed from σ. Axiom (RC2), a weak form of monotonicity, is easy to verify for σ using property (8.1). Axiom (RC3) states that if $x \notin A$ and $A \cup x \in \mathscr{F}$, then $x \in \sigma(A \cup y)$ implies $y \in \sigma(A \cup x)$. This generalizes the MacLane exchange axiom for matroid closures, where the requirement $A \cup x \in \mathscr{F}$ is replaced by $x \notin \sigma(A)$. The MacLane exchange axiom is in general not satisfied by greedoid closure.

It turns out to be convenient to consider also the *monotone closure operator* μ: $2^E \to 2^E$ obtained from σ by the following construction:

$$\mu(A) = \cap\{\sigma(X): A \subseteq \sigma(X),\ X \subseteq E\}.$$

It is easy to check that $A \subseteq \mu(A) \subseteq \sigma(A)$ and $\mu\mu(A) = \mu(A)$ for all $A \subseteq E$, and that $A \subseteq B$ implies $\mu(A) \subseteq \mu(B)$. So, μ is a closure operator in the usual sense. However, μ does not determine $\mathscr{F}$, since $\mu = \mathrm{id}$ for all full greedoids.

8.4.C. Rank and Closure Feasibility

We have seen that greedoid rank and closure lack some of the good behaviour of their matroid counterparts. However, loosely speaking, they behave better on certain subsets of the greedoid than on others. This motivates the introduction of two special feasibility concepts, which are trivial for matroids, but give useful structural information about greedoids.

Let $G = (E, \mathscr{F})$ be a greedoid, and define the *basis rank* of $A \subseteq E$ by

$$\beta(A) = \max\{|A \cap X| : X \in \mathscr{F}\}.$$

Equivalently, $\beta(A)$ is the maximal size of the intersection of A with a basis. It is clear that $\beta(A) \geqq r(A)$, for all $A \subseteq E$. For matroids the equality always holds, but this is not true for greedoids. For suppose that $A \subseteq B$, $A \notin \mathscr{F}$, $B \in \mathscr{F}$; then $r(A) < |A| = \beta(A)$, since $A = A \cap B$. Hence, $\beta(A) = r(A)$ for all subsets A if and only if G is a matroid.

A set $A \subseteq E$ is called *rank feasible* if $\beta(A) = r(A)$, that is, if $|A \cap X| \leqq r(A)$ for all $X \in \mathscr{F}$. The collection of all rank feasible subsets is denoted by $\mathscr{R}$ or $\mathscr{R}_G$. It is clear that $\mathscr{F} \subseteq \mathscr{R}$, and that for a full greedoid $\mathscr{F} = \mathscr{R}$ (because then $\beta(A) = |A|$). Also, we have seen that $\mathscr{R}_G = 2^E$ if and only if G is a matroid. The collection $\mathscr{R}_G$ is an interesting set system associated with the greedoid G. Many properties valid for $\mathscr{F}$ generalize to $\mathscr{R}$.

Here is a characterization of rank feasible sets.

8.4.3. Proposition. *Let* $(E, \mathscr{F})$ *be a greedoid,* $A \subseteq E$. *Then the following conditions are equivalent*:

(i) *A is rank feasible*;
(ii) $r(A \cup X) \leqq r(A) + |X|$, *for all* $X \subseteq E - A$;
(iii) $X \subseteq A \subseteq \mu(X)$, *for some* $X \in \mathscr{F}$.

Property (ii) shows that r has the unit-increase property on rank feasible sets. Also, it allows us to conclude that for $A, B \subseteq E$:

$$\beta(A \cup B) + r(A \cap B) \leqq \beta(A) + \beta(B),$$

which implies that on rank feasible sets, $r = \beta$ is semimodular. Property (iii) easily implies that $\mathscr{R}$ is an accessible set system. However, in general $(E, \mathscr{R})$ is not itself a greedoid.

In some situations it is useful to consider a stronger property than rank feasibility: $A \subseteq E$ is said to be *closure feasible* if $A \subseteq \sigma(X)$ implies $A \subseteq \mu(X)$ for all subsets $X \subseteq E$. Equivalently, A is closure feasible if $A \subseteq \sigma(X)$ and $X \subseteq Y \subseteq E$ imply $A \subseteq \sigma(Y)$. We denote the collection of closure feasible sets by $\mathscr{C}$ or $\mathscr{C}_G$. Here are some basic properties.

8.4.4. Proposition. *Every closure feasible set is rank feasible:* $\mathscr{C} \subseteq \mathscr{R}$. *Furthermore, the set system* $\mathscr{C}$ *is closed under union.*

8.4.5. Proposition. *The following conditions are equivalent:*

(i) $(E, \mathscr{F})$ *is an interval greedoid;*
(ii) $\mathscr{F} \subseteq \mathscr{C}$;
(iii) $\mathscr{C} = \mathscr{R}$.

Since $\mathscr{R}$ is an accessible set system, it follows from Propositions 8.2.7, 8.4.4, and 8.4.5 that $(E, \mathscr{C}) = (E, \mathscr{R})$ is an antimatroid if $(E, \mathscr{F})$ has the interval property.

8.4.D. Constructions

The basic matroid construction of deletion, contraction, truncation, and direct sum generalize to greedoids in the following way.

Let $G = (E, \mathscr{F})$ be a greedoid and $A \subseteq E$. Define

$$\mathscr{F} \backslash A = \{X \subseteq E - A\colon X \in \mathscr{F}\}, \tag{8.3}$$

and, if A is feasible,

$$\mathscr{F}/A = \{X \subseteq E - A\colon X \cup A \in \mathscr{F}\}. \tag{8.4}$$

It is not hard to check that the set systems obtained are in both cases greedoids on the ground set $E' = E - A$. $G \backslash A = (E', \mathscr{F} \backslash A)$ is said to be obtained from G by *deletion* of A, or by *restriction* to $E - A$, and $G/A = (E', \mathscr{F}/A)$ by *contraction* of A. Also, by a *minor* of $(E, \mathscr{F})$ we shall mean any restriction of a contraction, i.e. any greedoid of the form $(E - (A \cup A'), (\mathscr{F}/A)\backslash A')$, where $A \in \mathscr{F}$ and $A' \subseteq E - A$.

Observe in this connection that restriction and contraction commute:

$$(\mathscr{F}/A)\backslash A' = (\mathscr{F} \backslash A')/A = \{X \subseteq E - (A \cup A')\colon X \cup A \in \mathscr{F}\},$$

for $A \cap A' = \varnothing$, $A \in \mathscr{F}$ and $A' \subseteq E$.

Thus minors can equivalently be defined as contractions of restrictions. Minors of minors of a greedoid are again minors of the greedoid, and hence '*being a minor of*' defines a partial order on the set of isomorphism types of greedoids. For more about this, see section 8.9.A.

Let r, r' and r'' denote the rank functions of G, $G\backslash A$, and G/A, respectively. Then for all $X \subseteq E - A$:

$$r'(X) = r(X) \quad \text{and} \quad r''(X) = r(X \cup A) - r(A). \tag{8.5}$$

Definition (8.4) produces a greedoid only if A is feasible, since otherwise $\emptyset \notin \mathscr{F}/A$. It is possible to extend the definition of contraction $\mathscr{F}/A$ in the following cases: (i) $\mathscr{F}$ is an arbitrary greedoid and A is rank feasible, or (ii) $\mathscr{F}$ is an interval greedoid and A is an arbitrary subset. (However, the contraction $\mathscr{F}/A$ is *not* a meaningful concept for general $\mathscr{F}$ and A.) In case (i) one extends the rank formula (8.5), while in case (ii) one picks a basis B of A, contracts by B, and then deletes $A - B$. It is instructive to try to visualize the second case in terms of branching greedoids.

It is easy to formulate the ordered versions of deletion and contraction. For instance, the *contraction* of a greedoid language $(E, \mathscr{L})$ by the feasible word α is defined by

$$\mathscr{L}/\alpha = \{\beta \in E_s^* : \alpha\beta \in \mathscr{L}\}, \tag{8.6}$$

which is clearly again a greedoid language.

If $(E, \mathscr{F})$ is a greedoid of rank r and $0 \leqq k \leqq r$, then the *k-truncation*

$$\mathscr{F}^{(k)} = \{X \in \mathscr{F} : |X| \leqq k\} \tag{8.7}$$

is a greedoid as well.

There are two ways to define the sum of greedoids. Let $G_1 = (E_1, \mathscr{F}_1)$ and $G_2 = (E_2, \mathscr{F}_2)$ be two greedoids on disjoint ground sets. Then their *direct sum* is the greedoid $G_1 \oplus G_2 = (E_1 \cup E_2, \mathscr{F}_1 \oplus \mathscr{F}_2)$, where

$$\mathscr{F}_1 \oplus \mathscr{F}_2 = \{X_1 \cup X_2 \colon X_1 \in \mathscr{F}_1 \text{ and } X_2 \in \mathscr{F}_2\}. \tag{8.8}$$

Their *ordered sum* is the greedoid $G_1 \otimes G_2 = (E_1 \cup E_2, \mathscr{F}_1 \otimes \mathscr{F}_2)$, where

$$\mathscr{F}_1 \otimes \mathscr{F}_2 = \mathscr{F}_1 \cup \{B \cup X \colon B \text{ is a basis of } \mathscr{F}_1,\ X \in \mathscr{F}_2\}. \tag{8.9}$$

Clearly, $G_1 \oplus G_2$ and $G_1 \otimes G_2$ have the same family of bases. Also, the language of feasible words of $G_1 \oplus G_2$ is the shuffle product of the two component languages.

All constructions discussed in this section take interval greedoids into interval greedoids, and similarly for local poset greedoids. The same is true for antimatroids, except that the k-truncation of an antimatroid is in general only an interval greedoid. Conversely, the restriction of an interval greedoid to any feasible set is an antimatroid (cf. Proposition 8.2.8). A special operation for antimatroids, called *trace*, will be defined in section 8.7.C.

Unfortunately, the duality operation of matroid theory has no counterpart for greedoids. Only a weak notion of duality operation exists for general greedoids; see section 8.6.C.

8.4.E. Connectivity

The concept of the connectivity of greedoids is modeled to generalize the graph-theoretic connectivity of rooted graphs in the case of branching greedoids. In this context a rooted digraph $\Delta = (V, E, r)$ is called *connected* (or *1-connected*) if there is a directed path from the root to every vertex. More generally, Δ is *k-connected* if every vertex $v \in V$ can be reached from the root by a directed path after removal of at most $k-1$ vertices in $V - \{r, v\}$. Equivalently, by Menger's theorem, Δ is k-connected if there are k vertex-disjoint directed paths from the root to every vertex $v \in V - r$ such that (r, v) is not an arc in E. Similar definitions apply to rooted undirected graphs.

The digraph Δ is connected if and only if the associated vertex search greedoid is full. In contrast, the connectedness of Δ is not encoded in the directed branching greedoid of Δ.

The case of higher connectivity ($k > 1$) suggests the following definition.

8.4.6. Definition. Let $(E, \mathscr{F})$ be a greedoid of rank r, and let $X \in \mathscr{F}$. A set $A \subseteq E - X$ is called *free over* X if for every $B \subseteq A$, $X \cup B$ is feasible. The greedoid $(E, \mathscr{F})$ is called *k-connected* ($1 \leqq k \leqq r$) if for every $X \in \mathscr{F}$ there is a free set A over X of size $\min\{k, r - r(X)\}$. Equivalently, $(E, \mathscr{F})$ is k-connected if for every $X \in \mathscr{F}$ there is a $Y \in \mathscr{F}$ such that $X \subseteq Y$, $|Y - X| = \min\{k, r - r(X)\}$ and the interval $[X, Y]$ of the poset $(\mathscr{F}, \subseteq)$ is Boolean.

Obviously, every greedoid is 1-connected, and every k-connected greedoid is also $(k-1)$-connected for $k \geqq 2$. Matroids are r-connected (i.e. maximally connected), but this does not fully characterize matroids.

If A is free over $X \in \mathscr{F}$, then it is contained in the set

$$\Gamma(X) = E \backslash \sigma(X) = \{a \in E - X \colon X \cup a \in \mathscr{F}\}$$

of *continuations* of X. In antimatroids, we know (e.g. from Lemma 8.7.9) that the free sets over X are exactly the subsets of $\Gamma(X)$. Thus an antimatroid is k-connected if and only if for all $X \in \mathscr{F}$,

$$|\Gamma(X)| \geqq \min\{k, r - r(X)\}.$$

The following result shows that (for $k > 1$) Definition 8.4.6 describes a reasonable generalization of graph connectivity.

8.4.7. Proposition. *Let $\Delta = (V, E, r)$ be a connected rooted digraph. Then the following are equivalent for $k > 1$:*

(i) Δ *is k-connected;*
(ii) *the branching greedoid on* Δ *is k-connected;*
(iii) *the vertex search greedoid on* Δ *is k-connected.*

Proof. Since (ii)$\Leftrightarrow$(iii) is a special case of Proposition 8.8.9, we will only demonstrate (i)$\Leftrightarrow$(iii).

If the digraph Δ is not k-connected, then there is a cut set $A \subseteq V' = V - r$ of size $|A| < k$ that separates a vertex $v \in V' - A$ from the root. Consider the feasible set $X = \cup\{Y \in \mathscr{F} : Y \subseteq V' - A\}$ of the vertex search greedoid $(V', \mathscr{F})$. We have $|A| + r(X) < r = |V'|$, since $v \notin X \cup A$. But the free sets over X are contained in $\Gamma(X) \subseteq A$, and $|A| < \min\{k, r - r(X)\}$. Thus $(V', \mathscr{F})$ is not k-connected.

Conversely, if the vertex search greedoid is not k-connected, then there is a set $X \in \mathscr{F}$ such that $|\Gamma(X)| < \min\{k, r - r(X)\}$. But then $|X \cup \Gamma(X)| = r(X) + |\Gamma(X)| < r$, so $\Gamma(X)$ is a cut set of size less than k that separates all vertices of $V' - (X \cup \Gamma(X))$ from the root. $\square$

In concluding this section, we observe that the case of *undirected* rooted graphs and their associated greedoids can be reduced to the previous *directed* case by a standard graph-theoretic construction: for an undirected rooted graph $\Gamma = (V, E, r)$ let $\Delta = (V, E', r)$ be the rooted digraph on the same vertex set that has a pair of antiparallel arcs for every edge of Γ. Then Γ is k-connected if and only if Δ is k-connected, and the vertex search greedoids of Γ and Δ coincide. The branching greedoids of Γ and Δ differ already in the size of their ground sets, but the associated posets $(\mathscr{F}, \subseteq)$ are isomorphic. This proves the analog of Proposition 8.4.7 for undirected graphs via the observation that k-connectedness of a greedoid can be determined from the unlabeled poset $(\mathscr{F}, \subseteq)$ alone.

8.5. Optimization on Greedoids

As mentioned in the introduction, greedoids were originally developed to give a unified approach to the optimality of various *greedy algorithms* known in combinatorial optimization. Such algorithms can be loosely characterized as having locally optimal strategy and no backtracking.

In this section we will formulate a greedy algorithm for hereditary languages, define *compatible* objective functions on such languages, and then characterize greedoids as those languages on which the greedy algorithm is optimal for all compatible objective functions. The well known algorithmic characterization of matroids in terms of *linear* objective functions is here viewed in a broader context.

To illustrate the results, we will discuss Kruskal's and Prim's algorithms for minimal spanning trees and Dijkstra's shortest path algorithm as instances of greedoid optimization.

8.5.A. The Greedy Algorithm

In the following, let $(E, \mathcal{L})$ be a simple hereditary language over a finite ground set E. We do not assume *a priori* that $\mathcal{L}$ is pure. As usual, maximal words in $\mathcal{L}$ are called *basic*. We will be interested in the following optimization problem.

Given an objective function $\omega: \mathcal{L} \to \mathbb{R}$, find a basic word α that maximizes $\omega(\alpha)$.

The greedy approach to this problem is expressed by the following algorithm.

GREEDY: (1) Put $\alpha_0 := \varnothing$ and $i := 0$.
(2) Given α_i, choose $x_{i+1} \in E$ such that
(i) $\alpha_i x_{i+1} \in \mathcal{L}$
(ii) $\omega(\alpha_i x_{i+1}) \geqq \omega(\alpha_i y)$, for all $y \in E$ such that $\alpha_i y \in \mathcal{L}$.
(3) Put $\alpha_{i+1} := \alpha_i x_{i+1}$.
(4) If the word α_{i+1} is not basic, put $i := i + 1$ and go to (2).
(5) If α_{i+1} is basic, print $\alpha := \alpha_{i+1}$ and stop.

Whether GREEDY works (that is, whether the *greedy solution* α produced by GREEDY actually maximizes $\omega(\alpha)$) must obviously depend on both $\mathcal{L}$ and ω – they have to be 'compatible'.

8.5.1. Definition. An objective function $\omega: \mathcal{L} \to \mathbb{R}$ is *compatible* with $\mathcal{L}$ if it satisfies the following conditions: for $\alpha x \in \mathcal{L}$ such that $\omega(\alpha x) \geqq \omega(\alpha y)$ for every $\alpha y \in \mathcal{L}$ ('x is a best choice after α'),

(C1) $\alpha\beta x\gamma \in \mathcal{L}$ and $\alpha\beta z\gamma \in \mathcal{L}$ imply that $\omega(\alpha\beta x\gamma) \geqq \omega(\alpha\beta z\gamma)$ ('x is a best choice at every later stage'), and
(C2) $\alpha x\beta z\gamma \in \mathcal{L}$ and $\alpha z\beta x\gamma \in \mathcal{L}$ imply that $\omega(\alpha x\beta z\gamma) \geqq \omega(\alpha z\beta x\gamma)$ ('it is always better to choose x first and z later than the other way round').

Of course, if ω is *stable*, in the sense that $\omega(\alpha)$ only depends on the underlying set $\tilde{\alpha}$, then (C2) is vacuously satisfied.

The following main theorem characterizes greedoids algorithmically.

8.5.2. Theorem. *Suppose $(E, \mathcal{L})$ is a simple hereditary language. Then $(E, \mathcal{L})$ is a greedoid if and only if GREEDY gives an optimal solution for every compatible objective function on $\mathcal{L}$.*

Proof. (1) We will show that GREEDY works on interval greedoids – the proof for general greedoids is similar but more complicated. All examples discussed below involve interval greedoids and are therefore covered by this proof.

Let $(E, \mathscr{L})$ be an interval greedoid, ω a compatible objective function and γ a greedy solution. Choose an optimal solution δ so that the common prefix with γ is of maximal length, i.e. $|\alpha|$ is maximal with $\gamma = \alpha\gamma'$ and $\delta = \alpha\delta'$. We claim that $\gamma = \alpha = \delta$.

If this is not the case, then $\gamma' = x\gamma''$, $\delta' = y_1 y_2 \dots y_n$ where $x \neq y_1$. Now augment αx from $\beta = \alpha y_1 \dots y_n$, using the strong exchange property (L2′) for interval greedoids: $\alpha x y_1 \dots \hat{y}_k \dots y_n \in \mathscr{L}$, for some $1 \leqq k \leqq n$. Here, '$\hat{y}_k$' denotes that y_k is deleted.

For $1 \leqq i \leqq k-1$ define

$$\beta_i = \alpha y_1 \dots y_{i-1} x y_i \dots \hat{y}_k \dots y_n,$$

and let

$$\beta_k = \alpha y_1 \dots y_{k-1} x y_{k+1} \dots y_n.$$

We know that $\beta_1 = \alpha x y_1 \dots \hat{y}_k \dots y_n \in \mathscr{L}$, and augmenting $\alpha y_1 \dots y_i$ from $\beta_i \in \mathscr{L}$, using strong exchange (L2′), we get $\beta_{i+1} \in \mathscr{L}$. Hence, $\beta_1, \beta_2, \dots, \beta_k \in \mathscr{L}$.

Now, x is a 'best choice' after α (since γ is greedy), so conditions (C1) and (C2) give

$$\omega(\beta_1) \geqq \omega(\beta_2) \geqq \dots \geqq \omega(\beta_k),$$

and (C1) implies

$$\begin{aligned} \omega(\beta_k) &= \omega(\alpha y_1 \dots y_{k-1} x y_{k+1} \dots y_n) \\ &\geqq \omega(\alpha y_1 \dots y_{k-1} y_k y_{k+1} \dots y_n) = \omega(\delta). \end{aligned}$$

Hence, $\omega(\beta_1) \geqq \omega(\delta)$. But then β_1 is an optimal basic word having a longer common prefix αx with γ than does δ. This contradicts the choice of δ.

(2) For the converse, we define *generalized bottleneck functions* on $\mathscr{L}$: they are the objective functions of the form $\omega(x_1 x_2 \dots x_n) = \min\{f_1(x_1), \dots, f_n(x_n)\}$, where the $f_i\colon E \to \mathbb{R}$ $(1 \leqq i \leqq r)$ are functions satisfying $f_i(x) \leqq f_{i+1}(x)$ for every $x \in E$, $1 \leqq i < r$. Here r denotes the maximal length of a word in $\mathscr{L}$. Generalized bottleneck functions are compatible with all hereditary languages, as is easily checked.

