Oriented Matroids - From Matroids and Digraphs to Polyhedral Theory

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These notes are intended for participants of the MAA Shortcourse on Matroid Theory January 2011 in New Orleans. Therefore our intention is not to give an introduction into the theory of oriented matroids from scratch (as in [9]), but to recapture how they arise from matroids. Therefore, we assume basic knowledge of matroid theory.

For a gentle introduction into the theory of oriented matroids we recommend [1], the standard reference is [2].

1 Directed Planar Graphs and their Duals

1.1 Introduction

A *graph* $G = (V, E)$, where V is a finite set (of *vertices*) and $E \subseteq \binom{V}{2} \cup V$ is a finite set of *edges* (one- or two-element subsets of the vertices), may be considered as a symmetric, binary relation. If we drop the symmetry requirement we arrive at digraphs.

So, the difference between graphs and digraphs is that the arcs have an orientation from one end vertex to the other. The purpose of this section is to give an idea how we can save at least some of the orientation information to matroids, where we do no longer have vertices.

A main concern should be that the orientation is somewhat compatible with duality. Thus maybe we should start with a planar graph and its dual.

1.2 An example

Consider the following orientation of the dodekahedron (see Figure 1). How to choose the direction for the dual arcs? Here we have chosen the orientation such that the dual arc has the right-of-way, i.e. the primal arc points from the left to the right.

Figure 3 illustrates that directed circuits give rise to directed cuts and vice versa.

It seems that the orientation of the graph can be encoded as partitions of the circuits and partitions of the cuts into forward and backward arcs. If we