Chapter 39

Matroids

This chapter gives the basic definitions, examples, and properties of matroids. We use the shorthand notation

$$X + y := X \cup \{y\} \text{ and } X - y := X \setminus \{y\}.$$

39.1. Matroids

A pair $(S, \mathcal{I})$ is called a *matroid* if S is a finite set and $\mathcal{I}$ is a nonempty collection of subsets of S satisfying:

(39.1) (i) if $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$,
(ii) if $I, J \in \mathcal{I}$ and $|I| < |J|$, then $I + z \in \mathcal{I}$ for some $z \in J \setminus I$.

(These axioms are given by Whitney [1935].)

Given a matroid $M = (S, \mathcal{I})$, a subset I of S is called *independent* if I belongs to $\mathcal{I}$, and *dependent* otherwise. For $U \subseteq S$, a subset B of U is called a *base* of U if B is an inclusionwise maximal independent subset of U. That is, $B \in \mathcal{I}$ and there is no $Z \in \mathcal{I}$ with $B \subset Z \subseteq U$.

It is not difficult to see that, under condition (39.1)(i), condition (39.1)(ii) is equivalent to:

(39.2) for any subset U of S, any two bases of U have the same size.

The common size of the bases of a subset U of S is called the *rank* of U, denoted by $r_M(U)$. If the matroid is clear from the context, we write $r(U)$ for $r_M(U)$.

A set is called simply a *base* if it is a base of S. The common size of all bases is called the *rank* of the matroid. A subset of S is called *spanning* if it contains a base as a subset. So bases are just the inclusionwise minimal spanning sets, and also just the independent spanning sets. A *circuit* of a matroid is an inclusionwise minimal dependent set. A *loop* is an element s such that $\{s\}$ is a circuit. Two elements s, t of S are called *parallel* if $\{s, t\}$ is a circuit.

Nakasawa [1935] showed the equivalence of axiom system (39.1) with an ostensibly weaker system, which will be useful in proofs:

3

Unimodular Matroids

NEIL WHITE

3.1. Equivalent Conditions for Unimodularity

Unimodular matroids were defined in Chapter 1 as the class of matroids which may be coordinatized over every field. In Theorem 3.1.1 we give a number of equivalent characterizations of this class. Certainly the two most striking and powerful of these are Tutte's excluded minor characterization and Seymour's decomposition [conditions (8) and (9) of Theorem 3.1.1]. We first need some definitions and notation.

A coordinatization of $M(S)$ over $\mathbb{Q}$ given by $n \times N$ matrix A with integer entries, and $n < N$, is said to be *totally unimodular* if every $k \times k$ submatrix has determinant equal to 0 or ± 1, for all k, $1 \leqslant k \leqslant n$, and is said to be *locally unimodular* if every $n \times n$ submatrix has determinant equal to 0 or ± 1.

Let D be the bond-element incidence matrix of $M(S)$. That is, if $R_1, R_2, \ldots, R_m$ are the bonds of M and $S = \{x_1, x_2, \ldots, x_N\}$, then $D = (b_{ij})$, with $b_{ij} = 1$ if $x_j \in R_i$, and $b_{ij} = 0$ otherwise. Similarly, let E be the circuit-element incidence matrix of M. Suppose that it is possible to change some of the entries of D from 1 to -1 to get a matrix D', and similarly, change E to E', so that $D'(E')^t = 0$ over $\mathbb{Q}$ (where t denotes transpose). Then we say that M is *signable*. [This is closely related to the notion of orientability, considered in a chapter of White (1988).]

