SIAM J. DISCRETE MATH.
Vol. 20, No. 4, pp. 1042–1045
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ROTA'S BASIS CONJECTURE FOR PAVING MATROIDS*

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Abstract. Rota conjectured that, given n disjoint bases of a rank-n matroid M, there are n disjoint transversals of these bases that are all bases of M. We prove a stronger statement for the class of paving matroids.

Key words. Rota's basis conjecture, paving matroids

AMS subject classification. 05B35

DOI. 10.1137/060655596

1. Introduction.

We prove the following theorem.

THEOREM 1.1. *Let* $B_1, \ldots, B_n$ *be disjoint sets of size* $n \geq 3$, *and let* $M_1, \ldots, M_n$ *be rank-n paving matroids on* $\bigcup_i B_i$ *such that* B_i *is a basis of* M_i *for each* $i \in \{1, \ldots, n\}$. *Then there exist n disjoint transversals* $A_1, \ldots, A_n$ *of* $(B_1, \ldots, B_n)$ *such that* A_i *is a basis of* M_i *for each* $i \in \{1, \ldots, n\}$.

A *paving matroid* M is a matroid in which each circuit has size $r(M)$ or $r(M)+1$, where $r(M)$ is the rank of M. Theorem 1.1 implies Rota's basis conjecture for paving matroids.

CONJECTURE 1.2 (Rota (see [6])). *Given n disjoint bases* $B_1, \ldots, B_n$ *in a rank-n matroid* M, *there exist n disjoint transversals* $A_1, \ldots, A_n$ *of* $(B_1, \ldots, B_n)$ *that are all bases of* M.

For $n = 2$, Conjecture 1.2 follows immediately from basis exchange in matroids. Chan [2] proved the conjecture for $n = 3$. Wild [9] proved a stronger conjecture for the class of strongly base-orderable matroids, while more recently a slightly weaker result was proved for a general matroid (Ponomarenko [8]). Further partial results may be found in [1], [3], [4], [5], and [9].

Theorem 1.1 fails for both $n = 2$ and matroids in general. When $n = 2$, if we take $\mathcal{B}(M_1) = \{\{e,f\},\{e,g\},\{f,h\},\{g,h\}\}$ and $\mathcal{B}(M_2) = \{\{e,f\},\{e,h\},\{f,g\},\{g,h\}\}$, then $\{e,f\},\{g,h\}$ is the only pair of disjoint bases. In the second instance, if $r_{M_1}(E - B_1) = 0$, then there are no M_1-independent transversals of $(B_1, \ldots, B_n)$.

The remainder of this paper is taken up with the proof of the theorem. In section 2, we prove that Theorem 1.1 holds when $n = 3$. This result is used, in section 3, as the base case of an inductive proof of Theorem 1.1. The induction argument is surprisingly straightforward and can be read independently of section 2.

2. The case $n = 3$.

For basic concepts in matroid theory, the reader is referred to Oxley [7]. We follow the same notation as Oxley throughout this paper.

A closed set in a matroid is commonly known as a flat. We will primarily be interested in rank-2 flats, or *lines*. In the proof of Theorem 2.1, we make frequent use

*Received by the editors March 29, 2006; accepted for publication (in revised form) June 19, 2006; published electronically December 15, 2006. This research was partially supported by grants from the Natural Sciences and Engineering Research Council of Canada and the New Zealand Marsden Fund.

http://www.siam.org/journals/sidma/20-4/65559.html

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