5

Fractional Arboricity and Matroid Methods

The material in this chapter is motivated by two notions of the density of a graph. The *arboricity* and the *maximum average degree* of a graph G measure the concentration of edges in the "thickest" part of the graph.

5.1 Arboricity and maximum average degree

Suppose we wish to decompose the edges of a graph G into acyclic subsets, i.e., if $G = (V, E)$ we want to find $E_1, E_2, \ldots, E_k \subseteq E$ so that (1) each of the subgraphs (V, E_i) is acyclic and (2) $E = E_1 \cup E_2 \cup \cdots \cup E_k$. The smallest size of such a decomposition is called the *arboricity* (or *edge-arboricity*) of G and is denoted $\Upsilon(G)$. If G is connected, the arboricity is also the minimum number of spanning trees of G that include all edges of G.

One can think of arboricity as being a variant of the edge chromatic number. We are asked to paint the edges of G with as few colors as possible. In the case of edge chromatic number, we do not want to have two edges of the same color incident with a common vertex. In the case of arboricity, we do not want to have a monochromatic cycle.

There is an obvious lower bound on $\Upsilon(G)$. Since G has $\varepsilon(G)$ edges and each spanning acyclic subgraph has at most $\nu(G) - 1$ edges we have $\Upsilon(G) \geq \varepsilon(G)/(\nu(G) - 1)$. Moreover, since Υ is an integer, we have $\Upsilon(G) \geq \left\lceil \frac{\varepsilon(G)}{\nu(G)-1} \right\rceil$.

This bound is not very accurate if the graph is highly "unbalanced"; for example, consider the graph G consisting of a K_9 with a very long tail attached—say 100 additional vertices. We have $\nu(G) = 109$, $\varepsilon(G) = 136$, and therefore $\Upsilon(G) \geq \left\lceil \frac{136}{108} \right\rceil = 2$. The actual value of $\Upsilon(G)$ is larger since we clearly cannot cover the edges of K_9 with two trees; indeed, the arboricity of a graph is at least as large as the arboricity of any of its subgraphs. Thus we have

$$\Upsilon(G) \geq \max \left\lceil \frac{\varepsilon(H)}{\nu(H) - 1} \right\rceil$$

where the maximum is over all subgraphs of H with at least 2 vertices. Indeed, this improved lower bound gives the correct value.

Theorem 5.1.1

$$\Upsilon(G) = \max \left\lceil \frac{\varepsilon(H)}{\nu(H) - 1} \right\rceil$$

where the maximum is over all subgraphs of H with at least 2 *vertices.*

The proof of this theorem of Nash-Williams [137, 138] is presented in §5.4 below.