In Section 7.6 of White (1986) 1-sums, 2-sums, or (for binary matroids) 3-sums of two matroids $M_1(E_1)$ and $M_2(E_2)$ were defined as $P_x(M_1, M_2) - x$, where $P_x(M_1, M_2)$ is the generalized parallel connection across a flat x, and x is empty, a point, or a 3-point line (respectively). To avoid triviality we insist that $P_x(M_1, M_2) - x$ have larger cardinality than M_1 or M_2. For binary matroids, with which we are concerned here, an equivalent definition is to say that each of these three sums is the matroid $M_1 \triangle M_2$ on the symmetric difference $E_1 \triangle E_2$ which has as its cycles (i.e., disjoint unions of circuits) all subsets of the form $C_1 \triangle C_2$, where C_i is a cycle of M_i. Then

A Short Proof of Tutte's Characterization of Totally Unimodular Matrices

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Dedicated to Alan J. Hoffman on the occasion of his 65th birthday.

Submitted by Robert Bixby

ABSTRACT

We give, in terms of totally unimodular matrices, a short and easy proof of Tutte's characterization of regular matroids.

1. INTRODUCTION

We give a short and easy proof of the following well-known result of Tutte (1958, 1965, 1971):

TUTTE'S THEOREM. *Let A be a* $\{0,1\}$*-matrix. Then the following are equivalent*:

(i) *A has a totally unimodular signing*,
(ii) *A cannot be transformed to*

$$M(F_7) := \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

by applying (repeatedly) the following operations:

(1) *deleting rows or columns*,
permuting rows or columns,
taking the transposed matrix,
pivoting over GF(2).

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ON THEOREMS OF WHITNEY AND TUTTE

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Short proofs of two theorems are given: (i) Whitney's 2-isomorphism theorem characterizing all graphs with the same cycle matroid, and (ii) Tutte's excluded minor characterization of those binary matroids that are graphic. Graph connectivity plays an important role in both proofs.

1. Introduction

Familiarity with graph and matroid theory is assumed; see [1] and [10]. Where G is a graph with $S \subseteq E(G)$, $G[S]$ denotes the subgraph induced by S. A partition $\{S, T\}$ of $E(G)$ is a *k-separation* of G, for k a positive integer, if $|S| \geq k \leq |T|$ and $|V(G[S]) \cap V(G[T])| \leq k$. A graph is *n-connected*, for n a positive integer, if it has no k-separation for $k < n$; a 2-connected graph is *nonseparable*.

Let G be a nonseparable graph with 2-separation $\{S, T\}$ and let $V(G[S]) \cap V(G[T]) = \{x, y\}$. Let G' be the graph obtained from G by interchanging in $G[S]$ the incidences of the edges at x and y. Then G' is obtained from G by *reversing* $G[S]$. A graph obtainable from G by a sequence of reversals is *2-isomorphic* to G.

Let $M(G)$ denote the cycle matroid of G. Whitney [12] proved the following result.

(1.1) Let G and G' be nonseparable graphs. Then $M(G) = M(G')$ if and only if G and G' are 2-isomorphic.

Let K_5 and $K_{3,3}$ denote the Kuratowski graphs and let F_7 denote the Fano matroid. Denote the dual of a matroid M by M^*.

Tutte [8] proved the following result.

(1.2) Let M be a binary matroid. Then M is graphic if and only if M has no F_7, F_7^*, $M^*(K_5)$ or $M^*(K_{3,3})$ minor.

The purpose of this paper is to provide new short proofs of (1.1) and (1.2). It is hoped the present proofs will be more accessible and will provide additional insight into the results. Graph connectivity plays an important role in both proofs.

The paper is outlined as follows: (1.1) is proved in Section 2, some preliminary

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Oriented Matroids - From Matroids and Digraphs to Polyhedral Theory

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These notes are intended for participants of the MAA Shortcourse on Matroid Theory January 2011 in New Orleans. Therefore our intention is not to give an introduction into the theory of oriented matroids from scratch (as in [9]), but to recapture how they arise from matroids. Therefore, we assume basic knowledge of matroid theory.

For a gentle introduction into the theory of oriented matroids we recommend [1], the standard reference is [2].

1 Directed Planar Graphs and their Duals

1.1 Introduction

A *graph* $G = (V, E)$, where V is a finite set (of *vertices*) and $E \subseteq \binom{V}{2} \cup V$ is a finite set of *edges* (one- or two-element subsets of the vertices), may be considered as a symmetric, binary relation. If we drop the symmetry requirement we arrive at digraphs.

