Chapter 39

Matroids

This chapter gives the basic definitions, examples, and properties of matroids. We use the shorthand notation

$$X + y := X \cup \{y\} \text{ and } X - y := X \setminus \{y\}.$$

39.1. Matroids

A pair $(S, \mathcal{I})$ is called a *matroid* if S is a finite set and $\mathcal{I}$ is a nonempty collection of subsets of S satisfying:

(39.1) (i) if $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$,
(ii) if $I, J \in \mathcal{I}$ and $|I| < |J|$, then $I + z \in \mathcal{I}$ for some $z \in J \setminus I$.

(These axioms are given by Whitney [1935].)

Given a matroid $M = (S, \mathcal{I})$, a subset I of S is called *independent* if I belongs to $\mathcal{I}$, and *dependent* otherwise. For $U \subseteq S$, a subset B of U is called a *base* of U if B is an inclusionwise maximal independent subset of U. That is, $B \in \mathcal{I}$ and there is no $Z \in \mathcal{I}$ with $B \subset Z \subseteq U$.

It is not difficult to see that, under condition (39.1)(i), condition (39.1)(ii) is equivalent to:

(39.2) for any subset U of S, any two bases of U have the same size.

The common size of the bases of a subset U of S is called the *rank* of U, denoted by $r_M(U)$. If the matroid is clear from the context, we write $r(U)$ for $r_M(U)$.

A set is called simply a *base* if it is a base of S. The common size of all bases is called the *rank* of the matroid. A subset of S is called *spanning* if it contains a base as a subset. So bases are just the inclusionwise minimal spanning sets, and also just the independent spanning sets. A *circuit* of a matroid is an inclusionwise minimal dependent set. A *loop* is an element s such that $\{s\}$ is a circuit. Two elements s, t of S are called *parallel* if $\{s, t\}$ is a circuit.

Nakasawa [1935] showed the equivalence of axiom system (39.1) with an ostensibly weaker system, which will be useful in proofs: