A survey of χ-boundedness

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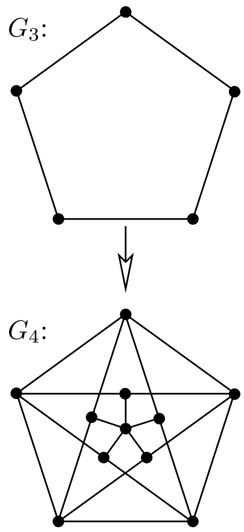

Constructing the Mycielski graph

Triangle-free graphs with high chromatic number

Here is a sequence of triangle-free graphs $G_3, G_4, \dots$ with

$$\chi(G_n) \;=\; n.$$

Start with $G_3 = C_5$, the 5-cycle; thus $\chi(G_3) = 3$. Suppose we have already constructed G_n on the vertex set V. The new graph G_{n+1} has the vertex set $V \cup V' \cup \{z\}$, where the vertices $v' \in V'$ correspond bijectively to $v \in V$, and z is a single other vertex. The edges of G_{n+1} fall into 3 classes: First, we take all edges of G_n; secondly every vertex v' is joined to precisely the neighbors of v in G_n; thirdly z is joined to all $v' \in V'$. Hence from $G_3 = C_5$ we obtain as G_4 the so-called *Mycielski graph*.

Clearly, G_{n+1} is again triangle-free. To prove $\chi(G_{n+1}) = n+1$ we use induction on n. Take any n-coloring of G_n and consider a color class C. There must exist a vertex $v \in C$ which is adjacent to at least one vertex of every other color class; otherwise we could distribute the vertices of C onto the $n-1$ other color classes, resulting in $\chi(G_n) \le n-1$. But now it is clear that v' (the vertex in V' corresponding to v) must receive the same color as v in this n-coloring. So, all n colors appear in V', and we need a new color for z.

Theorem 3. *For every $k \ge 2$, there exists a graph G with chromatic number $\chi(G) > k$ and girth $\gamma(G) > k$.*

The strategy is similar to that of the previous proofs: We consider a certain probability space on graphs and go on to show that the probability for $\chi(G) \le k$ is smaller than $\frac{1}{2}$, and similarly the probability for $\gamma(G) \le k$ is smaller than $\frac{1}{2}$. Consequently, there must exist a graph with the desired properties.

■ **Proof.** Let $V = \{v_1, v_2, \dots, v_n\}$ be the vertex set, and p a fixed number between 0 and 1, to be carefully chosen later. Our probability space $\mathcal{G}(n,p)$ consists of all graphs on V where the individual edges appear with probability p, independently of each other. In other words, we are talking about a Bernoulli experiment where we throw in each edge with probability p. As an example, the probability $\text{Prob}(K_n)$ for the complete graph is $\text{Prob}(K_n) = p^{\binom{n}{2}}$. In general, we have $\text{Prob}(H) = p^m(1-p)^{\binom{n}{2}-m}$ if the graph H on V has precisely m edges.

Let us first look at the chromatic number $\chi(G)$. By $\alpha = \alpha(G)$ we denote the *independence number*, that is, the size of a largest independent set in G. Since in a coloring with $\chi = \chi(G)$ colors all color classes are independent (and hence of size $\le \alpha$), we infer $\chi\alpha \ge n$. Therefore if α is small as compared to n, then χ must be large, which is what we want.

Suppose $2 \le r \le n$. The probability that a fixed r-set in V is independent

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ON THE CHROMATIC NUMBER OF SUBGRAPHS OF A GIVEN GRAPH

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ABSTRACT. It is shown that for arbitrary positive integers m,n there exists a $\phi(m,n)$ such that if $\chi(G) > \phi(m,n)$, then G contains either a complete subgraph of size m or a subgraph H with $\chi(H) = n$ containing no C_3. This gives an answer to a problem of Erdös and Hajnal.

Introduction. In [1], [2], [3] P. Erdös raised the following question: Is it true that to every n there is an $f(n)$ such that if $\chi(G) \geqslant f(n)$ then G contains a subgraph H in the sense defined below such that $\chi(H) = n$ and H contains no C_3 (the circuit of size 3) ? In this paper we prove this conjecture.

We prove the following

THEOREM. *For arbitrary positive integers m,n there exists a $\phi(m,n)$ such that if $\chi(G) \geqslant \phi(m,n)$ then G contains either a complete subgraph K_m of size m or a subgraph H with $\chi(H) \geqslant n$ containing no C_3.*

The conjecture of Erdös and Hajnal now follows easily. Tutte proved [5] that there exists a graph H with $\chi(H)$ arbitrarily large, which contains no C_3. Now given a graph G_1 with $\chi(G) \geqslant \phi(m,n)$, then either G contains K_m and we find a triangle-free subgraph of K_m with big chromatic number, or G contains H as above. We make no attempt here to find the smallest $\phi(m,n)$ with the desired property.