So, the difference between graphs and digraphs is that the arcs have an orientation from one end vertex to the other. The purpose of this section is to give an idea how we can save at least some of the orientation information to matroids, where we do no longer have vertices.

A main concern should be that the orientation is somewhat compatible with duality. Thus maybe we should start with a planar graph and its dual.

1.2 An example

Consider the following orientation of the dodekahedron (see Figure 1). How to choose the direction for the dual arcs? Here we have chosen the orientation such that the dual arc has the right-of-way, i.e. the primal arc points from the left to the right.

Figure 3 illustrates that directed circuits give rise to directed cuts and vice versa.

It seems that the orientation of the graph can be encoded as partitions of the circuits and partitions of the cuts into forward and backward arcs. If we

6 ORIENTED MATROIDS

Jürgen Richter-Gebert and Günter M. Ziegler

INTRODUCTION

The theory of *oriented matroids* provides a broad setting in which to model, describe, and analyze combinatorial properties of geometric configurations. Mathematical objects of study that appear to be disjoint and independent, such as *point and vector configurations*, *arrangements of hyperplanes*, *convex polytopes*, *directed graphs*, and *linear programs* find a common generalization in the language of oriented matroids.

The oriented matroid of a finite set of points P extracts relative position and orientation information from the configuration; for example, it can be given by a list of signs that encodes the orientations of all the bases of P. In the passage from a concrete point configuration to its oriented matroid, metrical information is lost, but many structural properties of P have their counterparts at the—purely combinatorial—level of the oriented matroid. (In computational geometry, the oriented matroid data of an unlabelled point configuration are sometimes called the *order type*.) From the oriented matroid of a configuration of points, one can compute not only that face lattice of the convex hull, but also the set of all its triangulations and subdivisions (cf. Chapter 16).

We first introduce oriented matroids in the context of several models and motivations (Section 6.1). Then we present some equivalent axiomatizations (Section 6.2). Finally, we discuss concepts that play central roles in the theory of oriented matroids (Section 6.3), among them *duality*, *realizability*, the study of *simplicial cells*, and the treatment of *convexity*.

6.1 MODELS AND MOTIVATIONS

This section discusses geometric examples that are usually treated on the level of concrete coordinates, but where an "oriented matroid point of view" gives deeper insight. We also present these examples as standard models that provide intuition for the behavior of general oriented matroids.

6.1.1 ORIENTED BASES OF VECTOR CONFIGURATIONS

GLOSSARY

Vector configuration X: A matrix $X = (x_1, \ldots, x_n) \in (\mathbb{R}^d)^n$, usually assumed to have full rank d.

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Chapter 40

The greedy algorithm and the independent set polytope

We now pass to algorithmic and polyhedral aspects of matroids. We show that the greedy algorithm characterizes matroids and that it implies a characterization of the independent set polytope (the convex hull of the incidence vectors of the independent sets).
Algorithmic and polyhedral aspects of the *intersection* of two matroids will be studied in Chapter 41.

40.1. The greedy algorithm

Let $\mathcal{I}$ be a nonempty collection of subsets of a finite set S closed under taking subsets. For any weight function $w : S \to \mathbb{R}$ we want to find a set I in $\mathcal{I}$ maximizing $w(I)$. The *greedy algorithm* consists of setting $I := \emptyset$, and next repeatedly choosing $y \in S \setminus I$ with $I \cup \{y\} \in \mathcal{I}$ and with $w(y)$ as large as possible. We stop if no such y exists.

