Discrete Mathematics 309 (2009) 3083–3091

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Pruning processes and a new characterization of convex geometries

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ARTICLE INFO

Article history:
Received 2 February 2008
Accepted 13 August 2008
Available online 20 September 2008

Keywords:
Antimatroid
Convex geometry
k-SAT
Pruning process
Removal process

ABSTRACT

We provide a new characterization of convex geometries via a multivariate version of an identity that was originally proved, in a special case arising from the k-SAT problem, by Maneva, Mossel and Wainwright. We thus highlight the connection between various characterizations of convex geometries and a family of removal processes studied in the literature on random structures.

1. Introduction

This article studies a general class of procedures in which the elements of a set are removed one at a time according to a given rule. We refer to such a procedure as a *removal process*. If every element which is removable at some stage of the process remains removable at any later stage, we call this a *pruning process*. The subsets that one can reach through a pruning process have the elegant combinatorial structure of a *convex geometry*. Our first goal is to highlight the role of convex geometries in the literature on random structures, where many pruning processes have been studied without exploiting their connection to these objects. Our second contribution is a proof that a generalization of a polynomial identity, first obtained for a specific removal process in [17], provides a new characterization of pruning processes and of convex geometries. To prove this result we also show how a convex geometry is equivalent to a particular kind of interval partition of the Boolean lattice.

Two equivalent families of combinatorial objects, known as *convex geometries* and *antimatroids*, were defined in the 1980s [8,11]. The fact that these objects can be characterized via pruning processes has been known since then. Some examples of pruning processes considered at that time are the removal of vertices of the convex hull of a set of points in $\mathbb{R}^n$, the removal of the leaves of a tree, and the removal of minimal elements of a poset. More recently various pruning processes have been studied in the literature on random structures, and referred to also as peeling, stripping, whitening, coarsening, identifying, etc. A typical example is the removal of vertices of degree less than k in the process of finding the k-core of a random (hyper)graph.

In [17], a surprising identity was proved to hold for a particular removal process which arises in the context of the k-SAT problem. In this paper, we answer the question posed by Mossel [23] of characterizing the combinatorial structures that satisfy (the multivariate version of) that identity: they are precisely the convex geometries or equivalently the pruning processes. That is the content of our main result, Theorem 3.1 and Corollary 3.2. It says that any pruning process has the following two properties, and that in fact either of these two properties characterizes pruning processes among removal processes.

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doi:10.1016/j.disc.2008.08.020