# Halfway to Rota's basis conjecture

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**Abstract**

In 1989, Rota made the following conjecture. Given $n$ bases $B_1, \dots, B_n$ in an $n$-dimensional vector space $V$, one can always find $n$ disjoint bases of $V$, each containing exactly one element from each $B_i$ (we call such bases *transversal bases*). Rota's basis conjecture remains wide open despite its apparent simplicity and the efforts of many researchers (for example, the conjecture was recently the subject of the collaborative "Polymath" project). In this paper we prove that one can always find $(1/2 - o(1))n$ disjoint transversal bases, improving on the previous best bound of $\Omega(n/\log n)$. Our results also apply to the more general setting of matroids.

## 1 Introduction

Given bases $B_1, \dots, B_n$ in an $n$-dimensional vector space $V$, a *transversal basis* is a basis of $V$ containing a single distinguished vector from each of $B_1, \dots, B_n$. Two transversal bases are said to be *disjoint* if their distinguished vectors from $B_i$ are distinct, for each $i$ (here "distinguished" means that two copies of the same vector appearing in two $B_i$s are considered distinct). In 1989, Rota conjectured (see [23, Conjecture 4]) that for any vector space $V$ over a characteristic-zero field, and any choice of $B_1, \dots, B_n$, one can always find $n$ pairwise disjoint transversal bases.

Despite the apparent simplicity of this conjecture, it remains wide open, and has surprising connections to apparently unrelated subjects. Specifically, it was discovered by Huang and Rota [23] that there are implications between Rota's basis conjecture, the Alon–Tarsi conjecture [2] concerning enumeration of even and odd Latin squares, and a certain conjecture concerning the supersymmetric bracket algebra.

Rota also observed that an analogous conjecture could be made in the much more general setting of *matroids*, which are objects that abstract the combinatorial properties of linear independence in vector spaces. Specifically, a finite matroid $M = (E, \mathcal{I})$ consists of a finite ground set $E$ (whose elements may be thought of as vectors in a vector space), and a collection $\mathcal{I}$ of subsets of $E$, called independent sets. The defining properties of a matroid are that:

- the empty set is independent (that is, $\emptyset \in \mathcal{I}$);
- subsets of independent sets are independent (that is, if $A' \subseteq A \subseteq E$ and $A \in \mathcal{I}$, then $A' \in \mathcal{I}$);
- if $A$ and $B$ are independent sets, and $|A| > |B|$, then an independent set can be constructed by adding an element of $A$ to $B$ (that is, there is $a \in A \backslash B$ such that $B \cup \{a\} \in \mathcal{I}$). This final property is called the *augmentation property*.

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