For general collections $\mathcal{I}$ of this kind this need not lead to an optimum solution. Indeed, matroids are precisely the structures where it always works, as the following theorem shows (Rado [1957] (necessity) and Gale [1968] and Edmonds [1971] (sufficiency)):

Theorem 40.1. *Let $\mathcal{I}$ be a nonempty collection of subsets of a set S, closed under taking subsets. Then the pair $(S, \mathcal{I})$ is a matroid if and only if for each weight function $w : S \to \mathbb{R}_+$, the greedy algorithm leads to a set I in $\mathcal{I}$ of maximum weight $w(I)$.*

Proof. *Necessity.* Let $(S, \mathcal{I})$ be a matroid and let $w : S \to \mathbb{R}_+$ be any weight function on S. Call an independent set I *good* if it is contained in a maximum-weight base. It suffices to show that if I is good, and y is an element in $S \setminus I$ with $I + y \in \mathcal{I}$ and with $w(y)$ as large as possible, then $I + y$ is good.

As I is good, there exists a maximum-weight base $B \supseteq I$. If $y \in B$, then $I + y$ is good again. If $y \notin B$, then there exists a base B' containing $I + y$ and contained in $B + y$. So $B' = B - z + y$ for some $z \in B \setminus I$. As $w(y)$ is chosen maximum and as $I + z \in \mathcal{I}$ since $I + z \subseteq B$, we know $w(y) \geq w(z)$.

Chapter 41

Matroid intersection

Edmonds discovered that matroids have even more algorithmic power than just that of the greedy method. He showed that there exist efficient algorithms also for *intersections* of matroids. That is, a maximum-weight common independent set in *two* matroids can be found in strongly polynomial time. Edmonds also found good min-max characterizations for matroid intersection.

Matroid intersection yields a motivation for studying matroids: we may apply it to two matroids from different classes of examples of matroids, and thus we obtain methods that exceed the bounds of any particular class.

We should note here that if $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ are matroids, then $(S, \mathcal{I}_1 \cap \mathcal{I}_2)$ need not be a matroid. (An example with $|S| = 3$ is easy to construct.)

Moreover, the problem of finding a maximum-size common independent set in *three* matroids is NP-complete (as finding a Hamiltonian circuit in a directed graph is a special case; also, finding a common transversal of three partitions is a special case).

41.1. Matroid intersection theorem

Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be two matroids, on the same set S. Consider the collection $\mathcal{I}_1 \cap \mathcal{I}_2$ of *common independent sets*. The pair $(S, \mathcal{I}_1 \cap \mathcal{I}_2)$ is generally *not* a matroid again.

Edmonds [1970b] showed the following formula, for which he gave two proofs — one based on linear programming duality and total unimodularity (see the proof of Theorem 41.12 below), and one reducing it to the matroid union theorem (see Corollary 42.1a and the remark thereafter). We give the direct proof implicit in Brualdi [1971e].

Theorem 41.1 (matroid intersection theorem)**.** *Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be matroids, with rank functions r_1 and r_2, respectively. Then the maximum size of a set in $\mathcal{I}_1 \cap \mathcal{I}_2$ is equal to*

$$\min_{U \subseteq S} (r_1(U) + r_2(S \setminus U)). \tag{41.1}$$

Chapter 44

Submodular functions and polymatroids

In this chapter we describe some of the basic properties of a second main object of the present part, the submodular function. Each submodular function gives a polymatroid, which is a generalization of the independent set polytope of a matroid. We prove as a main result the theorem of Edmonds [1970b] that the vertices of a polymatroid are integer if and only if the associated submodular function is integer.

44.1. Submodular functions and polymatroids

Let f be a *set function* on a set S, that is, a function defined on the collection $\mathcal{P}(S)$ of all subsets of S. The function f is called *submodular* if

$$f(T) + f(U) \geq f(T \cap U) + f(T \cup U) \tag{44.1}$$

for all subsets T, U of S. Similarly, f is called *supermodular* if $-f$ is submodular, i.e., if f satisfies (44.1) with the opposite inequality sign. f is *modular* if f is both submodular and supermodular, i.e., if f satisfies (44.1) with equality.

A set function f on S is called *nondecreasing* if $f(T) \leq f(U)$ whenever $T \subseteq U \subseteq S$, and *nonincreasing* if $f(T) \geq f(U)$ whenever $T \subseteq U \subseteq S$.

As usual, denote for each function $w : S \to \mathbb{R}$ and for each subset U of S,

$$w(U) := \sum_{s \in U} w(s). \tag{44.2}$$

So w may be considered also as a set function on S, and one easily sees that w is modular, and that each modular set function f on S with $f(\emptyset) = 0$ may be obtained in this way. (More generally, each modular set function f on S satisfies $f(U) = w(U) + \gamma$ (for $U \subseteq S$), for some unique function $w : S \to \mathbb{R}$ and some unique real number γ.)

In a sense, submodularity is the discrete analogue of convexity. If we define, for any $f : \mathcal{P}(S) \to \mathbb{R}$ and any $x \in S$, a function $\delta f_x : \mathcal{P}(S) \to \mathbb{R}$ by: $\delta f_x(T) := f(T \cup \{x\}) - f(T)$, then f is submodular if and only if δf_x is nonincreasing for each $x \in S$.

In other words:

Chapter 45

Submodular function minimization

This chapter describes a strongly polynomial-time algorithm to find the minimum value of a submodular function. It suffices that the submodular function is given by a value giving oracle.
One application of submodular function minimization is optimizing over the intersection of two polymatroids. This will be discussed in Chapter 47.

45.1. Submodular function minimization

It was shown by Grötschel, Lovász, and Schrijver [1981] that the minimum value of a rational-valued submodular set function f on S can be found in polynomial time, if f is given by a value giving oracle and an upper bound B is given on the numerators and denominators of the values of f. The running time is bounded by a polynomial in $|S|$ and $\log B$. This algorithm is based on the ellipsoid method: we can assume that $f(\emptyset) = 0$ (by resetting $f(U) := f(U) - f(\emptyset)$ for all $U \subseteq S$); then with the greedy algorithm, we can optimize over EP_f in polynomial time (Corollary 44.3b), hence the separation problem for EP_f is solvable in polynomial time, hence also the separation problem for

(45.1) $\qquad P := EP_f \cap \{x \mid x \leq \mathbf{0}\},$

and therefore also the optimization problem for P. Now the maximum value of $x(S)$ over P is equal to the minimum value of f (by (44.8), (44.9), and (44.34)).

Having a polynomial-time method to find the minimum value of a submodular function, we can turn it into a polynomial-time method to find a subset T of S minimizing $f(T)$: For each $s \in S$, we can determine if the minimum value of f over all subsets of S is equal to the minimum value of f over subsets of $S \setminus \{s\}$. If so, we reset $S := S \setminus \{s\}$. Doing this for all elements of S, we are left with a set T minimizing f over all subsets of (the original) S.

Grötschel, Lovász, and Schrijver [1988] showed that this algorithm can be turned into a strongly polynomial-time method. Cunningham [1985b] gave a

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Pruning processes and a new characterization of convex geometries

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ABSTRACT

We provide a new characterization of convex geometries via a multivariate version of an identity that was originally proved, in a special case arising from the k-SAT problem, by Maneva, Mossel and Wainwright. We thus highlight the connection between various characterizations of convex geometries and a family of removal processes studied in the literature on random structures.

1. Introduction

This article studies a general class of procedures in which the elements of a set are removed one at a time according to a given rule. We refer to such a procedure as a *removal process*. If every element which is removable at some stage of the process remains removable at any later stage, we call this a *pruning process*. The subsets that one can reach through a pruning process have the elegant combinatorial structure of a *convex geometry*. Our first goal is to highlight the role of convex geometries in the literature on random structures, where many pruning processes have been studied without exploiting their connection to these objects. Our second contribution is a proof that a generalization of a polynomial identity, first obtained for a specific removal process in [17], provides a new characterization of pruning processes and of convex geometries. To prove this result we also show how a convex geometry is equivalent to a particular kind of interval partition of the Boolean lattice.

Two equivalent families of combinatorial objects, known as *convex geometries* and *antimatroids*, were defined in the 1980s [8,11]. The fact that these objects can be characterized via pruning processes has been known since then. Some examples of pruning processes considered at that time are the removal of vertices of the convex hull of a set of points in $\mathbb{R}^n$, the removal of the leaves of a tree, and the removal of minimal elements of a poset. More recently various pruning processes have been studied in the literature on random structures, and referred to also as peeling, stripping, whitening, coarsening, identifying, etc. A typical example is the removal of vertices of degree less than k in the process of finding the k-core of a random (hyper)graph.

In [17], a surprising identity was proved to hold for a particular removal process which arises in the context of the k-SAT problem. In this paper, we answer the question posed by Mossel [23] of characterizing the combinatorial structures that satisfy (the multivariate version of) that identity: they are precisely the convex geometries or equivalently the pruning processes. That is the content of our main result, Theorem 3.1 and Corollary 3.2. It says that any pruning process has the following two properties, and that in fact either of these two properties characterizes pruning processes among removal processes.

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ROTA'S BASIS CONJECTURE FOR PAVING MATROIDS*

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Abstract. Rota conjectured that, given n disjoint bases of a rank-n matroid M, there are n disjoint transversals of these bases that are all bases of M. We prove a stronger statement for the class of paving matroids.

Key words. Rota's basis conjecture, paving matroids

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1. Introduction.

We prove the following theorem.

THEOREM 1.1. *Let $B_1, \dots, B_n$ be disjoint sets of size $n \geq 3$, and let $M_1, \dots, M_n$ be rank-n paving matroids on $\bigcup_i B_i$ such that B_i is a basis of M_i for each $i \in \{1, \dots, n\}$. Then there exist n disjoint transversals $A_1, \dots, A_n$ of $(B_1, \dots, B_n)$ such that A_i is a basis of M_i for each $i \in \{1, \dots, n\}$.*

A *paving matroid* M is a matroid in which each circuit has size $r(M)$ or $r(M)+1$, where $r(M)$ is the rank of M. Theorem 1.1 implies Rota's basis conjecture for paving matroids.

CONJECTURE 1.2 (Rota (see [6])). *Given n disjoint bases $B_1, \dots, B_n$ in a rank-n matroid M, there exist n disjoint transversals $A_1, \dots, A_n$ of $(B_1, \dots, B_n)$ that are all bases of M.*

For $n = 2$, Conjecture 1.2 follows immediately from basis exchange in matroids. Chan [2] proved the conjecture for $n = 3$. Wild [9] proved a stronger conjecture for the class of strongly base-orderable matroids, while more recently a slightly weaker result was proved for a general matroid (Ponomarenko [8]). Further partial results may be found in [1], [3], [4], [5], and [9].

Theorem 1.1 fails for both $n = 2$ and matroids in general. When $n = 2$, if we take $\mathcal{B}(M_1) = \{\{e, f\}, \{e, g\}, \{f, h\}, \{g, h\}\}$ and $\mathcal{B}(M_2) = \{\{e, f\}, \{e, h\}, \{f, g\}, \{g, h\}\}$, then $\{e, f\}, \{g, h\}$ is the only pair of disjoint bases. In the second instance, if $r_{M_1}(E - B_1) = 0$, then there are no M_1-independent transversals of $(B_1, \dots, B_n)$.

The remainder of this paper is taken up with the proof of the theorem. In section 2, we prove that Theorem 1.1 holds when $n = 3$. This result is used, in section 3, as the base case of an inductive proof of Theorem 1.1. The induction argument is surprisingly straightforward and can be read independently of section 2.

2. The case $n = 3$.

For basic concepts in matroid theory, the reader is referred to Oxley [7]. We follow the same notation as Oxley throughout this paper.

A closed set in a matroid is commonly known as a flat. We will primarily be interested in rank-2 flats, or *lines*. In the proof of Theorem 2.1, we make frequent use

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Halfway to Rota's basis conjecture

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Abstract

In 1989, Rota made the following conjecture. Given n bases $B_1, \dots, B_n$ in an n-dimensional vector space V, one can always find n disjoint bases of V, each containing exactly one element from each B_i (we call such bases *transversal bases*). Rota's basis conjecture remains wide open despite its apparent simplicity and the efforts of many researchers (for example, the conjecture was recently the subject of the collaborative "Polymath" project). In this paper we prove that one can always find $(1/2 - o(1))n$ disjoint transversal bases, improving on the previous best bound of $\Omega(n/\log n)$. Our results also apply to the more general setting of matroids.

1 Introduction

Given bases $B_1, \dots, B_n$ in an n-dimensional vector space V, a *transversal basis* is a basis of V containing a single distinguished vector from each of $B_1, \dots, B_n$. Two transversal bases are said to be *disjoint* if their distinguished vectors from B_i are distinct, for each i (here "distinguished" means that two copies of the same vector appearing in two B_is are considered distinct). In 1989, Rota conjectured (see [23, Conjecture 4]) that for any vector space V over a characteristic-zero field, and any choice of $B_1, \dots, B_n$, one can always find n pairwise disjoint transversal bases.

Despite the apparent simplicity of this conjecture, it remains wide open, and has surprising connections to apparently unrelated subjects. Specifically, it was discovered by Huang and Rota [23] that there are implications between Rota's basis conjecture, the Alon–Tarsi conjecture [2] concerning enumeration of even and odd Latin squares, and a certain conjecture concerning the supersymmetric bracket algebra.

Rota also observed that an analogous conjecture could be made in the much more general setting of *matroids*, which are objects that abstract the combinatorial properties of linear independence in vector spaces. Specifically, a finite matroid $M = (E, \mathcal{I})$ consists of a finite ground set E (whose elements may be thought of as vectors in a vector space), and a collection $\mathcal{I}$ of subsets of E, called independent sets. The defining properties of a matroid are that:

- the empty set is independent (that is, $\emptyset \in \mathcal{I}$);
- subsets of independent sets are independent (that is, if $A' \subseteq A \subseteq E$ and $A \in \mathcal{I}$, then $A' \in \mathcal{I}$);
- if A and B are independent sets, and $|A| > |B|$, then an independent set can be constructed by adding an element of A to B (that is, there is $a \in A \backslash B$ such that $B \cup \{a\} \in \mathcal{I}$). This final property is called the *augmentation property*.

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Connected rigidity matroids and unique realizations of graphs

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Abstract

A d-dimensional *framework* is a straight line realization of a graph G in $\mathbb{R}^d$. We shall only consider *generic* frameworks, in which the co-ordinates of all the vertices of G are algebraically independent. Two frameworks for G are *equivalent* if corresponding edges in the two frameworks have the same length. A framework is a *unique realization* of G in $\mathbb{R}^d$ if every equivalent framework can be obtained from it by an isometry of $\mathbb{R}^d$. Bruce Hendrickson proved that if G has a unique realization in $\mathbb{R}^d$ then G is $(d+1)$-connected and redundantly rigid. He conjectured that every realization of a $(d+1)$-connected and redundantly rigid graph in $\mathbb{R}^d$ is unique. This conjecture is true for $d = 1$ but was disproved by Robert Connelly for $d \geqslant 3$. We resolve the remaining open case by showing that Hendrickson's conjecture is true for $d = 2$. As a corollary we deduce that every realization of a 6-connected graph as a two-dimensional generic framework is a unique realization. Our proof is based on a new inductive characterization of 3-connected graphs whose rigidity matroid is connected.

1. Introduction

We shall consider finite graphs without loops, multiple edges or isolated vertices. A d-dimensional *framework* is a pair (G, p), where $G = (V, E)$ is a graph and p is a map

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5

Fractional Arboricity and Matroid Methods

The material in this chapter is motivated by two notions of the density of a graph. The *arboricity* and the *maximum average degree* of a graph G measure the concentration of edges in the "thickest" part of the graph.