Now, suppose that $\alpha, \beta \in \mathscr{L}$ and $|\alpha| = k > m = |\beta|$. We want to show that there is some $x \in \tilde{\alpha}$ such that $\beta x \in \mathscr{L}$. For this, let $A = \tilde{\alpha} \cup \tilde{\beta}$ and define a generalized bottleneck function ω by

$$f_1(x) = \dots = f_k(x) = \begin{cases} 0 & \text{if } x \notin A, \\ 1 & \text{if } x \in A, \end{cases}$$

$$f_{k+1}(x) = \dots = f_r(x) = \begin{cases} 1 & \text{if } x \notin A, \\ 2 & \text{if } x \in A. \end{cases}$$

Let $\delta = \alpha\delta'$ be a basic word extending α. Then $\omega(\delta) = 1$. Next, let $\gamma = \beta x_1 x_2 \dots x_p$ be a greedy solution extending β. Such solutions clearly exist. Since GREEDY is assumed to be optimal, we have

$$1 = \omega(\delta) \leqq \omega(\gamma) \leqq f_{m+1}(x_1),$$

which, since $m+1 \leqq k$, implies that $x_1 \in A$. Now, $\beta x_1 \in \mathscr{L}$, since $\mathscr{L}$ is hereditary, and therefore $x_1 \in A - \tilde{\beta} \subseteq \tilde{\alpha}$, since $\mathscr{L}$ is simple. This completes the proof. □

8.5.B. Examples

8.5.3. Example. (Matroid optimization) An objective function $\omega: \mathscr{L} \to \mathbb{R}$ is called *linear* if it is of the form

$$\omega(x_1 x_2 \ldots x_n) = \sum_{i=1}^{n} u(x_i)$$

for some given weight function u: $E \to \mathbb{R}$.

If $(E, \mathscr{L})$ is a matroid then all linear objective functions are compatible and hence can be greedily optimized. One easily checks condition (C1), and (C2) is clear since linear objective functions are stable.

For example, if $\Gamma = (V, E)$ is an undirected connected graph with weight function u: $E \to \mathbb{R}$, then a *minimal spanning tree* (i.e. a spanning tree $T \subseteq E$ minimizing $\sum_{e \in T} u(e)$) will be obtained by applying Kruskal's algorithm. From a greedoid point of view, we apply GREEDY to the linear objective function $\omega(T) = -\sum_{e \in T} u(e)$ on the graphic matroid $(E, \mathscr{L})$ associated with Γ.

8.5.4. Example. (Breadth-first-search) Let $(E, \mathscr{L})$ be the branching greedoid of a connected rooted digraph $\Delta = (V, E, r)$, and assume that d: $E \to \mathbb{R}^+$ is a length function on the arcs. Define an objective function ω: $\mathscr{L} \to \mathbb{R}$ by

$$\omega(x_1 x_2 \ldots x_n) = -\sum_{i=1}^{n} d(r, v_i),$$

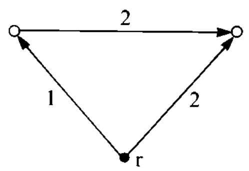
where the $v_i = \text{head}(x_i)$ are the nodes reached by the branching $x_1 \ldots x_n$, and $d(r, v_i)$ is the sum of the lengths of the arcs on the unique path from r to v_i in $x_1 x_2 \ldots x_n$.

We have to check that ω is compatible with $\mathscr{L}$: (C2) is again clear because ω is stable; the easy argument for (C1) is an instructive exercise.

Hence, the theorem implies that GREEDY, which executes Breadth-first-search on Δ, finds a spanning arborescence that minimizes the sum of the distances from the root. Such an arborescence must, in fact, also minimize each individual distance, since, as can easily be seen, there exist spanning arborescences that simultaneously minimize all the distances from the root to the other vertices. This particular instance of GREEDY gives the shortest path algorithm of Dijkstra.

It is noteworthy that the objective function $-\omega$ (corresponding to *Depth-first-search*) is *not* compatible with the branching greedoid language. For instance, GREEDY fails to optimize $-\omega$ for the greedoid shown in Figure 8.6.

Figure 8.6.



8.5.C. Linear Objective Functions

Linear objective functions (as defined in Example 8.5.3) cannot in general be greedily optimized over greedoids. However, for some special linear functions and for some special greedoids the situation is better.

Let $(E, \mathscr{F})$ be a greedoid, $\mathscr{R} \subseteq 2^E$ its collection of rank feasible sets, and u: $E \to \mathbb{R}$ a function. The linear objective function $\omega(x_1 x_2 \ldots x_n) = \sum_{i=1}^n u(x_i)$ is called *$\mathscr{R}$-compatible* if $\{x \in E\colon u(x) \geqq c\} \in \mathscr{R}$, for all $c \in \mathbb{R}$, that is, if all the *level sets* of u are rank feasible.

In the situation of the preceding paragraph, suppose that $c_1 > c_2 > \ldots > c_k$ are the values assumed by u, and let $C_i = \{x \in E\colon u(x) \geqq c_i\}$. Clearly, GREEDY will first pick a basis of C_1, then augment it to a basis of C_2, and so on. Hence, if B is a greedy basis then $|B \cap C_i| = \text{rank } C_i$, for $1 \leqq i \leqq k$. Since $C_i \in \mathscr{R}$, an arbitrary basis B' must satisfy $|B' \cap C_i| \leqq \text{rank } C_i$, $1 \leqq i \leqq k$. It easily follows that $\omega(B') \leqq \omega(B)$, i.e. we have proven the following.

8.5.5. Proposition. *Let $(E, \mathscr{F})$ be a greedoid. Then GREEDY is optimal for every $\mathscr{R}$-compatible linear objective function.*

As observed in section 8.4.C, $(E, \mathscr{F})$ is a matroid if and only if $\mathscr{R} = 2^E$, that is, if and only if every linear objective function is $\mathscr{R}$-compatible. So Proposition 8.5.5 again generalizes, but in a different way from Theorem 8.5.2, the fact that matroids have the property that all linear objective functions can be greedily optimized.

However, not every greedoid with that property is a matroid, as we shall now see.

8.5.6. Proposition. *Let $(E, \mathscr{F})$ be a greedoid. Then GREEDY is optimal for every linear objective function if and only if the hereditary closure $(E, \mathscr{H}(\mathscr{F}))$ is a matroid and every set that is closed in $(E, \mathscr{F})$ is also closed in $(E, \mathscr{H}(\mathscr{F}))$.*

An example of a greedoid that satisfies these conditions is the undirected branching greedoid of a connected rooted graph $\Gamma = (V, E, r)$, for which the hereditary closure is the corresponding graphic matroid. Greedy optimization of linear objective functions over this branching greedoid is equivalent to Prim's algorithm for finding a minimal spanning tree in Γ.

8.6. The Greedoid Polynomial

Every greedoid has an associated polynomial that reflects some of its combinatorial structure. In this section we will present the basic properties of this polynomial and also discuss the related notions of greedoid invariants and dual complexes.

8.6.A. A Greedoid 'Tutte' Polynomial

Let $G=(E,\mathscr{F})$ be a greedoid with $n=|E|$ and $r=\text{rank } G$. Give the underlying set E a total ordering Ω. This induces a total ordering of the set $\mathscr{B}_G$ of bases of G as follows: $B<_{\Omega}B'$ if the lexicographically first feasible permutation of B is lexicographically smaller than the lexicographically first feasible permutation of B'.

For instance, consider the branching greedoid of the directed graph shown in Figure 8.7. There are two bases: $B_1=\{a,b\}$ and $B_2=\{a,c\}$. If Ω is $a<b<c$ then $B_1<B_2$, but if Ω is $b<c<a$ then $B_2<B_1$.

Now, for a basis $B\in\mathscr{B}_G$ we shall say that $x\in E-B$ is *externally active in* B if $B<(B\cup x)-y$, for all $y\in B$ such that $(B\cup x)-y$ is a basis. Let $\text{ext}_{\Omega}(B)$ denote the set of externally active elements, and define

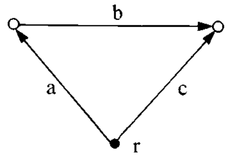
$$\lambda_{G,\Omega}(t)=\sum_{B\in\mathscr{B}_G} t^{|\text{ext}_{\Omega}(B)|}. \tag{8.10}$$

Let us again exemplify with the small branching greedoid above:

Ω	$\text{ext}_{\Omega}(B_1)$	$\text{ext}_{\Omega}(B_2)$	$\lambda_{G,\Omega}(t)$
$a<b<c$	$\{c\}$	$\varnothing$	$1+t$
$b<c<a$	$\varnothing$	$\{b\}$	$1+t$

The crucial combinatorial fact about this notion of external activity in bases is stated in the following lemma. Recall that a subset $A\subseteq E$ is said to be *spanning* if it contains a basis.

Figure 8.7.



8.6.1. Lemma. *For each spanning set A there exists a unique basis B such that $B \subseteq A \subseteq B \cup \mathrm{ext}_\Omega(B)$.*

The basis B with this property is the first one (in the ordering induced by Ω) that is contained in A. The partitioning of the set $\mathcal{S}$ of spanning sets into Boolean intervals implies part (iii) of the following theorem, which in turn implies part (i).

8.6.2. Theorem. Let $G = (E, \mathcal{F})$ *be a greedoid of rank r and cardinality n.*

(i) $\lambda_G(t) := \lambda_{G,\Omega}(t)$ *is independent of the ordering Ω of E.*

(ii) $\lambda_G(t)$ *is a monic polynomial of degree $n - r$ with non-negative integer coefficients.*

(iii) *If G has s_j spanning sets of size j, for $r \leqq j \leqq n$, then*

$$\lambda_G(1 + t) = \sum_{i=0}^{n-r} s_{r+i} t^i.$$

(iv)

$$\lambda_G(t) = \begin{cases} \lambda_{G/e}(t) & \text{if } \{e\} \in \mathcal{F} \text{ and } e \text{ is a coloop,} \\ \lambda_{G/e}(t) + \lambda_{G\backslash e}(t) & \text{if } \{e\} \in \mathcal{F} \text{ and } e \text{ is not a coloop,} \\ \lambda_{G_1}(t)\lambda_{G_2}(t) & \text{if } G \text{ is the direct or ordered sum of } G_1 \text{ and } G_2. \end{cases}$$

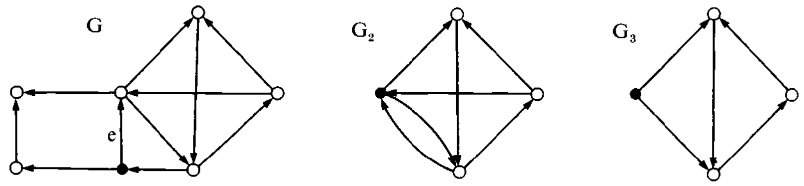
The polynomial $\lambda_G(t)$ is a greedoid counterpart to the Tutte polynomial. If G is a matroid with Tutte polynomial $T_G(x, y)$ and dual matroid G^*, then $T_G(1, t) = \lambda_G(t)$ and $T_G(t, 1) = \lambda_{G^*}(t)$.

Every greedoid of positive rank must have some feasible singleton. Therefore part (iv) of the theorem gives a recursive algorithm for computing the polynomial λ_G for any greedoid G. The algorithm will stop at the trivial greedoids of cardinality k and rank zero, whose polynomial is t^k. In general, if G has k loops and G' is obtained by deleting these loops, then $\lambda_G(t) = t^k \lambda_{G'}(t)$, since G is then the direct sum of G' and the loops. For instance, if G is of rank one with k feasible singletons, then $\lambda_G = t^{n-k}(1 + t + \dots + t^{k-1})$. Notice (e.g. from (8.10)) that if G is full (i.e. $E \in \mathcal{F}$) then $\lambda_G(t) = 1$.

Let us as a small example compute λ_G for the branching greedoid G in Figure 8.8a.

One sees that the arc e is a feasible coloop (i.e. it emanates from the root and lies in every spanning arborescence). Hence, it may be contracted away without affecting λ_G. Now, $G/e \cong G_1 \oplus G_2$, where G_1 is the branching greedoid in Figure 8.7 and G_2 that in Figure 8.8b. Hence, $\lambda_G = (1 + t)\lambda_{G_2}$, and deleting the two loops in G_2 (i.e. the arcs going into the root) we get $\lambda_{G_2} = t^2 \lambda_{G_3}$, where G_3 is the greedoid in Figure 8.8c. Using deletion–contraction or simple counting, part (iv) or part (iii) of Theorem 8.6.2 quickly gives $\lambda_{G_3} = 2t + t^2$. Hence, $\lambda_G(t) = (1 + t)t^2(2t + t^2) = 2t^3 + 3t^4 + t^5$.

Figure 8.8.



(a) (b) (c)

We have seen that the degree of $\lambda_G(t)$ as well as the coefficients of $\lambda_G(1+t)$ have direct combinatorial meaning. Also the subdegree of $\lambda_G(t)$ has an interesting interpretation in terms of a certain algorithmic property. By the *subdegree* of a non-zero polynomial $c_0 + c_1 t + \ldots + c_k t^k$ we mean the least integer d such that $c_d \neq 0$.

Suppose that we want to design an algorithm that for arbitrary subsets $A \subseteq E$ decides whether A is a spanning set in the greedoid $G = (E, \mathscr{F})$. Think of A as being represented by its incidence vector $\mathscr{X}_A$ (a 0–1-vector of length n with elements equal to unity in the positions corresponding to A), and suppose that the algorithm can read $\mathscr{X}_A$ by inspecting only one position at a time. If the best such algorithm can decide whether A is spanning or not after k inspections for all $A \subseteq E$, and if $k-1$ inspections will not suffice, then we say that k is the *argument complexity* of the spanning property in G.

For instance, it is easy to check that the argument complexity is 4 for the branching greedoid in Figure 8.8c. In other words, it would be redundant to inspect all five arcs in order to decide algorithmically whether a subset of arcs contains a directed path from the root to every other node.

8.6.3. Proposition. *The argument complexity of the spanning property in G is $n-d$, where d is the subdegree of $\lambda_G(t)$.*

The result can be made more precise, since the method of proof implies an explicit optimal algorithm that will decide whether an arbitrary subset A is spanning after at most $n-d$ inspections of $\mathscr{X}_A$. Briefly, here is what to do.

(1) Pick a feasible singleton $\{e\}$ in G and read the corresponding position $\mathscr{X}_e$ of $\mathscr{X}_A$.
(2) If $\mathscr{X}_e = 1$ and rank $G > 1$, put $G := G/e$ and go to (1).
(3) If $\mathscr{X}_e = 1$ and rank $G = 1$, print 'A spans' and stop.
(4) If $\mathscr{X}_e = 0$ and e is not a coloop, put $G := G\backslash e$ and go to (1).
(5) If $\mathscr{X}_e = 0$ and e is a coloop, print 'A does not span' and stop.

There is one more characterization of the subdegree of $\lambda_G(t)$, which for branching greedoids takes the following form: If G is the branching greedoid of a rooted directed graph and if d is the least number of edges that must be removed in order to obtain a spanning and acyclic (no directed cycles) subgraph, then d is the subdegree of $\lambda_G(t)$.

8.6.B. Invariants and Reliability

Suppose that ϕ is a function that associates some complex number with each greedoid $G = (E, \mathscr{F})$. For instance, $\phi(G)$ could be the number of feasible sets, the number of bases, or the cardinality of the ground set. Such a function ϕ is called the *invariant* if the following axioms are satisfied.

(I1)	$\phi(G) = \phi(G/e)$	if $\{e\}$ is a feasible coloop.
(I2)	$\phi(G) = \phi(G/e) + \phi(G\backslash e)$	if $\{e\}$ is feasible and not a coloop.
(I3)	$\phi(G) = \phi(G_1)\phi(G_2)$	if G is the direct or ordered sum of G_1 and G_2.
(I4)	$\phi(G_1) = \phi(G_2)$	if G_1 and G_2 are isomorphic.
(I5)	$\phi(G) \neq 0$	for at least one greedoid G.

Let G_0^n denote the (up to isomorphism) unique greedoid of rank zero and cardinality n. If $\phi(G_0^1) = z$, then $\phi(G_0^n) = z^n$. For $n \geqq 1$ this is a direct consequence of axiom (I3). For $n = 0$, $\phi(G_0^0) = 1$ follows from (I3) and (I5) taken together. Notice that this together with (I1) implies that $\phi(G) = 1$ for every full greedoid G.

There is a close connection between invariants and the polynomial $\lambda_G(t)$.

8.6.4. Proposition. *Every invariant ϕ is an evaluation of the greedoid polynomial. More precisely, if $\phi(G_0^1) = z \in \mathbb{C}$, then $\phi(G) = \lambda_G(z)$ for all greedoids G.*

Proof. By definition the invariant ϕ enjoys the same recursive properties as the polynomial evaluation $\lambda_G(z)$. Hence, the two will coincide for all greedoids if they coincide for greedoids of rank zero. But we have already seen that this is the case. □

Simple examples of greedoid invariants are the number of bases $(= \lambda_G(1))$ and the number of spanning sets $(= \lambda_G(2))$. The following probabilistic example is, however, more interesting.

Let $G = (E, \mathscr{F})$ be a greedoid of rank r and cardinality n. Suppose that each element of E is colored red with probability p and blue with probability $1 - p$, for some real number $0 < p < 1$. The coloring of each element is assumed to be independent of the coloring of the others. Let $\pi_G(p)$ denote the probability of the event that the set of blue elements is spanning in G: $\pi_G(p) = \text{Prob(blue spans)}$.

For instance, if G is the circuit matroid of a connected graph, then $\pi_G(p)$ is the probability that the blue edges will connect all vertices; if G is the branching greedoid of a directed rooted graph then $\pi_G(p)$ is the probability that each node can be reached along a path of blue edges from the root; if G is the k-truncation of a poset greedoid then $\pi_G(p)$ is the probability that the order filter generated by the red elements has size at most $n-k$; if G is the k-truncation of a Euclidean convex pruning greedoid then $\pi_G(p)$ is the probability that the convex hull of the red points has size at most $n-k$. In examples such as these one may think of the greedoid as some abstract stochastic system in which components may 'fail' independently of each other with probability p, and the question is to assess the probability that the system will still 'operate' in an appropriate sense (the 'damage' caused by failed components is sufficiently limited).

To analyze the function $\pi_G(p)$, define $\phi_p(G) = p^{r-n}(1-p)^{-r}\pi_G(p)$. We claim that $\phi_p(G)$ *is an invariant*. Only axiom (I2) will be verified here; verification of the other axioms is either similar or trivial.

Suppose that $\{e\}$ is feasible and not a coloop. Then Prob(blue spans) = Prob(e is blue and blue spans) + Prob(e is red and blue spans), or equivalently, $\pi_G(p) = (1-p)\pi_{G/e}(p) + p\pi_{G\backslash e}(p)$. Multiplication of this equation by $p^{r-n}(1-p)^{-r}$ gives $\phi_p(G) = \phi_p(G/e) + \phi_p(G\backslash e)$, so (I2) is satisfied.

Since ϕ_p is an invariant, and clearly $\phi_p(G_0^1) = p^{-1}$ (all subsets of G_0^1 are spanning), we conclude that $\phi_p(G) = \lambda_G(p^{-1})$ for all greedoids G. Hence, we have proven the following.

8.6.5. Proposition. *The probability that the set of blue elements is spanning in G is $p^{n-r}(1-p)^r\lambda_G(p^{-1})$.*

For instance, using our previous calculation we conclude that the blue arcs will reach every node of the graph in Figure 8.8a with probability $(1-p)^6(1+3p+2p^2)$.

8.6.C. Duality

What parts, if any, of the matroid duality operation remain valid for greedoids? The answer to this question will depend on what we mean by a duality operation $G \to G^*$ that sends a greedoid $G = (E, \mathscr{F})$ to some structured set system $G^* = (E, \mathscr{F}^*)$ on the same ground set. To get reasonably general positive answers we must unfortunately give up the requirement that the dual $(E, \mathscr{F}^*)$ itself is a greedoid. Then there are two weak notions of greedoid duality.

The first is the complementation construction $\mathscr{F}^c = \{E - X : X \in \mathscr{F}\}$. For antimatroids (Proposition 8.7.3), this leads to a dual object that is a convex geometry (it is a greedoid if and only if the original antimatroid is a poset greedoid). A pleasant property of this duality is that the original antimatroid can be uniquely recovered from its dual. A definite disadvantage is that the construction gives nothing of apparent interest for general greedoids, with the following exception.

8.6.6. Theorem. *If $(E, \mathscr{F})$ is a full Gaussian greedoid, then so is also $(E, \mathscr{F}^c)$.*

The second notion of duality, and the one that we will briefly discuss here, associates with an arbitrary greedoid a dual object that is a shellable simplicial complex. One price paid for the generality is that the original greedoid cannot be uniquely reconstructed from its dual. On the other hand, this duality operation commutes with deletion and contraction in the desired way and also generalizes some other matroid properties to arbitrary greedoids. Last but not least, it throws additional light on the greedoid polynomial.

In the following discussion hereditary set systems will be called *simplicial complexes*, and we will assume familiarity with the concepts of *shellability* and *shelling polynomial* of simplicial complexes. These and other related notions are defined and discussed in Chapter 7 of this book.

For a greedoid $G=(E, \mathscr{F})$, define its *dual complex* $G^*=(E, \mathscr{F}^*)$ by $\mathscr{F}^*=\{A\colon A\subseteq E-B$ for some basis $B\in\mathscr{B}_G\}$. So, $\mathscr{F}^*$ is the hereditary closure of the family of complements of the bases of G. It is therefore a pure simplicial complex. Clearly, if G is a matroid then G^* is the dual matroid in the usual sense.

As a simplicial complex, a matroid can be characterized either by the exchange property (by definition) or else by being sufficiently shellable (by Theorem 7.3.4). These two properties, exchange and shellability, go different ways in the more general picture: the former is found in all greedoids and the latter in their duals.

8.6.7. Theorem. *The dual complex G^* of a greedoid $G=(E, \mathscr{F})$ is shellable, and its shelling polynomial is $\lambda_G(t)$. Furthermore, if $e\in E$, then*

(i) $G^*\backslash e=(G/e)^*$, *if $\{e\}$ is feasible;*
(ii) $G^*/e=(G\backslash e)^*$, *if e is not a coloop.*

The deletion and contraction operation on simplicial complexes should be understood in the natural way: $G^*\backslash e=\{A\subseteq E-e\colon A\in\mathscr{F}^*\}$ and $G^*/e=\{A\subseteq E-e\colon A\cup e\in\mathscr{F}^*\}$ if $\{e\}\in\mathscr{F}^*$. The special requirements on the element e in (i) and (ii) ensure that in each case contraction is well defined.

Let us sketch the proof of Theorem 8.6.7 to the extent that its relevance for the greedoid polynomial becomes clear. Start by assigning a total ordering Ω to the ground set E. As explained in the first paragraph of section 8.6.A, this induces a total ordering of the set $\mathscr{B}$ of bases of G. The first main fact is that the corresponding ordering of the basis complements is a shelling order. In particular, G^* is shellable. The second main fact is that the restriction of the facet $E - B$ induced by this shelling order (as defined in section 7.2) is $E - (B \cup \text{ext}_\Omega(B))$. Consequently, $\lambda_G(t) = \lambda_{G,\Omega}(t)$, as defined in (8.10), is the shelling polynomial. Also, it follows that Lemma 8.6.1 is a special case of Proposition 7.2.2.

8.7. Antimatroids

Antimatroids were defined in section 8.2 as greedoids that have the interval property without upper bounds. They were characterized in Proposition 8.2.7 as accessible set systems closed under union and by a special exchange property. Several examples were given in section 8.3.B.

In this section we shall discuss some of the special structure of antimatroids, which makes them an exceptional class of greedoids. It turns out that antimatroids model some combinatorial properties of the *convex hull* operator in Euclidean spaces much like matroids model the combinatorial properties of the *linear span* operator.

8.7.A. The Duality with Convex Geometries

A *closure operator* on a finite set E is an increasing, monotone, and idempotent function τ: $2^E \rightarrow 2^E$. This means that for all $A, B \subseteq E$:

(CO1) $A \subseteq \tau(A)$;
(CO2) $A \subseteq B$ implies $\tau(A) \subseteq \tau(B)$;
(CO3) $\tau\tau(A) = \tau(A)$.

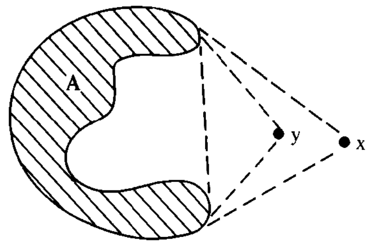
Fixed sets $A = \tau(A)$ are called *closed*, and it follows from the axioms that the family $\mathscr{C}$ of closed sets is preserved under intersection (i.e. $A, B \in \mathscr{C} \Rightarrow A \cap B \in \mathscr{C}$). Conversely, if $\mathscr{C}' \subseteq 2^E$ is a set system preserved under intersection then $\tau(A) = \cap\{C \in \mathscr{C}' : A \subseteq C\}$ is a closure operator, and this gives a one-to-one correspondence between closure operators and intersection-invariant set systems containing E. In particular, a closure operator can be specified by giving the family $\mathscr{C}$ of closed sets.

The closure operator $\tau(A) = \{x \in E: r(A \cup x) = r(A)\}$ of a matroid is characterized by the additional *MacLane exchange axiom*:

(E) If $x, y \notin \tau(A)$ and $y \in \tau(A \cup x)$, then $x \in \tau(A \cup y)$.

Now, let E be a finite subset of $\mathbb{R}^n$ and for subsets $A \subseteq E$ let $\tau(A) = E \cap \text{conv}(A)$, where conv($A$) denotes the convex hull of A in the usual sense of Euclidean geometry, i.e. $\tau(A) = \{x \in E: x = \Sigma\lambda_i a_i,\ a_i \in A,\ 0 \leqq \lambda_i \leqq 1,\ \Sigma\lambda_i = 1\}$. It is a very

Figure 8.9.



interesting fact that this convex hull closure τ satisfies a property opposite to (E), which we call the *anti-exchange axiom*:

(AE) If $x, y \notin \tau(A)$, $x \neq y$, and $y \in \tau(A \cup x)$, then $x \notin \tau(A \cup y)$.

An intuitive illustration of this axiom is given in Figure 8.9.

This leads to the following general definition.

8.7.1. Definition. A *convex geometry* is a pair (E, τ) where E is a finite set and τ is a closure operator on E satisfying the anti-exchange condition (AE).

To stay close to the geometric intuition it could have seemed natural to demand that $\tau(\emptyset) = \emptyset$, and even that $\tau(\{x\}) = \{x\}$ for all $x \in E$, in a convex geometry. However, it will soon appear that from a greedoid point of view such restrictions are unwise.

As the following characterization shows, convex geometries have several of the well known properties of Euclidean convexity, for instance with respect to the role of extreme points. For a general closure operator τ: $2^E \to 2^E$, a point $x \in A$ is called an *extreme point* of $A \subseteq E$ if $x \notin \tau(A - x)$. The set of extreme points of A is denoted by $\mathrm{ex}(A)$. Observe that for general closures (e.g. for matroid closures), $\mathrm{ex}(A) = \emptyset$ is possible for sets $A \nsubseteq \tau(\emptyset)$.

8.7.2. Proposition. *Let τ: $2^E \to 2^E$ be a closure operator on a finite set E. Then the following conditions are equivalent.*

(i) *(E, τ) is a convex geometry.*
(ii) *For all closed sets $A \subset B$ there exists $x \in B - A$ such that $A \cup x$ is closed.*
(iii) *For every closed set $A \subset E$ there exists $x \in E - A$ such that $A \cup x$ is closed.*
(iv) *All maximal chains of closed sets, $\tau(\emptyset) = A_0 \subset A_1 \subset \ldots \subset A_k = E$, have the same length $k = |E - \tau(\emptyset)|$.*
(v) *$A = \tau(\mathrm{ex}(A))$, for every closed set A.*
(vi) *Every $A \subseteq E$ has a unique minimal spanning subset (i.e. the family $\{S \subseteq A: \tau(S) = \tau(A)\}$ has an inclusion-wise least member).*

Proof. (i)$\Rightarrow$(ii). Suppose C is a minimal closed set such that $A \subset C \subseteq B$, and let $x \in C - A$. Then $A \cup x$ is closed. For, if $y \in \tau(A \cup x) - (A \cup x)$, then by the anti-exchange condition (AE) $x \notin \tau(A \cup y)$; hence $A \subset \tau(A \cup y) \subset C$, which contradicts the minimality of C.

(ii)$\Leftrightarrow$(iii). Condition (iii) is a specialization of (ii). Suppose now that (iii) holds, and let $A \subset B$ be closed sets. By repeated use of (iii) we can find a chain $A = A_0 \subset A_1 \subset \dots \subset A_s = E$ of closed sets A_i with $|A_i| = |A| + i$, $0 \leqq i \leqq s = |E - A|$. Since $B \cap A_0 = A$ and $B \cap A_s = B$, we can find some i such that $|B \cap A_i| = |A| + 1$. Since $B \cap A_i = A \cup x$ is closed (being the intersection of two closed sets), (ii) follows.

(ii)$\Rightarrow$(iv). A reformulation of (ii) is that an inclusion $A \subset B$ is a covering in the lattice of closed sets if and only if $|B - A| = 1$, from which (iv) immediately follows.

(iv)$\Rightarrow$(v). Let A be a closed set. One easily sees that $x \in \text{ex}(A)$ if and only if $A - x$ is closed. Hence, assuming (iv), we have that $\text{ex}(A) = \cup\{A - B\colon A$ covers B in the lattice of closed sets$\}$. Now, if $\tau(\text{ex}(A)) \subset A$, then $\tau(\text{ex}(A)) \subseteq B = A - x$, for some $x \in \text{ex}(A)$, which contradicts $x \in \tau(\text{ex}(A))$. Hence, $\tau(\text{ex}(A)) = A$.

(v)$\Rightarrow$(vi). Let $D = \text{ex}(\tau(A))$. Then, clearly, $D \subseteq S$ for every spanning subset $S \subseteq A$. Also, $\tau(D) = \tau(A)$, by (v).

(vi)$\Rightarrow$(i). Suppose that axiom (AE) fails, i.e. we have $x, y \notin \tau(A)$, $x \neq y$, $y \in \tau(A \cup x)$, and $x \in \tau(A \cup y)$. Then $\tau(A \cup x) = \tau(A \cup y)$. Let D be the unique minimal spanning subset of $\tau(A \cup x)$. Then, since $A \cup x$ and $A \cup y$ are spanning, we get $D \subseteq (A \cup x) \cap (A \cup y) = A$, and hence $\tau(D) \subseteq \tau(A) \subset \tau(A \cup x)$, a contradiction. □

Here are a few examples of convex geometries (E, τ).

(1) Let $P = (E, \leqq)$ be a finite poset, and for $A \subseteq E$ define $\tau(A) = \{x \in E\colon x \geqq y$ for some $y \in A\}$. The closed sets in this geometry are the *order filters* (or, *dual ideals*) of P.

(2) Let $P = (E, \leqq)$ again be a finite poset and take the *interval closure* $\tau(A) = \{x \in E\colon y_1 \leqq x \leqq y_2$ for some $y_1, y_2 \in A\}$.

(3) Let E be the edge set (or, vertex set) of a tree T and for $A \subseteq E$ let $\tau(A)$ be the smallest subtree of T that contains A. The closed sets of this geometry are the subtrees of T, and the extreme points of a subtree are its leaves.

(4) Let E be the arc set of an acyclic digraph (i.e. a directed graph with no directed cycles), and let $\tau(A)$ be the transitive closure of $A \subseteq E$. In particular, if E is the set of comparability relations of a poset, then the closed sets of this geometry can be identified with the subposets, and the extreme points of a subposet are its covering relations.

(5) Let E be a finite subset of $\mathbb{R}^n$, and for $A \subseteq E$ let $\tau(A) = E \cap \operatorname{conv}(A)$ be the Euclidean convex hull closure. This example, which we already used to motivate the anti-exchange axiom (AE), is particularly important for providing geometric intuition.

The preceding list of examples of convex geometries shows considerable overlap with the examples of antimatroids given in section 8.3.B. In fact, the two concepts are completely equivalent, in the sense of the following duality.

8.7.3. Proposition. *Let E be a finite set and $\mathscr{F} \subseteq 2^E$. Then $(E, \mathscr{F})$ is an antimatroid if and only if $\mathscr{F}^c = \{E - X \colon X \in \mathscr{F}\}$ is the family of closed sets of a convex geometry. Hence, there is a one-to-one correspondence $\mathscr{F} \leftrightarrow \mathscr{F}^c$ between antimatroids and convex geometries on E.*

Proof. Condition (iii) of Proposition 8.7.2 shows that a set system $\mathscr{C} \subseteq 2^E$ is the family of closed sets of a convex geometry if and only if $\mathscr{C}$ is closed under intersection and for every $A \in \mathscr{C}$ there exists $B \in \mathscr{C}$ such that $A \subset B$ and $|B| = |A| + 1$. This means precisely that the family of set complements is closed under union and accessible, i.e. it is an antimatroid. □

The duality with convex geometries is very useful and illuminating for the study of antimatroids. Examples are often easily recognized by their closure operator, and the geometric intuition provided by the dual point of view is most valuable. From now on we will say that a subset A of an antimatroid is *convex* if A is closed in the dual convex geometry, i.e. if $E - A$ is feasible.

8.7.B. Some Characterizations of Antimatroids

In this section we shall give some additional characterizations of antimatroids both as set systems and as languages.

Let E be a finite set and let H be a mapping that associates with each element $x \in E$ a set system $H(x) \subseteq 2^{E-x}$. This defines a left hereditary language:

$$\mathscr{L}_H = \{x_1 x_2 \dots x_k \in E_s^* \colon \text{for all } 1 \leqq i \leqq k \text{ there is a set } A \in H(x_i) \text{ such that } A \subseteq \{x_1, x_2, \dots, x_{i-1}\}\}. \tag{8.11}$$

The system $H = (H(x))_{x \in E}$ will be called an *alternative precedence system*, and the language $\mathscr{L}_H$ that it generates an *alternative precedence language*. In the context of scheduling and searching procedures it is often natural to obtain feasible sequences this way: an item x becomes legal once at least one 'alternative precendence set' has already been processed. Examples will be discussed after this result.

8.7.4. Proposition. *Let $\mathscr{L} \subseteq E_s^*$ be a finite simple language. Then the following conditions are equivalent.*

(i) *$(E, \mathscr{L})$ is an antimatroid.*

(ii) *$\mathscr{L}$ is an alternative precedence language.*

(iii) *$\varnothing \in \mathscr{L}$ and $\mathscr{L}$ satisfies the following exchange axiom:*

(A′) *for $\alpha, \beta \in \mathscr{L}$ such that $\tilde{\alpha} \nsubseteq \tilde{\beta}$, there is some $x \in \tilde{\alpha} - \tilde{\beta}$ such that $\beta x \in \mathscr{L}$.*

(Note that (A′) is the ordered version of axiom (A) in Proposition 8.2.7.)

Proof. (i)$\Rightarrow$(ii). Define an alternative precedence system by $H(x) = \{\tilde{\alpha}: \alpha x \in \mathscr{L}\}$. Then clearly $\mathscr{L} \subseteq \mathscr{L}_H$. Conversely, suppose $x_1 x_2 \ldots x_k \in \mathscr{L}_H$. By induction on k we may assume that $x_1 x_2 \ldots x_{k-1} \in \mathscr{L}$. By definition of $\mathscr{L}_H$ there exists some $\alpha \in \mathscr{L}$ such that $\alpha x_k \in \mathscr{L}$ and $\tilde{\alpha} \subseteq \{x_1, \ldots, x_{k-1}\}$. Since $\tilde{\mathscr{L}}$ is closed under union, we get that $\{x_1, \ldots, x_k\} = \{x_1, \ldots, x_{k-1}\} \cup \widetilde{\alpha x_k} \in \tilde{\mathscr{L}}$. Hence, by Proposition 8.2.3, $x_1 x_2 \ldots x_k \in \mathscr{L}$. So, $\mathscr{L}_H \subseteq \mathscr{L}$.

(ii)$\Rightarrow$(iii). Suppose $\alpha, \beta \in \mathscr{L} = \mathscr{L}_H$, and $\tilde{\alpha} \nsubseteq \tilde{\beta}$, $\alpha = x_1 x_2 \ldots x_k$. Let j be minimal such that $x_j \notin \tilde{\beta}$. Then for some $A \in H(x_j)$, $A \subseteq \{x_1, \ldots, x_{j-1}\} \subseteq \tilde{\beta}$; hence $\beta x_j \in \mathscr{L}_H$.

(iii)$\Rightarrow$(i). This was proven in the unordered version in Proposition 8.2.7. □

Let us find alternative precedence systems giving rise to some of the familiar antimatroids.

(1) Let $P = (E, \leqq)$ be a finite poset, and for $x \in E$ let $H(x) = \{\{y \in E: y < x\}\}$. Then $(E, \mathscr{L}_H)$ is the poset greedoid.

(2) Let $(V', \mathscr{L})$ be the vertex search greedoid of a rooted graph (V, E, r), $V' = V - r$. If $x \in V'$ is adjacent to the root let $H(x) = \varnothing$; otherwise let $H(x)$ consist of singletons, one for each neighbor of x. Then $\mathscr{L} = \mathscr{L}_H$.

(3) Let E be a finite subset of $\mathbb{R}^n$, and let $(E, \mathscr{L})$ be the convex pruning greedoid. For each $x \in E$ let $H(x)$ consist of the intersections of $E - x$ with closed halfspaces having x on the boundary. Then $\mathscr{L} = \mathscr{L}_H$.

The feasible sets of an antimatroid $(E, \mathscr{F})$ ordered by inclusion form a lattice, with lattice operations: $X \vee Y = X \cup Y$, and $X \wedge Y$ is the unique basis of $X \cap Y$. Lattices of this kind can be characterized in purely lattice-theoretical terms.

A finite lattice L is said to be *join-distributive* (or, *locally free*) if for every $x \in L - \{\hat{1}\}$ the interval $[x, j(x)]$ is Boolean, where $j(x)$ is the join of all elements in L that cover x. Clearly, every distributive lattice is join-distributive, and every join-distributive lattice is semimodular. In particular, every join-distributive lattice is *graded*, i.e. there exists a rank function $r: L \to \mathbb{N}$ satisfying $r(\hat{0}) = 0$ and $r(x) = r(y) + 1$ whenever x covers y.

8.7.5. Proposition. *Let $\mathscr{F} \subseteq 2^E$ be an accessible set system. Then the following conditions are equivalent.*

(i) *$(E, \mathscr{F})$ is an antimatroid.*
(ii) *$(\mathscr{F}, \subseteq)$ is a join-distributive lattice.*
(iii) *$(\mathscr{F}, \subseteq)$ is a semimodular lattice.*

Proof. (i)⇒(ii). The sets that cover $X \in \mathscr{F}$ in an antimatroid lattice $(\mathscr{F}, \subseteq)$ are of the form $X \cup x_i$, for some $x_i \in E - X$, $1 \leqq i \leqq t$. Since $\mathscr{F}$ is closed under unions, $X \cup \{x_{i_1}, x_{i_2}, \ldots, x_{i_v}\} \in \mathscr{F}$, for all $1 \leqq i_1 < i_2 < \ldots < i_v \leqq t$. Hence, $(\mathscr{F}, \subseteq)$ is join-distributive.

(ii)⇒(iii). Every join-distributive lattice is semimodular.

(iii)⇒(i). The assumption implies that cardinality is a semimodular rank function on the lattice $(\mathscr{F}, \subseteq)$, i.e. $X \subset Y$ is a covering only if $|X| + 1 = |Y|$ and for all $X, Y \in \mathscr{F}$: $|X \wedge Y| + |X \vee Y| \leqq |X| + |Y|$.

Suppose that $X, Y \in \mathscr{F}$ and $X \nsubseteq Y$. Take a saturated chain $X \wedge Y = A_0 \subset A_1 \subset \ldots \subset A_k = X$ of sets $A_i \in \mathscr{F}$, $|A_i| = |X \wedge Y| + i$. Let j be maximal such that $A_j \subseteq Y$. Then, clearly $A_j = Y \wedge A_{j+1}$, and by semimodularity $1 \leqq |Y \vee A_{j+1}| - |Y| \leqq |A_{j+1}| - |Y \wedge A_{j+1}| = 1$. Hence, for $x \in A_{j+1} - A_j \subseteq X - Y$ we have $Y \cup x = Y \cup A_{j+1} = Y \vee A_{j+1} \in \mathscr{F}$. Thus, axiom (A) has been verified and, by Proposition 8.2.7, $(E, \mathscr{F})$ is an antimatroid. □

Antimatroids are related to join-distributive lattices in a stronger sense than that expressed by the previous result. The two concepts are essentially equivalent.

8.7.6. Theorem. *A finite lattice L is join-distributive if and only if L is isomorphic to the lattice $(\mathscr{F}, \subseteq)$ of feasible sets of some antimatroid $(E, \mathscr{F})$.*

Proof. We shall merely sketch the construction, leaving the verification of a crucial lemma aside.

Let L be a finite graded lattice and let $M(L)$ denote the set of meet-irreducible elements in L. The lemma we need is that L is join-distributive if and only if the natural map T: $x \mapsto \{y \in M(L)$: $y \ngeqq x\}$ embeds L into the Boolean lattice $2^{M(L)}$ preserving both rank and joins.

Now, if L is a join-distributive lattice, let $\mathscr{F} := T(L) \subseteq 2^{M(L)}$. From the properties of T one concludes that $\mathscr{F}$ is accessible and closed under union, i.e. an antimatroid, and that T gives an isomorphism $L \cong (\mathscr{F}, \subseteq)$. □

This representation theorem is a natural extension of G. Birkhoff's theorem, which says that L is distributive if and only if L is isomorphic to the lattice of ideals of some poset. As a direct consequence of Birkhoff's theorem we derive the following.

8.7.7. Proposition. *A greedoid $(E, \mathscr{F})$ is a poset greedoid if and only if $\mathscr{F}$ is closed under both union and intersection.*

Although quite special, poset greedoids generate all other antimatroids as homomorphic images.

8.7.8. Proposition. *Let $f: P \to E$ be a function from a finite poset P to a finite set E, and let $\mathscr{F} = \{f(A) \subseteq E: A$ is an ideal in $P\}$. Then $(E, \mathscr{F})$ is an antimatroid. Furthermore, every antimatroid is induced in this way by a map from some poset.*

8.7.C. Circuits

Let $(E, \mathscr{F})$ be an antimatroid, and let τ denote the convex closure operator of the dual convex geometry. Recall the notions of X-free sets and extreme points defined in sections 8.4.E and 8.7.A respectively. Also, for $X \in \mathscr{F}$ let $\Gamma(X) = \{a \in E - X: X \cup a \in \mathscr{F}\}$.

For a subset $A \subseteq E$, we define the *trace*, $\mathscr{F}: A = \{X \cap A: X \in \mathscr{F}\}$. Since $\mathscr{F}: A$ is accessible and closed under union, $(E, \mathscr{F}: A)$ is again an antimatroid.

8.7.9. Lemma. *For $A \subseteq E$, the following conditions are equivalent.*

(i) $\mathscr{F}: A = 2^A$.
(ii) $A = \Gamma(X)$ *for some* $X \in \mathscr{F}$.
(iii) A *is free over* X *for some* $X \in \mathscr{F}$.
(iv) $A = \mathrm{ex}(C)$ *for some convex set* C.
(v) $a \notin \tau(A - a)$ *for all* $a \in A$.
(vi) $E - A$ *is closed* (i.e. $\sigma(E - A) = E - A$).

We leave the easy verification as an exercise.

A subset $A \subseteq E$ is called *free* if it satisfies the conditions in Lemma 8.7.9. Subsets of free sets are again free. Minimal non-free sets are called *circuits*. A 1-element circuit is the same thing as a loop. (Notice that this terminology is consistent with the matroid case. If $(E, \mathscr{F})$ is a matroid then the trace $\mathscr{F}: A$ equals the restriction to A; hence free (in the sense of (i)) means independent. Notice also that for general interval greedoids the trace operation does not necessarily produce a greedoid.)

Let C be a circuit in the antimatroid $(E, \mathscr{F})$. Then $a \in \tau(C - a)$ for some $a \in C$, by condition (v). Let $x \in C - a$, and put $B = C - \{a, x\}$. Then $a, x \notin \tau(B)$,. since $B \cup a$ and $B \cup x$ are free, and $a \in \tau(B \cup x)$. Hence, by anti-exchange, $x \notin \tau(B \cup a) = \tau(C - x)$.

We have shown that each circuit C has a unique element a such that $a \in \tau(C - a)$. This is called the *root* of C.

8.7.10. Lemma. *For $a \in C \subseteq E$, the following conditions are equivalent.*

(i) *C is a circuit with root a.*

(ii) *If $B \subseteq C$ and $x \in C - B$, then $x \in \tau(B) \Leftrightarrow B = C - a$.*

(iii) *$\mathscr{F} : C = 2^C - \{\{a\}\}$.*

Proof. (i)⇒(ii). For $B \cup x = C$ this has already been shown. If $B \cup x \neq C$, then $B \cup x$ is free, hence $x \notin \tau(B)$.

(ii)⇒(iii). Reformulate (ii) as follows: if $B \subseteq C$ then $B = \tau(B) \cap C \Leftrightarrow B \neq C - a$. Since $E - \tau(B) \in \mathscr{F}$, this implies (iii).

(iii)⇒(i). Every proper subset of C is clearly free. □

Let us exemplify these definitions, using the familiar antimatroids.

(1) For a poset greedoid, the free sets are the antichains, and the circuits are the pairs $\{a, b\} \subseteq E$ such that $a < b$. The root of such a circuit is b, the larger point.

(2) For the interval closure greedoid of a poset, the free sets are the unions of two antichains, and the circuits are the triples $a < b < c$, with the root in the middle.

(3) For the vertex pruning greedoid of a tree, the free sets are the sets of endpoints (leaves) of subtrees, and the circuits are the triples of vertices that lie on some path, the middle vertex being the root.

(4) For a Euclidean convex pruning greedoid on $E \subseteq \mathbb{R}^n$, a subset $A \subseteq E$ is free if every point of A is an extreme point of the convex hull of A. A circuit consists of the vertices of a simplex together with a point in the relative interior of the simplex. The interior point is the root of the circuit. So the size of a circuit is at least 3 and at most $n + 2$.

It is clear from these examples that the circuits of an antimatroid do not determine the greedoid. For instance, a poset $P = (E, \leqq)$ and the dual poset $P^* = (E, \geqq)$ in general have different poset greedoids, but these greedoids have the same circuits. However, an antimatroid *is* determined by its *rooted circuits* (C, a), i.e. pairs such that C is a circuit with root a.

8.7.11. Proposition. *Let $(E, \mathscr{F})$ be an antimatroid and $A \subseteq E$. Then $A \in \mathscr{F}$ if and only if $C \cap A \neq \{a\}$, for every rooted circuit (C, a).*

Proof. The condition is necessary for a feasible set, by condition (iii) of Lemma 8.7.10. To prove sufficiency, suppose that $A \notin \mathscr{F}$, or, equivalently, that $E - A$ is not convex. Let $x \in A \cap \tau(E - A)$, and let $D = \mathrm{ex}(\tau(E - A))$. That is, D is the unique minimal spanning subset of $E - A$ (cf. Proposition 8.7.2). Since $x \notin D$ and $x \in \tau(D)$, the set $D \cup x$ is not free, and hence contains some circuit C. Since D is free (being the set of extreme points of a convex set), we conclude that (C, x) is a rooted circuit. But $C \cap A = \{x\}$, since by construction $C - x \subseteq D \subseteq E - A$. □

The previous result suggests that a characterization of antimatroids in terms of rooted circuits might be possible. This is indeed so, as shown by the following axiomatization, which has a curious resemblance to the circuit axioms for matroids.

8.7.12. Theorem. *Let $\mathscr{C} \subseteq \{(C, a)\colon a \in C \subseteq E\}$ be a family of rooted subsets of a finite set E. Then $\mathscr{C}$ is the family of rooted circuits of an antimatroid if and only if the following two conditions hold.*

(CI1) *If $(C_1, a), (C_2, a) \in \mathscr{C}$, then $C_1 \not\subset C_2$.*
(CI2) *If $(C_1, a_1), (C_2, a_2) \in \mathscr{C}$, $a_1 \neq a_2$, and $a_1 \in C_1 \cap C_2$, then there exists $(C, a_2) \in \mathscr{C}$ such that $C \subseteq C_1 \cup C_2 - a_1$.*

8.8. Poset of Flats

The geometric lattice of a matroid has two different generalizations in greedoid theory, which coincide exactly for the class of interval greedoids. Neither of them determines the associated greedoid completely. Nevertheless, a substantial part of the structure theory of greedoids is captured by its order and lattice theoretic aspects.

8.8.A. Poset Representations and Flats

We will now construct the poset of flats of a greedoid as its 'most efficient' poset representation, define the poset of closed sets of a greedoid (which requires a more complicated ordering than inclusion), and then study the canonical map from the poset of flats to the poset of closed sets.

In section 8.2 we described how a greedoid $(E, \mathscr{F})$ can be described by the Hasse diagram of the poset $(\mathscr{F}, \subseteq)$, in which every edge (covering relation) $X \lessdot Y$ is labeled by the 1-element set $Y - X$. From this labeled poset, the language $\mathscr{L}$ can explicitly be read off: the words of $\mathscr{L}$ are uniquely given by the label sequences along the unrefinable chains in the poset $(\mathscr{F}, \subseteq)$ that start at the least element $\varnothing$.

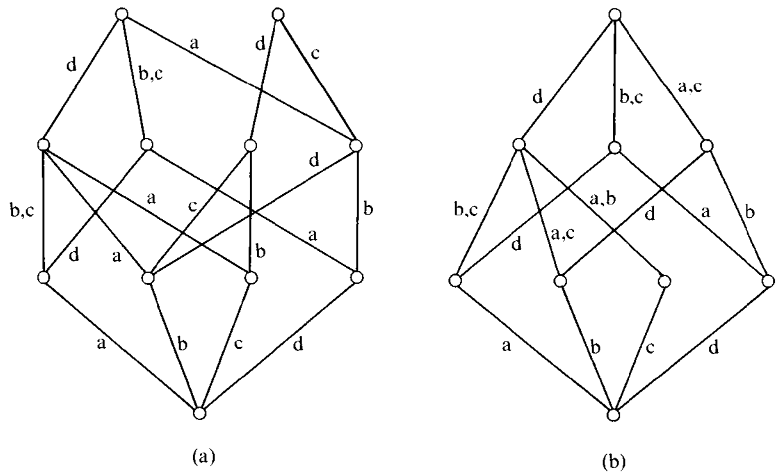
However, it is more efficient to allow larger sets of labels; often the greedoid can be given by the Hasse diagram of a smaller poset P with least element $\hat{0}$, whose edges $s \lessdot t$ are labeled by sets $\lambda(s \lessdot t) \subseteq E$ of alternative labels. A *poset representation* of a greedoid $(E, \mathscr{L})$ is such a set-labeled poset from which $\mathscr{L}$ arises (without repetitions) as the collection of words along unrefinable chains starting at $\hat{0}$ that pick exactly one letter from each label set. That is,

$$\mathscr{L} = \{x_1x_2 \ldots x_k : x_i \in \lambda(s_{i-1} \lessdot s_i) \text{ for } 1 \leqq i \leqq k, \text{ where } k \geqq 0 \text{ and } \hat{0} = s_0 \lessdot s_1 \lessdot \ldots \lessdot s_k \text{ is a chain in } P\}. \tag{8.12}$$

Here repetitions do not occur if and only if $\lambda(s \lessdot t_1) \cap \lambda(s \lessdot t_2) = \emptyset$ whenever $t_1 \neq t_2$ both cover $s \in P$.

For example, Figure 8.10 gives two poset representations of the greedoid that we have previously described in Figure 8.2. Set brackets for the label sets are again omitted.

Figure 8.10.



How do such poset representations arise? Whenever the same set of words can be read off above two different poset elements, these elements can be identified, resulting in a smaller poset and a more efficient poset representation. Now every poset element s corresponds to a set of words – the words in $\mathscr{L}$ that are coded along maximal chains from $\hat{0}$ to s. Thus two poset elements s_1 and s_2 can be identified if the words corresponding to s_1 and s_2 have the same continuations. This suggests the following construction for the most efficient (or universal) poset representation, using the contraction of greedoid languages as defined by (8.6).

8.8.1. Definition. Let $(E, \mathscr{L})$ be a greedoid. We define an equivalence relation on $\mathscr{L}$ by

$$\alpha \sim \beta \Leftrightarrow \mathscr{L}/\alpha = \mathscr{L}/\beta, \tag{8.13}$$

that is, α and β are equivalent if they have the same set of continuations.

The equivalence classes $[\alpha] \in \mathscr{L}/\sim$ with respect to this relation are the *flats* of the greedoid $\mathscr{L}$. The *poset of flats* of the greedoid $(E, \mathscr{L})$ is

$$\Phi = (\mathscr{L}/\sim, \leqq),$$

where the flats are ordered by

$$[\alpha] \leqq [\beta] \Leftrightarrow \alpha\gamma \sim \beta, \text{ for some } \gamma \in \mathscr{L}/\alpha. \tag{8.14}$$

The *labeled poset of flats* $\hat{\Phi}$ is the poset Φ together with the edge labeling λ of the Hasse diagram of Φ that associates to every covering relation $[\alpha] <\cdot [\beta]$ in Φ the set

$$\lambda([\alpha] < [\beta]) = \{x \in E \colon \alpha x \sim \beta\}. \tag{8.15}$$

The verification that $\leqq$ and λ are well defined by (8.14) and (8.15) is straightforward.

Since $\alpha \sim \beta$ implies $|\alpha| = |\beta|$, the rank function on $\mathscr{L}$ carries over to $\mathscr{L}/\sim$, with $r([\alpha]) = |\alpha|$, for all $\alpha \in \mathscr{L}$. This makes Φ into a graded poset of rank r; its unique minimal element is $[\varnothing]$ and its unique maximal element is the equivalence class of all basic words.

Now since words with the same support are always related ($\tilde{\alpha} = \tilde{\beta}$ implies $\alpha \sim \beta$), one can use the equivalence between greedoids and greedoid languages, as described in section 8.2, to give an equivalent definition of the poset of flats in set-theoretic terms. For this, let $(E, \mathscr{F})$ be a greedoid, and for X, $Y \in \mathscr{F}$ define $X \sim Y$ if $\mathscr{F}/X = \mathscr{F}/Y$. From this we get a poset of flats $\tilde{\Phi} = (\mathscr{F}/\sim, \leqq)$, where '$\leqq$' is the partial order induced by inclusion, that is, $[X] \leqq [Y]$ if and only if $X \cup Z \sim Y$ for some $Z \in \mathscr{F}/X$. Note that this in particular implies $[X] \leqq [Y]$ whenever $X \subseteq Y$.

8.8.2. Proposition. *The map given by* $[\alpha] \mapsto [\tilde{\alpha}]$ *is an isomorphism of posets* $\Phi \cong \tilde{\Phi}$.

This proposition (which, using Proposition 8.2.3, is easy to verify) shows that there is an essentially unique concept of the poset of flats $\Phi \cong \tilde{\Phi}$ and of the labeled poset of flats $\hat{\Phi}$.

The labeled poset $\hat{\Phi} = (\Phi, \lambda)$ is *universal* as a poset representation of $\mathscr{L}$, in the sense that for every poset representation (P, λ') of $\mathscr{L}$, there is a unique, order preserving, rank preserving surjective map

$$f \colon P \longrightarrow \Phi$$

such that for $s, t \in P$, $s <\cdot t$ implies that

$$\lambda'(s < t) \subseteq \lambda(f(s) < f(t)).$$

For example, consider again the greedoid depicted in Figure 8.2. Its labeled poset of flats is given by Figure 8.10b. The canonical maps from the poset representations in Figures 8.2b and 8.10a to the universal representation are easy to see.

Here are descriptions of the poset of flats Φ for some important classes of greedoids.

(1) In the case of matroids, Φ is (isomorphic to) the geometric lattice of flats, since for two independent sets X and Y, $X \sim Y$ holds exactly if X and Y have the same closure. The edge labeling of $\hat{\Phi}$ is then given by

$$\lambda([X] <\cdot [Y]) = \sigma(Y) - \sigma(X).$$

(2) If $(E, \mathcal{F})$ is a greedoid with only one basis, then $\Phi \cong (\mathcal{F}, \subseteq)$, and λ is the labeling by 1-element sets discussed in section 8.2. This includes the case of all antimatroids, so by Theorem 8.7.6 we see that the poset of flats of an antimatroid is a join-distributive lattice.

More generally, for every greedoid $(E, \mathcal{F})$ the canonical surjective poset map $f: (\mathcal{F}, \subseteq) \to \Phi$, defined by $f(X) = [X]$, is injective on each interval $[X, Y]$ in $\mathcal{F}$.

(3) Let $(E, \mathcal{F})$ be a branching greedoid on a rooted undirected or directed graph, as in section 8.3.C. Clearly, two branchings X and Y are related, $X \sim Y$, if and only if they reach the same set of vertices. Thus there is a bijection between $\mathcal{F}/\sim$ and feasible vertex sets. One sees from this that the poset of flats of the branching greedoid is isomorphic to the poset of feasible sets of the associated vertex search greedoid, and is hence a join-distributive lattice.

8.8.B. Poset of Closed Sets

From the example in Figure 8.10b we can see that for greedoids without the interval property the flats cannot be identified with closed sets – the greedoid has four flats, but only three closed sets of rank 1. What is the structure on the collection of closed sets? How does it relate to the poset of flats?

It is not natural to order the closed sets by inclusion – the resulting posets have little structure and do not seem to encode relevant information. An instructive example is the full greedoid with exactly one basic word: xy, whose poset of flats is a 3-element chain but whose closed sets are $\{x\}$, $\{y\}$ and $\{x, y\}$. Instead, examples such as this suggest ordering the closed sets by

$$A \leqq B \text{ if } B \text{ contains a basis of } A.$$

Equivalently, we could put

$$A \leqq B \text{ if } r(A \cap B) = r(A). \tag{8.16}$$

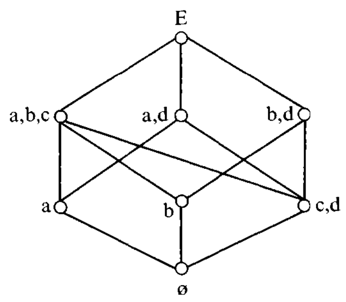
Clearly, this generalizes the matroid case. However, it turns out that for non-interval greedoids, the relation '$\leqq$' defined by (8.16) is not in general transitive. (For example, if $E = \{a, b, c, d, e\}$ and $(E, \mathcal{F})$ is the greedoid defined by $\mathcal{F} = 2^E - \{\{a, b\}, \{b, c, d\}\}$, then $\{a, b\}$, $\{b, c, d\}$, and $\{c, d, e\}$ are closed sets of ranks 1, 2, and 3, respectively. Here $\{a, b\} \leqq \{b, c, d\}$ and $\{b, c, d\} \leqq \{c, d, e\}$, but $r(\{a, b\} \cap \{c, d, e\}) = r(\emptyset) = 0 < 1 = r(\{a, b\})$.) We are therefore led to consider the transitive closure of the relation defined by (8.16).

8.8.3. Definition. The *poset of closed sets* of a greedoid is the set $\mathscr{C}\ell = \{\sigma(A) : A \subseteq E\}$ together with the partial order for which $A \leqq B$ holds if and only if there are closed sets $A_0 = A, A_1, \ldots, A_k = B$ such that for all i $(1 \leqq i \leqq k)$, A_i contains a basis of A_{i-1}, that is, $r(A_i \cap A_{i-1}) = r(A_{i-1})$.

The poset of closed sets has reasonable combinatorial properties. It is graded, and the poset and greedoid rank functions coincide for it. It is clear that $A \subseteq B$ implies $A \leqq B$, but not conversely.

For the greedoid of Figure 8.2, whose poset of flats is given by Figure 8.10b, the poset of closed sets is drawn in Figure 8.11.

Figure 8.11.



For matroids, antimatroids, and branching greedoids, the poset of flats Φ (as constructed before) and the poset of closed sets $\mathscr{C}\ell$ are canonically isomorphic. This is explained by the following result.

Let $(E, \mathscr{L})$ be a greedoid. If $\alpha \sim \beta$ for $\alpha, \beta \in \mathscr{L}$, then $\tilde{\alpha} \sim \tilde{\beta}$ by Proposition 8.8.2, from which $\sigma(\tilde{\alpha}) = \sigma(\tilde{\beta})$ follows. Hence, we have a well defined map

$$\phi : [\alpha] \mapsto \sigma(\tilde{\alpha})$$

from flats to closed sets.

8.8.4. Theorem. *The map $\phi : \Phi \rightarrow \mathscr{C}\ell$ is order preserving, rank preserving, and surjective. Furthermore, the following conditions are equivalent:*

(i) *ϕ is an isomorphism of posets;*
(ii) *ϕ is injective;*
(iii) *$(E, \mathscr{L})$ is an interval greedoid.*

The greedoid of Figure 8.2 again illustrates this: the map ϕ from Φ (Figure 8.10b) to $\mathscr{C}\ell$ (Figure 8.11) is obvious. The interval property fails (e.g. c, $dac \in \mathscr{L}$, but $dc \notin \mathscr{L}$) and ϕ is not injective ($[c] \neq [d]$, although $\phi([c]) = \phi([d]) = \{c, d\}$).

Observe that in general the composite map

$$\mathscr{F} \rightarrow \mathscr{F}/\sim \;\cong \Phi \rightarrow \mathscr{C}\ell$$

is the closure operator σ on $\mathscr{F}$. This illustrates the divergence of the concepts of flats and closed sets in non-interval greedoids: the closure operator factorizes over the flats.

