Chapter 40

The greedy algorithm and the independent set polytope

We now pass to algorithmic and polyhedral aspects of matroids. We show that the greedy algorithm characterizes matroids and that it implies a characterization of the independent set polytope (the convex hull of the incidence vectors of the independent sets).
Algorithmic and polyhedral aspects of the *intersection* of two matroids will be studied in Chapter 41.

40.1. The greedy algorithm

Let $\mathcal{I}$ be a nonempty collection of subsets of a finite set S closed under taking subsets. For any weight function $w : S \to \mathbb{R}$ we want to find a set I in $\mathcal{I}$ maximizing $w(I)$. The *greedy algorithm* consists of setting $I := \emptyset$, and next repeatedly choosing $y \in S \setminus I$ with $I \cup \{y\} \in \mathcal{I}$ and with $w(y)$ as large as possible. We stop if no such y exists.

For general collections $\mathcal{I}$ of this kind this need not lead to an optimum solution. Indeed, matroids are precisely the structures where it always works, as the following theorem shows (Rado [1957] (necessity) and Gale [1968] and Edmonds [1971] (sufficiency)):

Theorem 40.1. *Let $\mathcal{I}$ be a nonempty collection of subsets of a set S, closed under taking subsets. Then the pair $(S, \mathcal{I})$ is a matroid if and only if for each weight function $w : S \to \mathbb{R}_+$, the greedy algorithm leads to a set I in $\mathcal{I}$ of maximum weight $w(I)$.*

Proof. *Necessity.* Let $(S, \mathcal{I})$ be a matroid and let $w : S \to \mathbb{R}_+$ be any weight function on S. Call an independent set I *good* if it is contained in a maximum-weight base. It suffices to show that if I is good, and y is an element in $S \setminus I$ with $I + y \in \mathcal{I}$ and with $w(y)$ as large as possible, then $I + y$ is good.

As I is good, there exists a maximum-weight base $B \supseteq I$. If $y \in B$, then $I + y$ is good again. If $y \notin B$, then there exists a base B' containing $I + y$ and contained in $B + y$. So $B' = B - z + y$ for some $z \in B \setminus I$. As $w(y)$ is chosen maximum and as $I + z \in \mathcal{I}$ since $I + z \subseteq B$, we know $w(y) \geq w(z)$.