5.1 Arboricity and maximum average degree

Suppose we wish to decompose the edges of a graph G into acyclic subsets, i.e., if $G = (V, E)$ we want to find $E_1, E_2, \ldots, E_k \subseteq E$ so that (1) each of the subgraphs (V, E_i) is acyclic and (2) $E = E_1 \cup E_2 \cup \cdots \cup E_k$. The smallest size of such a decomposition is called the *arboricity* (or *edge-arboricity*) of G and is denoted $\Upsilon(G)$. If G is connected, the arboricity is also the minimum number of spanning trees of G that include all edges of G.

One can think of arboricity as being a variant of the edge chromatic number. We are asked to paint the edges of G with as few colors as possible. In the case of edge chromatic number, we do not want to have two edges of the same color incident with a common vertex. In the case of arboricity, we do not want to have a monochromatic cycle.

There is an obvious lower bound on $\Upsilon(G)$. Since G has $\varepsilon(G)$ edges and each spanning acyclic subgraph has at most $\nu(G) - 1$ edges we have $\Upsilon(G) \geq \varepsilon(G)/(\nu(G) - 1)$. Moreover, since Υ is an integer, we have $\Upsilon(G) \geq \left\lceil \frac{\varepsilon(G)}{\nu(G)-1} \right\rceil$.

This bound is not very accurate if the graph is highly "unbalanced"; for example, consider the graph G consisting of a K_9 with a very long tail attached—say 100 additional vertices. We have $\nu(G) = 109$, $\varepsilon(G) = 136$, and therefore $\Upsilon(G) \geq \left\lceil \frac{136}{108} \right\rceil = 2$. The actual value of $\Upsilon(G)$ is larger since we clearly cannot cover the edges of K_9 with two trees; indeed, the arboricity of a graph is at least as large as the arboricity of any of its subgraphs. Thus we have

$$\Upsilon(G) \geq \max \left\lceil \frac{\varepsilon(H)}{\nu(H) - 1} \right\rceil$$

where the maximum is over all subgraphs of H with at least 2 vertices. Indeed, this improved lower bound gives the correct value.

Theorem 5.1.1

$$\Upsilon(G) = \max \left\lceil \frac{\varepsilon(H)}{\nu(H) - 1} \right\rceil$$

where the maximum is over all subgraphs of H with at least 2 *vertices.*

The proof of this theorem of Nash-Williams [137, 138] is presented in §5.4 below.

8

Introduction to Greedoids

ANDERS BJÖRNER and GÜNTER M. ZIEGLER

8.1. Introduction

Greedoids were invented around 1980 by B. Korte and L. Lovász. Originally, the main motivation for proposing this generalization of the matroid concept came from combinatorial optimization. Korte and Lovász had observed that the optimality of a 'greedy' algorithm could in several instances be traced back to an underlying combinatorial structure that was not a matroid – but (as they named it) a 'greedoid'. In subsequent research greedoids have been shown to be interesting also from various non-algorithmic points of view.

The basic distinction between greedoids and matroids is that greedoids are modeled on the *algorithmic construction* of certain sets, which means that the *ordering of elements* in a set plays an important role. Viewing such ordered sets as words, and the collection of words as a formal language, we arrive at the general definition of a greedoid as a finite language that is closed under the operation of taking initial substrings and satisfies a matroid-type exchange axiom. It is a pleasant feature that greedoids can also be characterized in terms of set systems (the unordered version), but the language formulation (the ordered version) seems more fundamental.

Consider, for instance, the algorithmic construction of a spanning tree in a connected graph. Two simple strategies are: (1) pick one edge at a time, making sure that the current edge does not form a circuit with those already chosen; (2) pick one edge at a time, starting at some given node, so that the current edge connects a visited node with an unvisited node. These well known strategies are used respectively in Kruskal's and in Prim's minimal spanning tree algorithms. In both cases, the collection of feasible sequences of edges, i.e. sequences that are generated by the allowed strategy, forms a greedoid. However, in the first case, but not in the second, any permutation of a feasible sequence of edges is also feasible, so that ordering is irrelevant.