8.8.C. Interval Greedoids and Semimodular Lattices

Interval greedoids are intimately related to semimodular lattices both via the poset of feasible sets $(\mathscr{F}, \subseteq)$ and via the poset of flats Φ. In fact, it is reasonable to view interval greedoids as the combinatorial models for semimodular lattices much as matroids and antimatroids are the combinatorial models for geometic and join-distributive lattices respectively.

Here are some key facts about the lattice property and semimodularity in posets of feasible sets.

8.8.5. Proposition. *Let $(E, \mathscr{F})$ be a greedoid. Then*

(i) *$(E, \mathscr{F})$ is an antimatroid $\Leftrightarrow$ $(\mathscr{F}, \subseteq)$ is a semimodular lattice;*

(ii) *$(E, \mathscr{F})$ is an interval greedoid $\Leftrightarrow$ all closed intervals $[\varnothing, X]$ in $(\mathscr{F}, \subseteq)$ are semimodular lattices;*

(iii) *if $(E, \mathscr{F})$ is an interval greedoid, then $(\mathscr{F}, \subseteq)$ is a meet-semilattice.*

Proof. Part (i) is from Proposition 8.7.5, and part (ii) follows from it via Proposition 8.2.8.

For part (iii), let $\hat{\mathscr{F}}$ denote the poset $(\mathscr{F}, \subseteq)$ with a maximal element $\hat{1}$ adjoined. To see that meets (greatest lower bounds) exist in $(\mathscr{F}, \subseteq)$, it suffices (by a standard lattice-theoretical argument) to show that any pair $X, Y \in \mathscr{F}$ has a join $X \vee Y$ (least upper bound) in $\hat{\mathscr{F}}$. Now, if X and Y have an upper bound Z in $(\mathscr{F}, \subseteq)$, so that $X \cup Y \subseteq Z \in \mathscr{F}$, then from Propositions 8.2.7(ii) and 8.2.8(ii) we conclude that $X \cup Y \in \mathscr{F}$ and therefore $X \vee Y = X \cup Y$. If X and Y do not have an upper bound in $(\mathscr{F}, \subseteq)$, then $X \vee Y = \hat{1}$. □

We remark in connection with the preceding result that for non-interval greedoids the meet operation on $(\mathscr{F}, \subseteq)$ may or may not exist (both cases occur).

Some of the special structure of the poset $(\mathscr{F}, \subseteq)$ for interval greedoids carries over to the poset of flats Φ, and to its labeled version $\hat{\Phi}$.

8.8.6. Lemma. *Let $(E, \mathscr{L})$ be an interval greedoid, and $\alpha x, \alpha y \in \mathscr{L}$ with $[\alpha x] \neq [\alpha y]$. Then $\alpha xy, \alpha yx \in \mathscr{L}$ and $[\alpha xy] = [\alpha yx]$.*

Proof. This is a reformulation of the transposition property for interval greedoids, as observed in the last paragraph of section 8.3.G. □

The lemma shows that for interval greedoids, Φ is a *semimodular poset*: if $s \in \Phi$ is covered by two different elements $t_1, t_2 \in \Phi$, then there is an element $u \in \Phi$ covering both t_1 and t_2. More precisely, the following is true.

8.8.7. Theorem. *If $(E, \mathscr{F})$ is an interval greedoid, then the poset of flats Φ is a semimodular lattice. Conversely, every finite semimodular lattice arises as the poset of flats of some interval greedoid.*

Proof. (1) To prove the lattice property, we use the following simple lemma: a finite poset P having a least element is a lattice if, for $s_1, s_2, t \in P$, the join $s_1 \vee s_2$ exists whenever s_1 and s_2 both cover t.

Hence, here we only have to show that in the situation of Lemma 8.8.6, $[\alpha xy]$ is the only minimal upper bound for $[\alpha x]$ and $[\alpha y]$. Assume that on the contrary $[\alpha x\gamma] = [\alpha y\delta]$ is a different minimal upper bound (so that $[\alpha x\gamma] \ngeq [\alpha xy]$), with $\gamma = c_1 c_2 \dots c_k$. We will prove by induction that $\alpha x c_1 \dots c_i y \in \mathscr{L}$ and $\alpha x y c_1 \dots c_i \in \mathscr{L}$, for $0 \leqslant i \leqslant k$, which will lead to a contradiction.

The case $i = 0$ is clear. For $i > 0$ we know by induction that $\alpha x c_1 \dots c_{i-1} y \in \mathscr{L}$ and $\alpha x y c_1 \dots c_{i-1} \in \mathscr{L}$, and hence $[\alpha x c_1 \dots c_{i-1} y] = [\alpha x y c_1 \dots c_{i-1}] \geqslant [\alpha xy]$. On the other hand, for $\alpha x c_1 \dots c_i \in \mathscr{L}$ we have $[\alpha x c_1 \dots c_i] \ngeq [\alpha xy]$ by assumption. Hence, $[\alpha x c_1 \dots c_{i-1} y] \neq [\alpha x c_1 \dots c_i]$, and Lemma 8.8.6 implies that $\alpha x c_1 \dots c_i y \in \mathscr{L}$. Augmenting $\alpha x y c_1 \dots c_{i-1}$ from this yields $\alpha x y c_1 \dots c_i \in \mathscr{L}$, and the induction is complete.

We have in particular shown that $\alpha x c_1 \dots c_k y = \alpha x \gamma y \in \mathscr{L}$. Since $[\alpha x\gamma] = [\alpha y\delta]$ if follows that $\alpha y \delta y \in \mathscr{L}$, which is impossible since $\mathscr{L}$ is a simple language.

(2) For the converse, let L be a finite semimodular lattice, and let $E = J(L)$ be the set of join-irreducible elements of L, that is, the set of those lattice elements that cover exactly one element in L. We label the edges in the Hasse diagram of L by

$$\lambda(s <\cdot\, t) = \{p \in E \colon s \vee p = t\}.$$

Let $\mathscr{L}$ be the left hereditary language defined by the poset representation (L, λ) as in (8.12). Then $(E, \mathscr{L})$ is an interval greedoid, and L is isomorphic to its poset of flats. We leave the straightforward verification to the reader. □

Contrary to what one might expect, semimodularity of Φ does *not* characterize the interval greedoids. This is shown e.g. by the non-interval greedoid with 6 basic words, *xab*, *xba*, *yab*, *yba*, *xay*, and *yax*, whose poset of flats is a semimodular lattice (the unique non-modular semimodular lattice of rank 3 and order 7).

8.8.8. Lemma. *Let $(E, \mathscr{F})$ be an interval greedoid and $\hat{\Phi} = (\Phi, \lambda)$ the labeled poset of flats. If $t_1, t_2 \in \Phi$ and $t_1 \wedge t_2 <\cdot\, t_1$, then $t_2 <\cdot\, t_1 \vee t_2$ and*

$$\lambda(t_1 \wedge t_2 <\cdot\, t_1) \subseteq \lambda(t_2 <\cdot\, t_1 \vee t_2). \tag{8.17}$$

Proof. The first claim is true in any semimodular lattice. If $t_1 \wedge t_2 \lessdot t_2$, then (8.17) follows directly from Lemma 8.8.6. In general, (8.17) has to be proven by induction on $\operatorname{rank}(t_2) - \operatorname{rank}(t_1 \wedge t_2)$, using Lemma 8.8.6 repeatedly. □

We conclude with the following application of the results of this section.

Proof of Proposition 8.2.5 (necessity). Let $(E, \mathscr{L})$ be an interval greedoid and let $\alpha = a_1 a_2 \dots a_k$, $\beta = b_1 b_2 \dots b_l \in \mathscr{L}$, $k > l$. Now, α determines an unrefinable chain $\hat{0} = s_0 \lessdot s_1 \lessdot \dots \lessdot s_k$ in Φ by $s_i = [a_1 a_2 \dots a_i]$ for $0 \leqq i \leqq k$. Let $t = [\beta]$. Since, by Theorem 8.8.7, Φ is a semimodular lattice, each step in the following chain is an equality or a covering:

$$t = t \vee s_0 \leqq t \vee s_1 \leqq \dots \leqq t \vee s_k.$$

Now let $1 \leqq i_1 < i_2 < \dots < i_m \leqq k$ be the sequence of those indices i_j for which

$$t \vee s_{i_j - 1} \lessdot t \vee s_{i_j}.$$

Thus we have an unrefinable chain

$$t \lessdot t \vee s_{i_1} \lessdot t \vee s_{i_2} \lessdot \dots \lessdot t \vee s_{i_m} = t \vee s_k.$$

Clearly, $m = \operatorname{rank}(t \vee s_k) - \operatorname{rank}(t) \geqq k - l$.

Now, $a_i \in \lambda(s_{i-1} \lessdot s_i)$ implies by Lemma 8.8.8 that $a_{i_j} \in \lambda(t \vee s_{i_j - 1} \lessdot t \vee s_{i_j})$ for $1 \leqq j \leqq m$. Hence, the definition of Φ allows us to read off $\beta a_{i_1} a_{i_2} \dots a_{i_m} \in \mathscr{L}$, where $\alpha' = a_{i_1} a_{i_2} \dots a_{i_m}$ is a subword of α, of length $|\alpha'| = m \geqq k - l$. □

8.8.D. Poset Properties

The unlabeled poset $(\mathscr{F}, \subseteq)$ carries important but incomplete information about a greedoid $(E, \mathscr{F})$. We will here discuss which greedoid properties and invariants are *poset properties*, that is, completely determined by the abstract poset $(\mathscr{F}, \subseteq)$ and not requiring explicit knowledge of the set system $\mathscr{F}$.

We have seen that $(\mathscr{F}, \subseteq)$ is a finite graded poset of rank r and size $|\mathscr{F}|$, with minimum element $\hat{0} = \varnothing$ and $|\mathscr{B}|$ maximal elements. The number of unrefinable chains from $\hat{0}$ to some $X \in \mathscr{F}$ is $|\mathscr{L}|$. From this it is clear that r, $|\mathscr{F}|$, $|\mathscr{B}|$, $|\mathscr{L}|$, and the number of basic words are poset properties.

As a contrast, $|E|$ and $|\cup\mathscr{F}|$ are *not* poset properties. This follows e.g. from the observation, made in section 8.4.E, that the branching greedoids of a rooted graph and its associated digraph have isomorphic posets $(\mathscr{F}, \subseteq)$. Thus, any greedoid property that distinguishes the two branching greedoids cannot be a poset property. For example, although $\lambda(1)$ is the number of bases, the greedoid polynomial $\lambda(t)$ is not a poset invariant, and neither are the evaluations $\lambda(2)$ and $\lambda(0)$ (i.e. the number of spanning sets and the Euler characteristic of the dual complex).

The interval property and that of being a matroid, antimatroid, poset, or local poset greedoid are all poset properties. This follows from the following information, which was gathered in earlier sections:

$(E, \mathscr{F})$	$(\mathscr{F}, \subseteq)$
interval greedoid $\Leftrightarrow$	all intervals are semimodular lattices
local poset greedoid $\Leftrightarrow$	all intervals are distributive lattices
matroid $\Leftrightarrow$	all intervals are Boolean lattices
antimatroid $\Leftrightarrow$	semimodular lattice
poset greedoid $\Leftrightarrow$	distributive lattice

Next we note that k-connectivity is a poset property by definition. In fact, we can define a ranked poset P of rank r with minimal element $\hat{0}$ to be *k-connected* if for every $X \in P$ there is an element $Y \geq X$ in P such that the interval $[X, Y]$ of P is a Boolean lattice of rank $\min\{k, r - r(X)\}$. With this, a greedoid $(E, \mathscr{F})$ is k-connected if and only if the poset $(\mathscr{F}, \subseteq)$ of feasible sets is k-connected.

8.8.9. Proposition. *For a k-connected greedoid $(E, \mathscr{F})$, the poset Φ of flats is also k-connected. If $(E, \mathscr{F})$ is an interval greedoid, then the converse is also true.*

Proof. The first part follows from the remark, made near the end of section 8.8.A, that every restriction of the natural map $(\mathscr{F}, \subseteq) \to \Phi$ to an interval is injective. The second part follows from Lemma 8.8.8. □

Observe that this proposition implies in particular the equivalence (ii)$\Leftrightarrow$(iii) of Proposition 8.4.7, since the poset of flats of a branching greedoid equals the poset of feasible sets of the associated vertex search greedoid.

8.9. Further Topics

8.9.A. Excluded Minor Characterizations

For each of the main classes of greedoids arising among the examples, there is a representation problem: how do we recognize whether an abstractly given greedoid is isomorphic to some greedoid in that class? This problem is in most cases unsolved. For instance, no effective way is known for telling whether a given antimatroid can be represented as the convex pruning greedoid of a point set in $\mathbb{R}^n$.

There are two main ways of characterizing a class of greedoids, either by structural conditions or by excluded minors. The following are examples of structural characterizations: Let $G = (E, \mathscr{F})$ be a greedoid. Then

(1) G is an interval greedoid if and only if $X, Y \subseteq Z$ implies $X \cup Y \in \mathscr{F}$, for all $X, Y, Z \in \mathscr{F}$;
(2) G is a local poset greedoid if and only if $X, Y \subseteq Z$ implies $X \cup Y$, $X \cap Y \in \mathscr{F}$, for all $X, Y, Z \in \mathscr{F}$.

These statements are merely reformulations of information from Propositions 8.2.7, 8.2.8, and 8.7.7. The following result is more substantial.

8.9.1. Theorem. *G is a directed branching greedoid if and only if G is a local poset greedoid and* $\sigma(X) \cap \sigma(Y) \subseteq \sigma(X \cup Y) \subseteq \sigma(X) \cup \sigma(Y)$, *for all* $X, Y \in \mathscr{F}$.

Hereditary classes of greedoids, i.e. classes closed under taking minors, can be specified by listing the minimal non-members. (Here 'minimal' is to be understood as referring to the partial ordering of isomorphism classes of greedoids induced by the relation 'is a minor of', as discussed in section 8.4.D.) These minimal non-members are the *excluded minors* of the hereditary class, and clearly a greedoid is a member of the class precisely when none of its minors is among the excluded ones. Examples of hereditary classes are interval greedoids, local poset greedoids, directed and undirected branching greedoids, polymatroid greedoids, antimatroids, and matroids.

There is no *a priori* reason to expect the list of excluded minors for a hereditary class to be finite or effectively describable. For the classes that were mentioned the following can be said.

Let $E = \{a, b\}$ and $\mathscr{F} = 2^E - \{\{b\}\}$, $\mathscr{F}' = 2^E - \{\{a, b\}\}$. Clearly, $G = (E, \mathscr{F})$ is not a matroid and $G' = (E, \mathscr{F}')$ is not an antimatroid, so they are among the excluded minors for these classes. In fact, they are the only excluded minors.

8.9.2. Proposition.

(i) *A greedoid is a matroid if and only if it has no minor isomorphic to G.*

(ii) *A greedoid is an antimatroid if and only if it has no minor isomorphic to G′.*

(iii) *The classes of interval greedoids, directed branching greedoids, undirected branching greedoids, local poset greedoids, and polymatroid greedoids cannot be characterized by a finite set of excluded minors.*

Proof. (i) In a greedoid that is not a matroid there are feasible sets having non-feasible subsets. Pick one such feasible set X of minimal cardinality. Then for some $a \in X$ the subset $X - a$ is non-feasible, and accessibility gives that $X - b$ is feasible for some $b \in X$. The choice of X implies that $Y = X - \{a, b\}$ is feasible. Now, restriction to X and contraction by Y produces a minor isomorphic to G.

(ii) The lack of minors of type G' is equivalent to the following property: if X, Y and $X \cap Y$ are feasible and $|X| = |Y| = |X \cap Y| + 1$, then $X \cup Y$ is feasible. This implies that the poset of feasible sets is a semimodular lattice, which by Proposition 8.7.5 implies that the greedoid is an antimatroid.

(iii) For $k = 1, 2, \ldots$, let α be a simple word of length k not containing the letters x and y. Let G_k denote the full greedoid with exactly two basic words: $x\alpha y$ and $y\alpha x$. The following facts are easy to verify: (1) G_k lacks the interval property; (2) every proper minor of G_k is a branching greedoid (both directed and undirected). It follows that G_k is a minimal non-member for each of the five hereditary classes, and hence that $(G_k)_{k\geq 1}$ is an infinite list of excluded minors. □

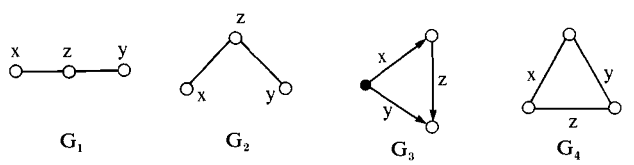
In spite of what has just been shown in turns out that undirected branching greedoids and local poset greedoids *can* be characterized by finite sets of excluded minors. The requirement for this is that attention must be restricted to the class of interval greedoids only.

Let $E = \{x, y, z\}$ and define greedoids $G_i = (E, \mathscr{F}_i)$, $i = 1, 2, 3, 4$, by (with some obvious simplifications of set notation)

$$\mathscr{F}_1 = 2^E - \{z\},$$
$$\mathscr{F}_2 = 2^E - \{z, xz, yz\},$$
$$\mathscr{F}_3 = 2^E - \{z, yz, xyz\},$$
$$\mathscr{F}_4 = 2^E - \{xyz\}.$$

In Figure 8.12 these greedoids G_1–G_4 are represented as a vertex pruning of a tree, a poset, a directed branching greedoid, and a graphic matroid, respectively.

Figure 8.12.



8.9.3. Theorem. *Let G be an interval greedoid. Then*

(i) *G is a local poset greedoid if and only if G has no minor isomorphic to* G_1.
(ii) *G is an undirected branching greedoid if and only if G has no minor isomorphic to* G_1, G_2, G_3, *or* G_4.

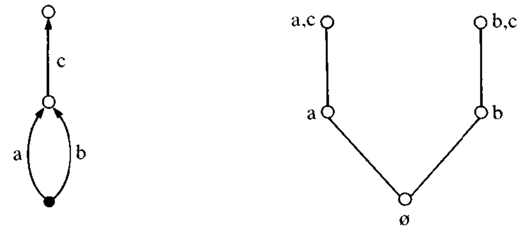
8.9.B. Diameter of the Basis Graph

Let $(E, \mathscr{F})$ be a greedoid of rank r, and $\mathscr{B}$ its set of bases. Two bases X, Y are *adjacent* if they differ in exactly one element and their intersection is feasible, that is, $|X \cap Y| = |X| - 1$ and $X \cap Y \in \mathscr{F}$. By definition, the *basis graph* of $(E, \mathscr{F})$ has vertex set $\mathscr{B}$, and two bases are joined by an edge whenever they are adjacent.

We ask under which conditions the basis graph of a greedoid is connected and what can be said about the diameter of basis graphs.

For matroids, the feasibility condition $X \cap Y \in \mathscr{F}$ is always true. Then the exchange axiom (G2) produces a sequence of adjacent bases between X and Y, which shows that the basis graph of a matroid is always connected and

Figure 8.13.



(a) (b)

has diameter at most r. On the other hand, the distance between any two disjoint bases of a matroid is exactly r.

Questions of the connectedness and the diameter for basis graphs of general greedoids are less trivial. For example the digraph of Figure 8.13a has a branching greedoid (Figure 8.13b) with disconnected basis graph; the intersection of its two bases is not feasible. The branching greedoid of the digraph in Figure 8.3a has rank $r = 2$, whereas its basis graph is a path of length 3; here the diameter of the basis graph is larger than the rank.

In general, the following can be said.

8.9.4. Theorem. *Let $(E, \mathscr{F})$ be a 2-connected greedoid of rank r.*

(i) *The basis graph of $(E, \mathscr{F})$ is connected.*

(ii) *The diameter of the basis graph is at most $2^r - 1$. This bound is sharp.*

(iii) *If $(E, \mathscr{F})$ is a branching greedoid of rank $r > 0$, then the diameter of the basis graph is at most $r^2 - r + 1$. The bound is best-possible.*

Note that the branching greedoid of Figure 8.3 is 2-connected and serves as an extremal example for the case $r = 2$. In fact, this case can be used to prove connectedness of the basis graph, and the bound in (ii) on its diameter, by induction on r. Sharpness of the bound is then established by an explicit construction.

To prove (iii), one has to exploit the surjective map of section 8.8.A from the branching greedoid $\mathscr{F}$ to its poset of flats Φ, corresponding to the vertex search greedoid of the graph. The poset Φ – a semimodular, coatomic lattice – has special properties that can be lifted back to $\mathscr{F}$ and used there to construct paths in the basis graph of $\mathscr{F}$.

In general, higher connectivity of a greedoid decreases the possible diameter of its basis graph, although most arguments for k-connected greedoids with $k \geqq 3$ require the interval property.

8.9.5. Proposition. *Let $(E, \mathscr{F})$ be a k-connected interval greedoid of rank r, where $2 \leqq k \leqq r$. Then the diameter of its basis graph is at most $2^{r-k+1} \cdot k - 1$.*

This bound is not sharp in general. However, for $k = 2$ it reduces to the sharp bound of Theorem 8.9.4(ii). For $k = r$ it states that the diameter of the basis graph of an r-connected interval greedoid is at most $2r - 1$. This bound is sharp, even for branching greedoids. It is, however, still higher than the bound r for matroids.

8.9.C. Non-simple Greedoids: Chip Firing Games and Coxeter Groups

The way in which a greedoid is defined in section 8.2 requires that all feasible words are *simple* (i.e. have no letter occurring more than once). If this requirement is dropped, one gets a more general notion of greedoids, and such *non-simple* greedoids arise naturally in some examples.

Let E be a finite alphabet and $\mathscr{L} \subseteq E^*$ a finite language. Consider these axioms (the notation is explained in section 8.2):

(L1) if $\alpha = \beta\gamma$ and $\alpha \in \mathscr{L}$, then $\beta \in \mathscr{L}$;
(L2) if $\alpha, \beta \in \mathscr{L}$ with $|\alpha| > |\beta|$, then α contains a letter x such that $\beta x \in \mathscr{L}$;
(L2′) if $\alpha, \beta \in \mathscr{L}$ with $|\alpha| > |\beta|$, then α contains a subword α' of length $|\alpha'| = |\alpha| - |\beta|$ such that $\beta\alpha' \in \mathscr{L}$.

In this section only (and in Exercises 8.36–8.38) we shall use the following definitions: $(E, \mathscr{L})$ is a *greedoid* if it satisfies (L1) and (L2), and a *strong greedoid* if it satisfies (L1) and (L2′). Thus, what was called a 'greedoid' and an 'interval greedoid' in section 8.2 would be called respectively a 'simple greedoid' and a 'simple strong greedoid' here.

Much of the theory of simple greedoids, as developed in previous sections, breaks down for non-simple greedoids. This is to a large extent due to the lack of an unordered, or set-theoretic, version. However, some parts of the theory that rely only on the ordered, or language-theoretic, version, survive the generalization. In particular, each greedoid $(E, \mathscr{L})$ has a *poset of flats* Φ (and a labelled poset of flats $\hat{\Phi}$), defined exactly as in Definition 8.8.1.

A proper subclass of greedoids, for which there is an unordered version, is given by the following definition: a finite language $(E, \mathscr{L})$ is a *polygreedoid* if it satisfies (L1) and

(L2″) If $\alpha, \beta \in \mathscr{L}$, $|\alpha| > |\beta|$, then there is some letter x, occurring more times in α than in β, such that $\beta x \in \mathscr{L}$.

Clearly, all simple greedoids are polygreedoids. Also, the polygreedoids for which all permutations of a feasible word are feasible are equivalent to the 'integral polymatroids' of J. Edmonds.

The two exchange axioms (L2′) and (L2″) are logically independent. Any simple greedoid without the interval property satisfies (L2″) but not (L2′).

Conversely, an example will be given after Proposition 8.9.7 of a greedoid that satisfies (L2′) but not (L2″).

Define the *support* $\tilde{\alpha}$ of a word α as the *multiset* of letters in α. For example, if α = loophole then $\tilde{\alpha} = \{e, h, l^2, o^3, p\}$. The support of a language $\mathscr{L}$ is the multiset system $\tilde{\mathscr{L}} = \{\tilde{\alpha}\colon \alpha \in \mathscr{L}\}$.

In general, a greedoid $\mathscr{L}$ cannot be uniquely recovered from its support $\tilde{\mathscr{L}}$, but this *is* the case if $\mathscr{L}$ is a polygreedoid. The multiset systems that are the supports of polygreedoids can be characterized by accessibility and a suitable exchange axiom, and this permits an equivalent unordered formulation; see Exercise 8.37 for the precise statement (which extends Proposition 8.2.3). Because of this special property, polygreedoids occupy a middle ground between simple greedoids and general greedoids, and several facts from simple greedoid theory have straightforward extensions to polygreedoids.

Let us look at two situations where non-simple greedoids arise.

Suppose that we have a finite connected rooted graph (V, E, r) and an integer $k > 0$. This gives rise to the following *chip firing game*.

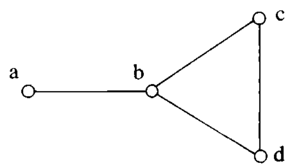
Think of the graph as drawn on a desk top. We have k chips that during the game are placed on and moved around among the vertices. By a *chip configuration* we mean a multiset $A\colon V \to \mathbb{N}$, $|A| = k$, which denotes that $A(v)$ chips are lying on vertex v.

When the game starts all k chips lie in a pile on the root vertex r (other initial positions work equally well). At this time or any later time a *legal move* consists in firing a legal vertex. By this is meant the following. For a given chip configuration A, a vertex v is *legal* if $A(v) \geqq \deg(v)$, i.e. if there are at least as many chips on v as there are neighbors. To *fire* v then means to remove $\deg(v)$ chips from v and distribute them along the adjacent edges to v's neighbors, one to each. The game will *terminate* when a chip configuration is reached that permits no further legal move.

For instance, consider the graph in Figure 8.14, and let $k = 4$. If b is the root, the game will terminate after 2 moves. If a is the root the game will terminate after 6 moves. Finally, if c or d is the root the game will go on for ever, no matter how it is played.

The chip firing game determines a language $\mathscr{L} \subseteq V^*$ of legal firing sequences: $x_1 x_2 \ldots x_k \in \mathscr{L}$ if x_i is a legal vertex in the chip configuration

Figure 8.14.



obtained after firing $x_1, x_2, \ldots, x_{i-1}$, for $1 \leqq i \leqq k$. This language is hereditary (i.e. satisfies (L1)), and in general not simple. For instance, if b is the root of the graph in Figure 8.14 and $k = 6$, then $babcdbaa \in \mathscr{L}$ but $babcb \notin \mathscr{L}$.

Assume from now on that the chip firing game terminates for some sequence of legal moves.

8.9.6. Proposition. *The chip firing language* $(V, \mathscr{L})$ *is a strong polygreedoid. Its poset of flats is isomorphic to the poset of chip configurations.*

It is easy to verify axiom (L2″), and an approach to axiom (L2′) is suggested in Exercise 8.38. The poset of chip configurations consists of the legal configurations (those which can occur in a game) ordered as follows: $A \leqq B$ if and only if some sequence of legal moves transforms A into B.

The preceding result contains some combinatorial information about the chip firing game that is not *a priori* evident. For instance, one sees that every maximal sequence of legal moves has the same length, and also that there is a unique final chip configuration to which all such sequences lead.

The other example where non-simple greedoids arise comes from group theory. Let W be a group, S a generating subset, and assume that all elements in S are of order 2 (i.e. $s^{-1} = s$ for all $s \in S$). Then every group element $w \in W$ can be expressed as a product $w = s_1 s_2 \ldots s_k$, $s_i \in S$, and we call such an expression *reduced* if k is minimal, i.e. if w cannot be obtained as the product of a shorter sequence of generators. The reduced expressions can be thought of as words in the alphabet S, and the collection of all reduced expressions for all group elements forms a language $\mathscr{L} \subseteq S^*$. The language of reduced expressions $(S, \mathscr{L})$ has the following strong heredity property: if $\alpha = \beta\gamma$, $\alpha \in \mathscr{L}$, then, $\beta, \gamma \in \mathscr{L}$. In particular, it satisfies axiom (L1).

The pair (W, S) is called a *Coxeter group* if all relations among the generators are implied by pairwise relations of the form $(st)^{m(s,t)} = e$, $s, t \in S$. Examples of finite Coxeter groups are the symmetry groups of regular convex polytopes and the Weyl groups of simple complex Lie groups. In fact, every finite Coxeter group is of either of these types or a direct product of such.

The set W of group elements in a Coxeter group has a well known partial ordering, called *weak order* (or *weak Bruhat order*), which can be defined as follows: for $u, w \in W$, $u \leqq w$ if some reduced expression for u $(u = s_1 \ldots s_i)$ can be extended to a reduced expression for w $(w = s_1 \ldots s_i \ldots s_k, i \leqq k)$. The weak ordering of W is a graded lattice, if W is finite.

The symmetric group Σ_n of all permutations of $\{1, 2, \ldots, n\}$, together with the generating set of all adjacent transpositions $S = \{(i, i+1): 1 \leqq i \leqq n-1\}$, is a Coxeter group. For example, if $n = 3$ and $S = \{a, b\}$ one sees that the language of reduced expressions is $\mathscr{L} = \{\varnothing, a, b, ab, ba, aba, bab\}$, and the weak order is the hexagon lattice.

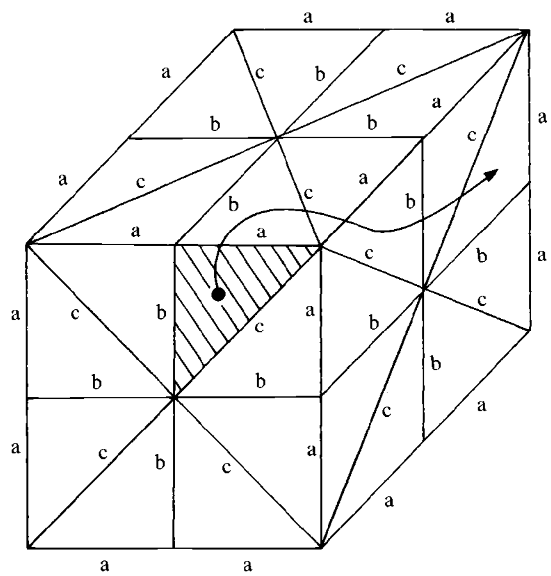
8.9.7. Proposition. *The language of reduced expressions* $(S, \mathscr{L})$ *of a finite Coxeter group* (W, S) *is a strong greedoid. Its poset of flats is isomorphic to the weak ordering of* W.

Coxeter group languages are not in general polygreedoids. For example, in the language of Σ_3 given above, '*ba*' can be augmented from '*aba*' only by '*b*', which occurs exactly once in both words.

The Coxeter group greedoids have an interesting geometric interpretation. In fact, they could be defined geometrically with no reference to group theory. We will sketch this geometric picture in one tangible special case only.

Let (W, S) be the symmetry group of the three-dimensional cube C. This is a Coxeter group of order 48, $|S| = 3$, and its language $(S, \mathscr{L})$ of reduced expressions is of rank 9 with 42 basic words. Let Δ be the barycentric subdivision of C's boundary. Then Δ consists of 48 triangles, and we pick one of these as a *root*; see Figure 8.15 where the root triangle is shaded.

Figure 8.15.



Consider now all *walks* from the root triangle T_0, by which we mean sequences of triangles $(T_0, T_1, \ldots, T_k)$ such that T_{i-1} and T_i are adjacent (share an edge) for $1 \leqq i \leqq k$. If the edges of Δ are labeled by a, b, and c as in Figure 8.15, then there is an obvious one-to-one correspondence between the walks from T_0 and the set S^* of all words in the alphabet $S = \{a, b, c\}$. For instance, the walk indicated in the figure corresponds to the word '*acabc*'.

Call a walk $(T_0, T_1, \ldots, T_k)$ *geodesic* if no shorter walk from T_0 to T_k exists. The *geodesic language* $\mathscr{L}' \subseteq S^*$ consists of all words that correspond to geodesic walks from T_0. It is clear by symmetry that $\mathscr{L}'$ does not depend on the choice of T_0.

The basic fact now is that $\mathscr{L}'$ *is isomorphic* to $\mathscr{L}$, i.e. the Coxeter group greedoid of the cube group is the same thing as the geodesic language of the subdivided cube. Similarly, the greedoid of any finite Coxeter group can be obtained as the geodesic language of a simplicial sphere, which in the case of polytopal groups is the subdivision of the boundary of the corresponding regular polytope.

The examples we have discussed in this section show that even when the alphabet E is finite it makes sense to consider *infinite* greedoids $\mathscr{L} \subseteq E^*$. A non-terminating chip firing game played on a finite graph, and also an infinite Coxeter group (W, S) with finite S (e.g. an affine Weyl group or the symmetry group of a sufficiently regular tesselation of $\mathbb{R}^d$), gives rise to such an infinite non-simple greedoid.

8.10. Notes and Comments

Section 8.1

Greedoids were introduced by Korte & Lovász (1981) and their basic properties were developed in Korte & Lovász (1983, 1984a). The theory has since then been extensively developed by its creators and others. A book-length exposition will appear in Korte, Lovász & Schrader (1991).

There had been some earlier attempts to develop order dependent versions of matroids, by Dunstan, Ingleton & Welsh (1972) and by Faigle (1979, 1980), but the more comprehensive work of Korte & Lovász seems originally to have been independent of these antecedents. In Korte & Lovász (1985a) Faigle's structures are shown to correspond to a certain class of interval greedoids (cf. section 8.3.E).

This chapter was written in 1986–7, and it covers most of the basic properties of greedoids known at that time. The forthcoming monograph by Korte, Lovász & Schrader (1991) will presumably be more comprehensive.

Section 8.2

Most of the material here is from the early papers of Korte & Lovász (1981, 1984a). However, Proposition 8.2.5 (due to Björner and Lovász) is from Björner (1985), and Proposition 8.2.7 is from Björner (1985) and Korte & Lovász (1984b). Interval greedoids without loops were studied under the name *selectors* in Crapo (1984); see also Korte & Lovász (1985c).

Antimatroids have been written about under several names: *alternative precedence structure* (*APS*) *greedoids*, *upper interval greedoids*, *anti-exchange greedoids*, *shelling structures*, and *locally free selectors*. Of these, the name *shelling structure* is very unfortunate, since *shelling* has a precise and well established meaning in combinatorics that finds use also in greedoid theory (cf. section 8.6.C). To add to the confusion, in some papers convex geometries (the dual objects to antimatroids) are called 'antimatroids'.

While on the subject of names, it could be mentioned that the suitability of the word *greedoid* itself has been heatedly debated. If already the name *matroid* is 'ineffably cacophonic', as was claimed by Crapo & Rota (1970), then *greedoid* is undoubtedly much worse. Alternative names which have been proposed for related structures include *exchange language* (Björner, 1985), *selector* (Crapo, 1984), and *exchange system* (Brylawski & Dieter, 1988). However, the name *greedoid* is distinctive and catchy, albeit slightly frivolous, and there is no doubt that it is here to stay.

Section 8.3

Section 8.3.*A*: Twisted matroids were defined in Björner (1985), and slimmed matroids were defined in Korte & Lovász (1984c) where also several procedures for 'slimming' a matroid were discussed. A notion of 'trimmed' matroids, equivalent to the notion of 'meet' defined in Exercise 8.11, appears in Korte & Lovász (1985b, 1989b).

Section 8.3.*B*: These standard examples of antimatroids and many others are described in e.g. Björner (1985), Edelman & Jamison (1985), and Korte & Lovász (1984a, 1984b). See also the comments for section 8.7.

Sections 8.3.*C and* 8.3.*D*: Branching greedoids originate in Korte & Lovász (1981, 1984a); polymatroid greedoids and local poset greedoids in Korte & Lovász (1985b).

Section 8.3.*E*: Faigle geometries were defined by Faigle (1979, 1980). The connection with greedoids was studied in Korte & Lovász (1985a).

Sections 8.3.*F and* 8.3.*G*: For retract greedoids, see Crapo (1984) and Korte & Lovász (1985c, 1986a). Transposition greedoids and dismantling greedoids were defined in Korte & Lovász (1986a). The original example of dismantling sequences is due to Duffus & Rival (1978).

Section 8.3.*H*: Gaussian greedoids are due to Goecke (1986, 1988). The concept was rediscovered by Serganova, Bagotskaya, Levit & Losev (1988). Axiomatic as well as algorithmic characterizations of this class of greedoids are known; see Goecke (1986, 1988) and Exercise 8.34. Medieval marriage greedoids were defined in Korte & Lovász (1986a). The name was coined by J. Edmonds in reference to some generic 'medieval' king, in whose opinion a sequence of suitors is feasible if and only if they will marry his daughters in decreasing order of age.

Section 8.3.*I*: Figure 8.5 is adapted from information in Korte & Lovász (1985e, 1986a).

Several examples of greedoids are known which have not been discussed here. For instance, Korte & Lovász (1984a, 1986a) have described several other classes of greedoids arising in graph theory: ear decomposition greedoids, blossom greedoids (Edmonds' matching algorithm), perfect elimination greedoids, series–parallel reduction greedoids, etc. Goecke, Korte & Lovász (1989) provide an extensive survey of examples.

Section 8.4

All material in sections 8.4.A–8.4.D comes from Korte & Lovász (1983), except that Theorem 8.4.1 was proven in Korte & Lovász (1984b). An alternative closure operation, called *kernel closure*, which is idempotent, monotone for interval greedoids, but not in general increasing, is defined and studied in Schmidt (1985a); see Exercise 8.9.

Two-connectivity in greedoids was defined and studied in Korte & Lovász (1985d), k-connectivity in Björner, Korte & Lovász (1985). The connectivity properties of branching greedoids are studied in more detail in Ziegler (1988).

Section 8.5

The material in sections 8.5.A and 8.5.B is from Korte & Lovász (1981, 1984a); see also Goecke, Korte & Lovász (1989). It should be said that there exist optimal discrete algorithms that are of a greedy nature, but which do not come from an underlying greedoid structure. For instance, no greedoid can be discerned behind the greedy algorithm for knapsack problems of Magazine, Nemhauser & Trotter (1975), also treated in Hu & Lenard (1976).

The algorithms of Dijkstra, Kruskal, and Prim are discussed in every book on combinatorial optimization; see Tarjan (1983) and the interesting historical discussion in Graham & Hell (1985). Korte & Lovász (1981, 1984a) show that some machine scheduling algorithms of Lawler also fit into the greedoid framework – the question is of optimizing some generalized bottleneck function over a poset greedoid.

The fact that Depth-first-search is not compatible with branching greedoids was pointed out by Korte & Lovász (1981, 1984a). They remark that the problem is NP-hard, since it includes the problem of finding a Hamiltonian path.

The results about linear objective functions in section 8.5.C are from Korte & Lovász (1984c). The optimization of linear objective functions over greedoids is also discussed in Brylawski (1991), Faigle (1985), Goecke (1986, 1988), Goecke, Korte & Lovász (1989), Goetschel (1986), and Serganova, Bagotskaya, Levit & Losev (1988). In connection with linear objective functions Gaussian greedoids have special properties; see the cited papers by Brylawski and by Goecke, and also Exercise 8.34.

Bagotskaya, Levit & Losev (1988, 1990a) define structures called 'fibroids', designed to incorporate some optimization features of dynamic programming. Fibroids contain Gaussian greedoids as a special case. Optimization over (W, P)-matroids (see the remarks below for section 8.9) is discussed in Zelevinsky & Serganova (1989). See also the work of Bouchet (1987).

Section 8.6

The material in this section is from sections 5–6 of Björner, Korte & Lovász (1985). From a matroid-theoretic point of view sections 8.6.A and 8.6.B extend parts of the theory of Tutte polynomials and Tutte–Grothendieck invariants to all greedoids. For these topics see Chapter 6 and section 7.3 of this book.

A general discussion of the concept of argument complexity can be found in Chapter 8 of Bollobás (1978). The $d = 0$ case of Proposition 8.6.3 appears in Björner, Korte & Lovász (1985).

There is a large literature on the reliability analysis of stochastic networks and other systems. See Colbourn (1987) for more information and references in this area.

The lack of a fully-fledged duality operation on the class of all greedoids has been noticed by several authors. It appears that it is only when the demand for total symmetry between primal and dual is abandoned that some interesting remnants of duality in greedoids can be discerned. Interesting axiomatic discussions of duality (for matroids and some other set systems) appear in Kung (1983) and in Bland & Dietrich (1987, 1988).

A 2-variable greedoid 'Tutte' polynomial, defined by the corank-nullity formula

$$f_G(t, z) = \sum_{A \subseteq E} t^{r(E) - r(A)} z^{|A| - r(A)},$$

has been studied by Gordon & McMahon (1989); see also Gordon & Traldi (1989), Chaudhuri & Gordon (1991), and Gordon (1990). Its relationship to the polynomial studied here is $\lambda_G(t) = f_G(0, t-1)$, as can be seen from Theorem 8.6.2(iii).

Section 8.7

Convex geometries were independently discovered by Edelman (1980) and Jamison (1982), and later studied by them jointly. Edelman & Jamison (1985) gives a good overview of their work on convex geometries, and all section 8.7A is from that paper except the duality with antimatroids (Proposition 8.7.3), which was first observed by Björner (1985). All examples of antimatroids that have been mentioned in the text were originally known to Edelman and Jamison as examples of convex geometries. The example of convex hull closure in $\mathbb{R}^n$ (and hence, dually, convex pruning antimatroids) was generalized to oriented matroids in Edelman (1982).

