Chapter 41

Matroid intersection

Edmonds discovered that matroids have even more algorithmic power than just that of the greedy method. He showed that there exist efficient algorithms also for *intersections* of matroids. That is, a maximum-weight common independent set in *two* matroids can be found in strongly polynomial time. Edmonds also found good min-max characterizations for matroid intersection.

Matroid intersection yields a motivation for studying matroids: we may apply it to two matroids from different classes of examples of matroids, and thus we obtain methods that exceed the bounds of any particular class.

We should note here that if $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ are matroids, then $(S, \mathcal{I}_1 \cap \mathcal{I}_2)$ need not be a matroid. (An example with $|S| = 3$ is easy to construct.)

Moreover, the problem of finding a maximum-size common independent set in *three* matroids is NP-complete (as finding a Hamiltonian circuit in a directed graph is a special case; also, finding a common transversal of three partitions is a special case).

41.1. Matroid intersection theorem

Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be two matroids, on the same set S. Consider the collection $\mathcal{I}_1 \cap \mathcal{I}_2$ of *common independent sets*. The pair $(S, \mathcal{I}_1 \cap \mathcal{I}_2)$ is generally *not* a matroid again.

Edmonds [1970b] showed the following formula, for which he gave two proofs — one based on linear programming duality and total unimodularity (see the proof of Theorem 41.12 below), and one reducing it to the matroid union theorem (see Corollary 42.1a and the remark thereafter). We give the direct proof implicit in Brualdi [1971e].

Theorem 41.1 (matroid intersection theorem)**.** *Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be matroids, with rank functions r_1 and r_2, respectively. Then the maximum size of a set in $\mathcal{I}_1 \cap \mathcal{I}_2$ is equal to*

$$\min_{U \subseteq S} (r_1(U) + r_2(S \setminus U)). \tag{41.1}$$