Chapter 44

Submodular functions and polymatroids

In this chapter we describe some of the basic properties of a second main object of the present part, the submodular function. Each submodular function gives a polymatroid, which is a generalization of the independent set polytope of a matroid. We prove as a main result the theorem of Edmonds [1970b] that the vertices of a polymatroid are integer if and only if the associated submodular function is integer.

44.1. Submodular functions and polymatroids

Let f be a *set function* on a set S, that is, a function defined on the collection $\mathcal{P}(S)$ of all subsets of S. The function f is called *submodular* if

(44.1) $$f(T) + f(U) \geq f(T \cap U) + f(T \cup U)$$

for all subsets T, U of S. Similarly, f is called *supermodular* if $-f$ is submodular, i.e., if f satisfies (44.1) with the opposite inequality sign. f is *modular* if f is both submodular and supermodular, i.e., if f satisfies (44.1) with equality.

A set function f on S is called *nondecreasing* if $f(T) \leq f(U)$ whenever $T \subseteq U \subseteq S$, and *nonincreasing* if $f(T) \geq f(U)$ whenever $T \subseteq U \subseteq S$.

As usual, denote for each function $w : S \to \mathbb{R}$ and for each subset U of S,

(44.2) $$w(U) := \sum_{s \in U} w(s).$$

So w may be considered also as a set function on S, and one easily sees that w is modular, and that each modular set function f on S with $f(\emptyset) = 0$ may be obtained in this way. (More generally, each modular set function f on S satisfies $f(U) = w(U) + \gamma$ (for $U \subseteq S$), for some unique function $w : S \to \mathbb{R}$ and some unique real number γ.)

In a sense, submodularity is the discrete analogue of convexity. If we define, for any $f : \mathcal{P}(S) \to \mathbb{R}$ and any $x \in S$, a function $\delta f_x : \mathcal{P}(S) \to \mathbb{R}$ by: $\delta f_x(T) := f(T \cup \{x\}) - f(T)$, then f is submodular if and only if δf_x is nonincreasing for each $x \in S$.

In other words: