3

Unimodular Matroids

NEIL WHITE

3.1. Equivalent Conditions for Unimodularity

Unimodular matroids were defined in Chapter 1 as the class of matroids which may be coordinatized over every field. In Theorem 3.1.1 we give a number of equivalent characterizations of this class. Certainly the two most striking and powerful of these are Tutte's excluded minor characterization and Seymour's decomposition [conditions (8) and (9) of Theorem 3.1.1]. We first need some definitions and notation.

A coordinatization of $M(S)$ over $\mathbb{Q}$ given by $n \times N$ matrix A with integer entries, and $n < N$, is said to be *totally unimodular* if every $k \times k$ submatrix has determinant equal to 0 or ± 1, for all k, $1 \leqslant k \leqslant n$, and is said to be *locally unimodular* if every $n \times n$ submatrix has determinant equal to 0 or ± 1.

Let D be the bond-element incidence matrix of $M(S)$. That is, if $R_1, R_2, \ldots, R_m$ are the bonds of M and $S = \{x_1, x_2, \ldots, x_N\}$, then $D = (b_{ij})$, with $b_{ij} = 1$ if $x_j \in R_i$, and $b_{ij} = 0$ otherwise. Similarly, let E be the circuit-element incidence matrix of M. Suppose that it is possible to change some of the entries of D from 1 to -1 to get a matrix D', and similarly, change E to E', so that $D'(E')^t = 0$ over $\mathbb{Q}$ (where t denotes transpose). Then we say that M is *signable*. [This is closely related to the notion of orientability, considered in a chapter of White (1988).]

In Section 7.6 of White (1986) 1-sums, 2-sums, or (for binary matroids) 3-sums of two matroids $M_1(E_1)$ and $M_2(E_2)$ were defined as $P_x(M_1, M_2) - x$, where $P_x(M_1, M_2)$ is the generalized parallel connection across a flat x, and x is empty, a point, or a 3-point line (respectively). To avoid triviality we insist that $P_x(M_1, M_2) - x$ have larger cardinality than M_1 or M_2. For binary matroids, with which we are concerned here, an equivalent definition is to say that each of these three sums is the matroid $M_1 \triangle M_2$ on the symmetric difference $E_1 \triangle E_2$ which has as its cycles (i.e., disjoint unions of circuits) all subsets of the form $C_1 \triangle C_2$, where C_i is a cycle of M_i. Then