We have included only a limited number of references for antimatroids

90(mainly those taking a greedoid point of view). See Edelman & Jamison (1985) and Edelman (1986) for more extensive bibliographies (stressing the convex geometry or lattice-theoretic point of view).

Theorem 8.7.6 showing the equivalence of antimatroids and join-distributivity is due to Edelman (1980), in a dual version for convex geometries. The result was rediscovered by Crapo (1984) using another terminology. The characterization of antimatroids as alternative precedence languages is due to Korte & Lovász (1984a, b). The characterization by exchange axiom (A) as well as by semimodularity of $\mathscr{F}$ is from Björner (1985). Proposition 8.7.7 is from Korte & Lovász (1985b), and Proposition 8.7.8 from Korte & Lovász (1986b); see also Edelman & Saks (1986). For Birkhoff's theorem, see Birkhoff (1967).

All of section 8.7.C is from Korte & Lovász (1984b), except for Theorem 8.7.12 which is due to Dietrich (1987).

Section 8.8

The concept of poset representations of greedoids, the definition of the poset of flats, and basic properties of Φ and $\hat{\Phi}$ are from Björner (1985). The poset of closed sets was first defined for interval greedoids (as in Exercise 8.26) by Korte & Lovász (1983), and then in general (as in Definition 8.8.3) by Björner, Korte & Lovász (1985). The relationship between flats and closed sets (Theorem 8.8.4), and also Proposition 8.8.5, is from Björner, Korte & Lovász (1985).

A notion of a 'lattice of flats' for selectors was defined by Crapo (1984); this can be shown to be equivalent to our poset of flats $\hat{\Phi}$ for the case of interval greedoids. Theorem 8.8.7 (in terms of selectors) is due to Crapo (1984). The lattice-theoretic lemma used in our proof is from Björner, Edelman & Ziegler (1990). A different proof for the lattice property (but not semimodularity), using the poset of closed sets, was given in Korte & Lovász (1983).

Poset properties were first studied in Ziegler (1988), from where Proposition 8.8.9 is taken.

Section 8.9

The excluded minor characterizations of local poset greedoids and undirected branching greedoids (Theorem 8.9.3) are due to Korte & Lovász (1985b) and Schmidt (1985a, 1988), respectively. See also Goecke & Schrader (1990) for a shorter proof. The characterization of directed branching greedoids in Theorem 8.9.1 appears in Schmidt (1985b).

The minor poset of isomorphism classes has long been studied for matroids and particularly for graphs. From work of N. Robertson and P. Seymour it is known that infinite antichains do not exist in the minor poset of all graphs. However, infinite antichains do exist in the minor poset of all matroids; see White (1986), p. 155. Also, infinite antichains of branching greedoids exist (cf. Exercise 8.27).

The basis graph of a greedoid was introduced in Korte & Lovász (1985d), where it was shown that a 2-connected greedoid has a connected basis graph. Basis graphs of matroids had earlier been studied by Maurer (1973). The bounds on the diameter for greedoids (Theorems 8.9.4 and 8.9.5) are due to Ziegler (1988). A higher-dimensional analog of the basis graph, the *basis polyhedron*, was investigated by Björner, Korte & Lovász (1985), and it was shown that a k-connected greedoid has a (topologically) $(k-2)$-connected basis polyhedron.

Non-simple greedoids were first studied in Björner (1985), and this was mainly motivated by the example of Coxeter group greedoids. See Björner (1984, 1985) for more information about Coxeter groups and their weak partial ordering, and for references to the extensive literature about these topics. The greedoids of graph chip firing games were discovered by Björner, Lovász & Shor (1988). The greedoid rank (length of the game) in terms of the size of the graph has been studied by Tardos (1988) and Eriksson (1989). Polygreedoids appeared in Björner (1985); for integral polymatroids see e.g. White (1987), p. 181. Faigle (1985) also discusses non-simple greedoids.

A different connection between greedoids and Coxeter groups was discovered by Gel'fand & Serganova (1987a, b). For each finite Coxeter group (W, S) and each subset P of the generating set S, they define a class of subsets of the family of left cosets $W^P = W/\langle P\rangle$, the members of which they call *(W, P)-matroids*. The definition involves a certain minimality condition in terms of Bruhat order on W^P. For the symmetric group $W = \Sigma_n$ and $P = \{(i, i+1): 1 \leqslant i \leqslant n-1,\ i \neq k\}$, the (W, P)-matroids are precisely the ordinary matroids of rank k given by their bases (as sets). The characterization of matroids obtained this way is equivalent to that of Gale (1968): for every ordering of the ground set there is a point-wise minimal basis. For $P = \{(i, i+1): k+1 \leqslant i \leqslant n-1\}$ the (W, P)-matroids are the Gaussian greedoids of rank k given by their basic words. Taking W to be the symmetry group of a cube and for a certain choice of P, the (W, P)-matroids coincide with the symmetric matroids of Bouchet (1987). (For those who undertake to read the papers of Gel'fand and Serganova, let us point out that part (b) of Theorem 2 in (1987a) and the definition of Bruhat order given there are incorrectly stated.) Many details about (W, P)-matroids can also be found in Zelevinsky & Serganova (1989).

Exercises

The following propositions, theorems, and lemmas which were stated without proof or with incomplete proof in the text, make suitable exercises: 8.2.3, 8.2.7, 8.2.8, 8.4.1–8.4.5, 8.5.6, 8.6.1–8.6.3, 8.7.8, 8.7.9, 8.8.2, 8.8.4, and 8.9.6.

8.1. Show that a set system $(E, \mathscr{F})$ is a greedoid if and only if it satisfies the two axioms:

(G1″) $\varnothing \in \mathscr{F}$ and for all $X, Y \in \mathscr{F}$ such that $Y \subset X$ there is an $x \in X - Y$ such that $X - x \in \mathscr{F}$.

(B) For any subset $A \subseteq E$ all maximal feasible subsets of A have the same cardinality.

8.2. (Korte & Lovász, 1986a) Let $\mathscr{F} \subseteq 2^E$, and consider the following axiom.

(G2″) If $A \subseteq E$, $x, y, z \in E - A$ such that $A \cup x$, $A \cup y$, $A \cup x \cup z \in \mathscr{F}$, and $A \cup x \cup y \notin \mathscr{F}$, then $A \cup y \cup z \in \mathscr{F}$.

Show that the axioms (G1) and (G2″) together define greedoids.

8.3. Prove the following sharpening of the strong exchange property (L2′) of Proposition 8.2.5 for interval greedoids $(E, \mathscr{L})$:

let $\alpha, \beta \in \mathscr{L}, \alpha = x_1 x_2 \ldots x_n$, and $|\alpha| - |\beta| = k > 0$; among all strings $(i_1, i_2, \ldots, i_k)$ such that $\beta x_{i_1} x_{i_2} \ldots x_{i_k} \in \mathscr{L}$, the lexicographically first one satisfies $i_1 < i_2 < \ldots < i_k$.

8.4. (Korte & Lovász, 1984a) Let E be finite and $\mathscr{P} \subseteq 2^E$ a system of non-empty sets such that $|A| = |B|$ and $|A - B| = 1$ imply $A \cap B \in \mathscr{P}$, for all $A, B \in \mathscr{P}$.

(a) Show that $(E, 2^E - \mathscr{P})$ is a greedoid (called a *paving greedoid*).

(b) Show that $(E, 2^E - \mathscr{P})$ in general lacks the transposition property.

8.5. (Crapo, 1984; Korte & Lovász, 1985c) Let $(E, \mathscr{F})$ be a greedoid and $\mathscr{A} = \{\cup \mathscr{F}' : \mathscr{F}' \subseteq \mathscr{F}\}$. Show that

(a) $(E, \mathscr{A})$ is an antimatroid,

(b) $\mathscr{F} \subseteq \mathscr{A} \subseteq \mathscr{R}$, if $(E, \mathscr{F})$ is an interval greedoid,

(c) both inclusions in part (ii) can be strict.

8.6. Let $\Gamma = (V, E, r)$ be a finite, connected undirected graph with root $r \in V$. Let $(E, \mathscr{F}_b)$ be the branching greedoid on Γ, and $(V - r, \mathscr{F}_s)$ the vertex search greedoid.

(a) The join irreducibles (elements covering exactly one element) in the poset $(\mathscr{F}_b, \subseteq)$ are the paths in Γ starting at r. Hence, the join irreducibles form an order ideal in $\mathscr{F}_b$.

(b) The join irreducibles in $(\mathscr{F}_s, \subseteq)$ are the induced paths (i.e. without chords) in Γ, starting at r.

(c) The meet irreducibles (elements covered by exactly one element) in $(\mathscr{F}_b, \subseteq)$ correspond to the bridges in Γ.

(d) The meet irreducibles in $(\mathscr{F}_s, \subseteq)$ of corank at least 2 correspond to cut vertices in Γ; those of corank 1 correspond to non-cut-vertices.

8.7. (Korte & Lovász, 1983) Show that the rank closure operator σ of a greedoid G is monotone only if G is a matroid.

8.8. (Schmidt, 1985a, b)

(a) Prove that directed and undirected branching greedoids satisfy $\sigma(X) \cap (Y) \subseteq \sigma(X \cup Y)$, for $X, Y \in \mathscr{F}$.

(b) Show also that $\sigma(X \cup Y) \subseteq \sigma(X) \cup \sigma(Y)$ holds for directed branching greedoids, but not in general for undirected branching greedoids.

8.9. (Schmidt, 1985a, b) The closure operator for greedoids, which is given by

$$\sigma(A) = \cup\{X \subseteq E: r(A \cup X) = r(A)\},$$

has some shortcomings (cf. section 8.4.B). As an alternative, the *kernel closure operator* λ: $2^E \rightarrow 2^E$, defined by

$$\lambda(A) = \cup\{X \in \mathscr{F}: r(A \cup X) = r(A)\},$$

has been proposed. Define the *kernel* of a subset $A \subseteq E$ by

$$\ker(A) = \cup\{X \in \mathscr{F}: X \subseteq A\}.$$

(a) Give a graph-theoretic description of $\sigma(A)$, $\lambda(A)$, and $\ker(A)$ for a set A in a branching greedoid. Exemplify with a branching greedoid that $A \subseteq \lambda(A)$ may fail.

(b) Show that the operators σ, λ, ker: $2^E \to 2^E$ satisfy the relations:

$$\lambda\sigma = \ker\,\sigma = \lambda,$$
$$\sigma\lambda = \sigma\,\ker = \sigma.$$

(c) Deduce that

$$\lambda^2 = \lambda \text{ (i.e. } \lambda \text{ is idempotent)},$$
$$\lambda\sigma\lambda = \lambda,$$
$$\sigma\lambda\sigma = \sigma.$$

(d) There is a canonical bijection between σ-closed sets and λ-closed sets, and

$$r(\lambda(A)) = r(\sigma(A)) = r(A), \quad \text{for all } A \subseteq E.$$

(e) In an interval greedoid

$$\beta(\lambda(A)) = r(A), \quad \text{for all } A \subseteq E.$$

(f) The kernel closure operator λ is monotone if and only if $(E, \mathscr{F})$ is an interval greedoid.

(g) For σ-closed sets $A, B \in \mathscr{C}\ell$, $A \leqq B$ implies $\lambda(A) \subseteq \lambda(B)$. The converse holds for interval greedoids, but fails in general.

8.10. (Korte & Lovász, 1983) Show that the monotone closure operator μ of full greedoid $(E, \mathscr{F})$ satisfies $\mu(A) = A$, for all $A \subseteq E$.

8.11. (Korte & Lovász, 1989b) Let $(E, \mathscr{M})$ and $(E, \mathscr{A})$ be respectively a matroid and an antimatroid on the same ground set E, with closure operators $\sigma_{\mathscr{M}}$ and $\sigma_{\mathscr{A}}$. Define a language $(E, \mathscr{L})$ by

$$\mathscr{L} = \{x_1x_2 \ldots x_k \in E^*: x_i \notin \sigma_{\mathscr{A}}(\sigma_{\mathscr{M}}(\{x_1, \ldots, x_{i-1}\})) \text{ for all } 1 \leqq i \leqq k\}.$$

(a) Show that $(E, \mathscr{L})$ is an interval greedoid. The corresponding set system, denoted by $(E, \mathscr{M} \wedge \mathscr{A})$, is called the *meet* of $(E, \mathscr{M})$ and $(E, \mathscr{A})$.

(b) Verify that $\mathscr{M} \cap \mathscr{A} \subseteq \mathscr{M} \wedge \mathscr{A} \subseteq \mathscr{M}$.

(c) Show that $(E, \mathscr{M} \cap \mathscr{A})$ is not a greedoid in general, but that if it is a greedoid, then $(E, \mathscr{M} \cap \mathscr{A}) = (E, \mathscr{M} \cap \mathscr{A}^*) = (E, \mathscr{M} \wedge \mathscr{A}^*)$ for the antimatroid $\mathscr{A}^* = \{\cup\mathscr{F}' : \mathscr{F}' \subseteq \mathscr{M} \cap \mathscr{A}\}$.

(d) Show that if $(E, \mathscr{M})$ is any matroid, and $(E, \mathscr{A})$ is a poset greedoid, then $(E, \mathscr{M} \wedge \mathscr{A})$ is a Faigle geometry.

(e) Show that every directed branching greedoid arises as a meet.

(f) Show that every polymatroid greedoid arises as a meet.

8.12. (Korte & Lovász, 1983) For a greedoid $(E, \mathscr{F})$, let $A_1, \ldots, A_n \subseteq E$. A *feasible system of representatives* for $\{A_1, \ldots, A_n\}$ in $(E, \mathscr{F})$ is a set $X \in \mathscr{F}$ for which there is a bijection $\phi: X \to \{A_1, \ldots, A_n\}$ with $x \in \phi(x)$ for all $x \in X$.

Suppose that $(E, \mathscr{F})$ is an interval greedoid and $A_1, \ldots, A_n$ are rank feasible.

Show that $\{A_1, ..., A_n\}$ has a feasible system of representatives if and only if

$$r(A_{i_1} \cup ... \cup A_{i_k}) \geqq k,$$

for all $1 \leqq i_1 < ... < i_k \leqq n$.

8.13. Let $K \subseteq \mathbb{R}^n$ be a convex body, and $E \subseteq \mathbb{R}^n - K$ a finite set. Let $\mathscr{F} \subseteq 2^E$ consist of those subsets A that are disjoint from the convex hull of $K \cup (E - A)$.
(a) Prove that $(E, \mathscr{F})$ is an antimatroid.
(b) Prove that the class of such antimatroids is hereditary (i.e. closed under taking minors).

8.14. (Björner, Korte & Lovász, 1985) Say that a greedoid $(E, \mathscr{F})$ is *weakly k-connected* if $r(E - A) = r(E)$ for all $A \subseteq E$ with $|A| < k$.
(a) Show that an undirected branching greedoid (or graphic matroid) is weakly k-connected if and only if the underlying graph is k-edge-connected.
(b) Show that the number of bases in a weakly k-connected greedoid of rank r is at least $\binom{k+r-1}{r}$.

8.15. (Korte & Lovász, 1989b) Let $(E, \mathscr{F})$ be a greedoid and A a closure feasible subset of E. Show that $(E, \mathscr{F}')$ is a full greedoid, but not in general an antimatroid, for

$$\mathscr{F}' = \mathscr{F} \cup \{B \subseteq E: \sigma(B) \supseteq A\}.$$

As a special case, conclude that every greedoid is a truncation of a full greedoid.

8.16. Let $(E, \mathscr{F})$ be a greedoid of rank r. Show that $(E, \mathscr{F})$ is the r-truncation of an antimatroid if and only if for all $X, Y \in \mathscr{F}$, $|X \cup Y| \leqq r$ implies $X \cup Y \in \mathscr{F}$. In this case, construct the smallest and the largest antimatroid on E whose r-truncation is $(E, \mathscr{F})$.

8.17. (Björner, Korte & Lovász, 1985) Let $(E, \mathscr{F})$ be an interval greedoid. Show that
(a) $(E, \mathscr{F})$ is a matroid if and only if $\{x\} \in \mathscr{F}$ for all $x \in \cup \mathscr{F}$;
(b) if $A \in \mathscr{F}$, then the free sets over A are the independent sets of a matroid.

8.18. (Korte & Lovász, 1986a) Show that an accessible set system with the transposition property (TP) of section 8.3.G is a greedoid.

8.19. (Korte & Lovász, 1983; Goecke, 1986) Prove that a greedoid $(E, \mathscr{F})$ has the interval property if and only if $\mathscr{F}/X_1 = \mathscr{F}/X_2$ for all subsets $A \subseteq E$ and all bases X_1 and X_2 of A.

8.20. One might have hoped for the following weak form of greedoid duality: if $\mathscr{B} \subseteq 2^E$ is the set of bases of a greedoid then $\{E - B: B \in \mathscr{B}\}$ is the set of bases of some other greedoid, not necessarily unique. Show that this is false.

8.21. Let $\tau: 2^E \to 2^E$ be a closure operator on a finite set E. Show that τ satisfies the anti-exchange condition if and only if every closed set other than $\tau(\emptyset)$ has at least one extreme point.

8.22. (Korte & Lovász, 1984b) Let E be a finite set, and for each $x \in E$ let $H(x) \subseteq 2^{E-x}$ be some set system. Define a left hereditary language

$$\mathscr{L}^H = \{x_1 x_2 \ldots x_k \in E_s^*: \text{for all } 1 \leqq i \leqq k \text{ and all } A \in H(x_i),\ A \nsubseteq \{x_{i+1}, \ldots, x_k\}\}.$$

(a) Show that $(E, \mathscr{L}^H)$ is an antimatroid.
(b) Show that every antimatroid arises in this way.
(c) Suppose that $(E, \mathscr{L}_K)$ is an alternative precedence language determined by some system K. For each $K(x)$ describe $H(x)$ so that $\mathscr{L}^H = \mathscr{L}_K$.

8.23. For a full antimatroid $(E, \mathscr{F})$, let Δ consist of those subsets of E that are free and convex. Show that
(a) Δ is a simplicial complex,
(b) $\sum(-1)^i f_i = 0$, where f_i is the number of sets in Δ of cardinality i,
(c) Δ is contractible (in the topological sense).
(d) Let h be the maximum cardinality of a set in Δ. Show that h is the *Helly number* of $(E, \mathscr{F})$, meaning that h is the least integer such that, for any family of convex sets, if each subfamily of size h has non-empty intersection then the whole family has non-empty intersection.

(Part (b) is an unpublished theorem of J. Lawrence; see Edelman & Jamison (1985). The proof of Theorem 7.4 in Björner, Korte & Lovász (1985) can be adapted to prove the contractibility of Δ. Part (d) is due to A. Hoffman and R. Jamison; see Edelman & Jamison (1985).)

8.24. (a) Show that the greedoid polynomial of the branching greedoid of a rooted connected graph is independent of the root.
(b) Show that the analogous statement for the branching greedoid of a strongly connected digraph is false.

8.25. (Björner, 1985) Show that the poset representations of a greedoid form a lattice when they are ordered by $(P_1, \lambda_1) \leqq (P_2, \lambda_2)$ if and only if there is a rank-preserving poset map $f: P_1 \to P_2$ satisfying $\lambda_1(x <\!\cdot\, y) \subseteq \lambda_2(f(x) <\!\cdot\, f(y))$ for $x, y \in P_1$ and $x <\!\cdot\, y$. Show that every such map is necessarily surjective. Identify the universal representation (poset of flats) in terms of this lattice.

8.26. (Korte & Lovász, 1983) According to Theorems 8.8.4 and 8.8.7, the poset of closed sets $(\mathscr{C}\ell, \leqq)$ of an interval greedoid is a semimodular lattice. Show that its meet operation is given by

$$A \wedge B = \sigma(A \cap B), \quad A, B \in \mathscr{C}\ell.$$

8.27. Construct an infinite sequence of branching greedoids G_i, $i = 1, 2, \ldots$, such that G_i is not a minor of G_j for all $i \neq j$.

8.28. Define a graph over the set $\mathscr{B}$ of bases of a greedoid G by letting (B_1, B_2) be an edge when $|B_1 - B_2| = 1$, $B_1, B_2 \in \mathscr{B}$. (This graph contains the basis graph as a subgraph.) Prove the bounds

$$r - |A \cap B| \leqq d(A, B) \leqq r - r(A \cap B),$$

for the graph distance $d(A, B)$ between two bases A and B, where $r = \text{rank } G$. In particular, the diameter of the graph is at most r.

8.29. Define the *basic word graph* of a greedoid $(E, \mathscr{L})$ as follows. The vertices are the basic words, and two basic words are adjacent if the corresponding maximal chains in the poset $(\mathscr{F}, \subseteq)$ differ in exactly one element (equivalently, if one arises from the other by exchanging two consecutive letters or by exchanging the last letter).
(a) (Korte & Lovász, 1985d) Show that the basic word graph is connected if $(E, \mathscr{L})$ is 2-connected.
(b) If $(E, \mathscr{L})$ is an antimatroid of rank r, show that the basic word graph is connected and has diameter at most $\binom{r}{2}$. This bound is best-possible.

8.30. (Korte & Lovász, 1984b) Show that convex pruning greedoids have the following property, not shared by general antimatroids: if $(A \cup x, x)$ and $(A \cup y, y)$ are rooted circuits, then there exists a unique subset $A' \subseteq A$ such that $(A' \cup x \cup y, y)$ is a rooted circuit.

8.31. For an antimatroid $(E, \mathscr{F})$, let c be the minimum and C the maximum size of a circuit.

(a) (Björner, Korte & Lovász, 1985) Show that $(E, \mathscr{F})$ is k-connected if and only if $C \geq k+1$.

(b) (Björner & Lovász, 1987) Show that $C-1$ is the *Carathéodory number* of $(E, \mathscr{F})$, i.e. the least integer such that if x lies in the convex hull of $A \subseteq E$, then there is some subset $A' \subseteq A$ of size at most $C-1$ such that x lies in the convex hull of A'.

8.32. (Korte & Lovász, 1984c) Let $(E, \mathscr{F})$ be a greedoid. Given a linear objective function, the *worst-out greedy algorithm* starts with the complete ground set E and at each step eliminates the worst possible element so that the remaining set is still spanning (contains a basis). Show that every linear objective function can be optimized over $(E, \mathscr{F})$ by the worst-out greedy algorithm if and only if the hereditary closure $(E, \mathscr{H}(\mathscr{F}))$ is a matroid.

8.33. (a) For an interval greedoid, show that every $\mathscr{R}$-compatible linear objective function is compatible in the sense of Definition 8.5.1.

(b) Give an example of a non-interval greedoid and a linear function that is $\mathscr{R}$-compatible but not compatible.

8.34. (Serganova, Bagotskaya, Levit & Losev, 1988) Let $\mathscr{F} \subseteq 2^E$ be an accessible set system. Show that the following are equivalent.

(a) $(E, \mathscr{F})$ is a Gaussian greedoid.

(b) For any linear objective function the greedy algorithm constructs a sequence of sets $A_i \in \mathscr{F}$, $i = 1, \ldots, r$, such that A_i is optimal in the class $\mathscr{F}_i = \{X \in \mathscr{F} : |X| = i\}$ for all $i = 1, \ldots, r = \max\{|X| : X \in \mathscr{F}\}$.

(c) For $X, Y \in \mathscr{F}$, $|X| = |Y| + 1$, there is an $x \in X - Y$ such that $Y \cup x \in \mathscr{F}$ and $X - x \in \mathscr{F}$.

(d) For $X, Y \in \mathscr{F}$, $|X| > |Y|$, there is a subset $A \subset X - Y$, $|A| = |X| - |Y|$, such that $Y \cup A \in \mathscr{F}$ and $X - A \in \mathscr{F}$.

Furthermore, show that the condition $|X| = |Y| + 1$ in (c) cannot be relaxed to $|X| > |Y|$.

(The equivalence of (a), (b), and (c) is proved in the cited source. The same authors have subsequently proved (personal communication) the equivalence of (c) and (d), two versions of what they call the 'fork axiom'. The equivalence of (a) and (b) was also observed by Brylawski (1991).)

8.35. Does there exist a non-Gaussian greedoid $(E, \mathscr{F})$ for which $\mathscr{H}(\mathscr{F}_i)$, the hereditary closure of the feasible i-sets, is a matroid, for $i = 0, 1, \ldots, r = \text{rank}(\mathscr{F})$?

The remaining three exercises concern non-simple greedoids (section 8.9.C).

8.36. (Björner, 1985, extending Korte and Lovász, 1984a)

(a) Show that the greedy algorithm will optimize any compatible objective function $w\colon \mathscr{L} \to \mathbb{R}$ over a polygreedoid $(E, \mathscr{L})$.

(b) Show that the greedy algorithm will optimize any generalized bottleneck function (defined in the proof of Theorem 8.5.2) over a greedoid, whether simple or not.

8.37. (Björner, 1985; extending Korte & Lovász, 1984a) For multisets A, B: $E \to \mathbb{N}$ define *inclusion* $A \subseteq B$ by $A(e) \leqq B(e)$ for all $e \in E$, and *cardinality* $|A| = \sum_{e \in E} A(e)$. Identify elements $e \in E$ with their characteristic functions χ_e: $E \to \{0, 1\}$. For a finite non-empty multiset system $\mathscr{F} \subseteq \mathbb{N}^E$, E finite, consider the following axioms.

(P1) For all $A \in \mathscr{F}$, $A \neq \emptyset$, there exists $B \subseteq A$ such that $|B| = |A| - 1$ and $B \in \mathscr{F}$.

(P2) If A, $B \in \mathscr{F}$ and $|A| > |B|$, then there exists an element $e \in E$ such that $A(e) > B(e)$ and $B + e \in \mathscr{F}$.

Prove the following.

(a) If $(E, \mathscr{L})$ is a polygreedoid, then the support $\tilde{\mathscr{L}}$ satisfies axioms (P1) and (P2).

(b) If $(E, \mathscr{F})$ is a multiset system satisfying (P1) and (P2) then the language

$$\mathscr{L}(\mathscr{F}) = \{x_1 x_2 \ldots x_k \in E^*: \widetilde{x_1 \ldots x_i} \in \mathscr{F} \text{ for } 1 \leqq i \leqq k\}$$

is a polygreedoid.

(c) These operations are mutually inverse: $\mathscr{L}(\tilde{\mathscr{L}}) = \mathscr{L}$ and $\widetilde{\mathscr{L}(\mathscr{F})} = \mathscr{F}$.

8.38. A finite language $\mathscr{L} \subseteq E^*$, not necessarily simple, is called an A-*language* if it is hereditary (axiom (L1)) and satisfies the following axiom.

(L2‴) If α, αx, $\alpha\beta \in \mathscr{L}$ and the letter x does not occur in β, then $\alpha x \beta$, $\alpha\beta x \in \mathscr{L}$ and $\alpha x \beta \gamma \in \mathscr{L}$ if and only if $\alpha\beta x \gamma \in \mathscr{L}$, for all $\gamma \in E^*$.

Prove the following (cf. section 8.9.C).

(a) Every A-language is a strong greedoid.

(b) Every graph chip firing language is an A-language.

(c) A simple A-language is the same thing as an antimatroid.

References

Bagotskaya, N. V., Levit, V. E. & Losev, I. S. (1988). On a generalization of matroids that preserves the applicability of the method of dynamic programming (in Russian), in *Sistemy Peredachi i Obrabotki Informatsii, Chast* 2, pp. 33–6, Institut Problem Peredachi Informatsii, Academy of Sciences, Moscow.

Bagotskaya, N. V., Levit, V. E. & Losev, I. S. (1990a). Fibroids, *Automatics and Telemechanics* (translation of Russian journal *Automatika i Telemechanika*) (to appear).

Bagotskaya, N. V., Levit, V. E. & Losev, I. S. (1990b). Branching greedy algorithm and its combinatorial structure, preprint.

Birkhoff, G. (1967). *Lattice Theory*, Amer. Math. Soc. Colloq. Publ. XXV, 3rd edition. American Mathematical Society, Providence, Rhode Island.

Björner, A. (1984). Orderings of Coxeter groups, in C. Greene (ed.), *Proceedings of the A.M.S.–N.S.F. Conference on Combinatorics and Algebra* (Boulder, 1983), Contemporary Mathematics **34**, pp. 175–95. American Mathematical Society, Providence, Rhode Island.

Björner, A. (1985). On matroids, groups and exchange languages, in L. Lovász & A. Recski (eds), *Matroid Theory and its Applications* (Szeged, 1982), Colloq. Math. Soc. János Bolyai **40**, pp. 25–60. North Holland, Amsterdam and Budapest.

Björner, A. & Lovász, L. (1987). Pseudomodular lattices and continuous matroids, *Acta Sci. Math. Szeged* **51**, 295–308.

Björner, A., Edelman, P. & Ziegler, G. M. (1990). Hyperplane arrangements with a lattice of regions, *Discrete and Comp. Geom.* **5**, 263–88.

Björner, A., Korte, B. & Lovász, L. (1985). Homotopy properties of greedoids, *Adv. Appl. Math.* **6**, 447–94.

Björner, A., Lovász, L. & Shor, P. W. (1988). Chip-firing games on graphs, *Europ. J. Comb.* (to

appear).
Bland, R. G. & Dietrich, B. L. (1987). A unified interpretation of several combinatorial dualities, preprint, Cornell University.
Bland, R. G. & Dietrich, B. L. (1988). An abstract duality, *Discrete Math.* **70**, 203–8.
Bollobás, B. (1978). *Extremal Graph Theory*, Academic Press, New York and London.
Bouchet, A. (1987). Greedy algorithm and symmetric matroids, *Math. Progr.* **38**, 147–59.
Boyd, E. A. (1987a). Optimization problems on greedoids, Ph.D. Thesis, M.I.T.
Boyd, E. A. (1987b). An algorithmic characterization of antimatroids, preprint, Rice University.
Brylawski, T. H. (1991). Greedy families for linear objective functions, *Studies Appl. Math.* **84**, 221–9.
Brylawski, T. H. & Dieter, E. (1988). Exchange systems, *Discrete Math.* **69**, 123–51.
Chang, G. J. (1986). MPP-greedoids, Preprint 86436-OR, Institut für Operations Research, Bonn.
Chaudhary, S. & Gordon, G. (1991). Tutte polynomials for trees, *J. Graph Theory* (to appear).
Colbourn, C. J. (1987). *The Combinatorics of Network Reliability*, Oxford University Press.
Crapo, H. (1984). Selectors. A theory of formal languages, semimodular lattices, branchings and shelling processes, *Adv. Math.* **54**, 233–77.
Crapo, H. & Rota, G.-C. (1970). *Combinatorial Geometries*, M.I.T. Press, Cambridge, Mass.
Dietrich, B. L. (1986). A unifying interpretation of several combinatorial dualities, Ph.D. Thesis, Cornell University.
Dietrich, B. L. (1987). A circuit set characterization of antimatroids, *J. Comb. Theory Ser. B* **43**, 314–21.
Dietrich, B. L. (1989). Matroids and antimatroids – A survey, *Discrete Math.* **78**, 223–37.
Ding, L.-Y. & Yue, M.-Y. (1987). On a generalization of the Hall–Rado theorem to greedoids, *Asia–Pacific J. Oper. Res.* **4**, 28–38.
Dress, A. W. M. & Wenzel, W. (1990). Valuated matroids – a new look at the greedy algorithm, *Appl. Math. Letters* **3**, 33–5.
Duffus, D. & Rival, I. (1978). Crowns in dismantlable partially ordered sets, in *Combinatorics* (Keszthely, 1976), Colloq. Math. Soc. János Bolyai **18**, Vol. I, pp. 271–92. North-Holland, Amsterdam.
Dunstan, F. D. J., Ingleton, A. W. & Welsh, D. J. A. (1972). Supermatroids, in *Combinatorics, Proceedings of the Conference on Combinatorial Mathematics* (Oxford 1972), pp. 72–122. The Institute of Mathematics and Its Applications, Southend-on-Sea, U.K.
Edelman, P. (1980). Meet-distributive lattices and the anti-exchange closure. *Algebra Universalis* **10**, 290–9.
Edelman, P. (1982). The lattice of convex sets of an oriented matroid. *J. Comb. Theory Ser. B* **33**, 239–44.
Edelman, P. (1986). Abstract convexity and meet-distributive lattices, *Contemporary Math.* **57**, 127–50.
Edelman, P. & Jamison, R. (1985). The theory of convex geometries, *Geom. Dedicata* **19**, 247–70.
Edelman, P. & Saks, M. (1986). Combinatorial representation and convex dimension of convex geometries, *Order* **5**, 23–32.
Edmonds, J. (1971). Matroids and the greedy algorithm, *Math. Progr.* **1**, 127–36.
Eriksson, K. (1989). No polynomial bound for the chip firing game on directed graphs, *Proc. Amer. Math. Soc.* (to appear).
Faigle, U. (1979). The greedy algorithm for partially ordered sets, *Discrete Math.* **28**, 153–9.
Faigle, U. (1980). Geometries on partially ordered sets, *J. Comb. Theory Ser. B* **28**, 26–51.
Faigle, U. (1985). On ordered languages and the optimization of linear functions by greedy algorithms, *J. Assoc. Comp. Mach.* **32**, 861–70.
Faigle, U. (1987). Exchange properties of combinatorial closure spaces, *Discrete Appl. Math.* **15**, 240–60.
Faigle, U., Goecke, O. & Schrader, R. (1986). Church–Rosser decomposition in combinatorial structures, Preprint 86426-OR, Institut für Operations Research, Bonn.
Gale, D. (1968). Optimal assignments in an ordered set: an application of matroid theory,

J. Comb. Theory **4**, 176–80.
Gel'fand, I. M. & Serganova, V. V. (1987a). On the general definition of a matroid and a greedoid, *Soviet Mathematics Doklady* **35**, 6–10.
Gel'fand, I. M. & Serganova, V. V. (1987b). Combinatorial geometries and torus strata on homogeneous compact manifolds, *Russian Math. Surveys* **42**, 133–68.
Goecke, O. (1986). Eliminationsprozesse in der kombinatorischen Optimierung – ein Beitrag zur Greedoidtheorie, Dissertation, Universität Bonn.
Goecke, O. (1988). A greedy algorithm for hereditary set systems and a generalization of the Rado–Edmonds characterization of matroids, *Discrete Appl. Math.* **20**, 39–49.
Goecke, O., Korte, B. & Lovász, L. (1989). Examples and algorithmic properties of greedoids, in B. Simione (ed.), *Combinatorial Optimization*, Lecture Notes in Mathematics 1403, pp. 113–61. Springer-Verlag, New York.
Goecke, O. & Schrader, R. (1990). Minor characterization of undirected branching greedoids – a short proof, *Discrete Math.* **82**, 93–9.
Goetschel, R. H. Jr. (1986). Linear objective functions on certain classes of greedoids, *Discrete Appl. Math.* **14**, 11–16.
Gordon, G. (1990). A Tutte polynomial for partially ordered sets, preprint, Lafayette College.
Gordon, G. & McMahon, E. (1989). A greedoid polynomial which distinguishes rooted arborescences, *Proc. Amer. Math. Soc.* **107**, 287–98.
Gordon, G. & Traldi, L. (1989). Polynomials for directed graphs, preprint, Lafayette College.
Graham, R. L. & Hell, P. (1985). On the history of the minimum spanning tree problem, *Ann. History of Computing* **7**, 43–57.
Hu, T. C. & Lenard, M. L. (1976). Optimality of a heuristic algorithm for a class of knapsack problems, *Oper. Res.* **24**, 193–6.
Iwamura, K. (1988a). Contraction greedoids and a Rado–Hall type theorem, Preprint 88502-OR, Institut für Operations Research, Bonn.
Iwamura, K. (1988b). Primal–dual algorithms for the lexicographically optimal base of a submodular polyhedron and its relation to a poset greedoid, Preprint 88507-OR, Institut für Operations Research, Bonn.
Iwamura, K. & Goecke, O. (1985). An application of greedoids to matroid theory, preprint, Josai University, Sakado, Saitama, Japan.
Jamison, R. E. (1982). A perspective on abstract convexity: Classifying alignments by varieties, in D. C. Kay and M. Breem (eds), *Convexity and Related Combinatorial Geometry*, pp. 113–50. Dekker, New York.
Korte, B. & Lovász, L. (1981). Mathematical structures underlying greedy algorithms, in *Fundamentals of Computation Theory* (Szeged, 1981), Lecture Notes in Computer Science **117**, pp. 205–9. Springer-Verlag, New York.
Korte, B. & Lovász, L. (1983). Structural properties of greedoids, *Combinatorica* **3**, 359–74.
Korte, B. & Lovász, L. (1984a). Greedoids, a structural framework for the greedy algorithm, in W. R. Pulleyblank (ed.), *Progress in Combinatorial Optimization, Proceedings of the Silver Jubilee Conference on Combinatorics* (Waterloo, 1982), pp. 221–43. Academic Press, London, New York, and San Francisco.
Korte, B. & Lovász, L. (1984b). Shelling structures, convexity and a happy end, in B. Bollobás (ed.), *Graph Theory and Combinatorics, Proceedings of the Cambridge Combinatorial Conference in Honour of Paul Erdős* (1983), pp. 219–32. Academic Press, New York and London.
Korte, B. & Lovász, L. (1984c). Greedoids and linear objective functions, *SIAM J. Algebraic and Discrete Math.* **5**, 229–38.
Korte, B. & Lovász, L. (1985a). Posets, matroids and greedoids, in L. Lovász & A. Recski (eds), *Matroid Theory and its Applications* (Szeged, 1982), Colloq. Math. Soc. Janós Bolyai **40**, pp. 239–65. North Holland, Amsterdam and Budapest.
Korte, B. & Lovász, L. (1985b). Polymatroid greedoids, *J. Comb. Theory Ser. B* **38**, 41–72.
Korte, B. & Lovász, L. (1985c). A note on selectors and greedoids, *Europ. J. Comb.* **6**, 59–67.
Korte, B. & Lovász, L. (1985d). Basis graphs of greedoids and two-connectivity, *Math. Progr.*

Study **24**, 158–65.

Korte, B. & Lovász, L. (1985e). Relations between subclasses of greedoids, *Z. Oper. Res. Ser. A* **29**, 249–67.

Korte, B. & Lovász, L. (1986a). Non-interval greedoids and the transposition property, *Discrete Math.* **59**, 297–314.

Korte, B. & Lovász, L. (1986b). Homomorphisms and Ramsey properties of antimatroids, *Discrete Appl. Math.* **15**, 283–90.

Korte, B. & Lovász, L. (1986c). On submodularity in greedoids and a counterexample, *Sezione di Matematica Applicata*, Dipartimento di Matematica, Universitá di Pisa **127**, 1–13.

Korte, B. & Lovász, L. (1989a). The intersection of matroids and antimatroids, *Discrete Math.* **73**, 143–57.

Korte, B. & Lovász, L. (1989b). Polyhedral results for antimatroids, in *Combinatorial Mathematics: Proceedings of the Third International Conference* (1985), Annals of the New York Academy of Sciences **555**, pp. 283–95.

Korte, B., Lovász, L. & Schrader, R. (1991). *Greedoid Theory*, Algorithms and Combinatorics **4**, Springer-Verlag (to appear).

Kung, J. P. S. (1983). A characterization of orthogonal duality in matroid theory, *Geom. Dedicata* **15**, 69–72.

Magazine, M., Nemhauser, G. L. & Trotter, L. E. (1975). When the greedy solution solves a class of knapsack problems, *Oper. Res.* **23**, 207–17.

Maurer, S. B. (1973). Matroid basis graphs I, II, *J. Comb. Theory Ser. B* **14**, 216–40; **15**, 121–45.

Schmidt, W. (1985a). Strukturelle Aspekte in der kombinatorischen Optimierung: Greedoide auf Graphen, Dissertation, Universität Bonn.

Schmidt, W. (1985b). Greedoids and searches in directed graphs, *Discrete Math.* (to appear).

Schmidt, W. (1985c). A min-max theorem for greedoids, Preprint 85396-OR, Institut für Operations Research, Bonn.

Schmidt, W. (1988). A characterization of undirected branching greedoids, *J. Comb. Theory Ser. B* **45**, 160–84.

Schrader, R. (1986). Structural theory of discrete greedy procedures, Habilitationsschrift, Universität Bonn.

Serganova, V. V., Bagotskaya, N. V., Levit, V. E. & Losev, I. S. (1988). Greedoids and the greedy algorithm (in Russian), in *Sistemy Peredachi i Obrabotki Informatsii, Chast 2*, pp. 49–52. Institut Problem Peredachi Informatsii, Academy of Sciences, Moscow.

Tardos, G. (1988). Polynomial bound for a chip firing game on graphs, *SIAM J. Discrete Math.* **1**, 397–8.

Tarjan, R. E. (1983). *Data structures and network algorithms*, C.B.M.S–N.S.F. Regional Conference Series in Applied Mathematics **44**, Society for Industrial and Applied Mathematics, Philadelphia.

Welsh, D. J. A. (1976). *Matroid Theory*, Academic Press, London.

White, N. (ed.) (1986). *Theory of Matroids*, Cambridge University Press.

White, N. (ed.) (1987). *Combinatorial Geometries*, Cambridge University Press.

Zelevinsky, A. V. & Serganova, V. V. (1989). Combinatorial optimization on Weyl groups, greedy algorithms and generalized matroids (in Russian), preprint, Nauchnyi Soviet po Kompleksnoi Probleme 'Kibernetika', Academy of Sciences, Moscow.

Ziegler, G. M. (1988). Branchings in rooted graphs and the diameter of greedoids, *Combinatorica* **8**, 214–37.