Notation. Given a graph $G = (V,E)$ we shall sometimes write $V = V(G)$, $E = E(G)$. H is a subgraph of G if $V(H) \subset V(G)$ and $E(H) \subset E(G)$. We shall denote this fact by $H \subset G$. Let G be a graph and let the set $V(G)$ be linearly ordered by $<$. By a left neighborhood of a vertex v we shall understand the graph $L(v,G)$ where

$$V(L(v,G)) = \{v';v' < v, \{v',v\} \in E(G)\},$$
$$E(L(v,G)) = E(G)/V(L(v,G)).$$

PROOF OF THE THEOREM. In the proof we use the following proposition of Zykov [4]. Let G be a graph with $\chi(G) \geqslant (p-1)(q-1)+1$. Let $E(G) = E_1 \cup E_2$ be a partition of $E(G)$. Then either $\chi((V(G),E_1)) \geqslant p$ or $\chi((V(G),E_2)) \geqslant q$. (This follows since $p-1$ coloring and a $(q-1)$ coloring

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Hajós' Graph-Coloring Conjecture: Variations and Counterexamples

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For each integer $n \geqslant 7$, we exhibit graphs of chromatic number n that contain no subdivided K_n as a subgraph. However, we show that a graph with chromatic number 4 contains as a subgraph a subdivided K_4 in which each triangle of the K_4 is subdivided to form an odd cycle.

1. INTRODUCTION

In this paper, unless otherwise stated, we follow the notation of [4]. A *subdivided* K_n is a graph obtained by replacing edges $\{x, y\}$ of the complete graph K_n with $x - y$ paths. We refer to the vertices where such paths meet as *nodes* of the subdivided K_n. A node has degree $n - 1$.

The Hajós conjecture asserts that a graph with chromatic number n has a subdivided K_n as a subgraph.

For $n = 1$ and 2 this is trivial, and for $n = 3$, it is clear, because a 3-chromatic graph contains an odd cycle, which is a subdivided K_3. The case $n = 4$ of the conjecture was proved by Dirac [1].

2. COUNTEREXAMPLES

Let $\Sigma(G)$, the *subdivision number* of a graph G, denote the largest integer n such that G contains a subdividison of K_n as a subgraph. The Hajós conjecture asserts that $\Sigma(G) \geqslant \chi(G)$. Let $L(G)$, the *line graph* of G, be the graph with vertices $V(L(G)) = E(G)$, where e, $e' \in E(G)$ are adjacent in $L(G)$ whenever e and e' are incident at a vertex of G.

Let ${}_kG$ denote the multigraph obtained by replacing each edge $\{x, y\}$ of G with k edges joining x and y.

We first consider the case where G is an odd cycle.

PROPOSITION 1. *For all $k \geqslant 1$, if $n \geqslant 2$, then*

$$\Sigma(L({}_kC_{2n+1})) = 2k + 1.$$

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04a

An even better Density Increment Theorem and its application to Hadwiger's Conjecture

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Dedicated to the memory of Robin Thomas

Abstract

In 1943, Hadwiger conjectured that every graph with no K_t minor is $(t-1)$-colorable for every $t \geq 1$. In the 1980s, Kostochka and Thomason independently proved that every graph with no K_t minor has average degree $O(t\sqrt{\log t})$ and hence is $O(t\sqrt{\log t})$-colorable. Recently, Norin, Song and the author showed that every graph with no K_t minor is $O(t(\log t)^{\beta})$-colorable for every $\beta > 1/4$, making the first improvement on the order of magnitude of the $O(t\sqrt{\log t})$ bound. More recently, the author showed that every graph with no K_t minor is $O(t(\log t)^{\beta})$-colorable for every $\beta > 0$; more specifically, they are $t \cdot 2^{O((\log\log t)^{2/3})}$-colorable. In combination with that work, we show in this paper that every graph with no K_t minor is $O(t(\log\log t)^6)$-colorable.

1 Introduction

All graphs in this paper are finite and simple. Given graphs H and G, we say that G has an *H minor* if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges. We denote the complete graph on t vertices by K_t.

In 1943 Hadwiger made the following famous conjecture.

Conjecture 1.1 (Hadwiger's conjecture [Had43]). *For every integer $t \geq 1$, every graph with no K_t minor is $(t-1)$-colorable.*

Hadwiger's conjecture is widely considered among the most important problems in graph theory and has motivated numerous developments in graph coloring and graph minor theory. For an overview of major progress on Hadwiger's conjecture, we refer the reader to [NPS19], and to the recent survey by Seymour [Sey16] for further background.

The following is a natural weakening of Hadwiger's conjecture, which has been considered by several researchers.

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04b

Hadwiger's conjecture

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Abstract
This is a survey of Hadwiger's conjecture from 1943, that for all $t \geq 0$, every graph either can be t-coloured, or has a subgraph that can be contracted to the complete graph on $t+1$ vertices. This is a tremendous strengthening of the four-colour theorem, and is probably the most famous open problem in graph theory.

1 Introduction

The four-colour conjecture (or theorem as it became in 1976), that every planar graph is 4-colourable, was the central open problem in graph theory for a hundred years; and its proof is still not satisfying, requiring as it does the extensive use of a computer. (Let us call it the 4CT.) We would very much like to know the "real" reason the 4CT is true; what exactly is it about planarity that implies that four colours suffice? Its statement is so simple and appealing that the massive case analysis of the computer proof surely cannot be the book proof.

So there have been attempts to pare down its hypotheses to a minimum core, in the hope of hitting the essentials; to throw away planarity, and impose some weaker condition that still works, and perhaps works with greater transparency so we can comprehend it. This programme has not yet been successful, but it has given rise to some beautiful problems.

Of these, the most far-reaching is Hadwiger's conjecture. (One notable other attempt is Tutte's 1966 conjecture [78] that every 2-edge-connected graph containing no subdivision of the Petersen graph admits a "nowhere-zero 4-flow", but that is

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05

An Update on the Four-Color Theorem

Robin Thomas

Every planar map of connected countries can be colored using four colors in such a way that countries with a common boundary segment (not just a point) receive different colors. It is amazing that such a simply stated result resisted proof for one and a quarter centuries, and even today it is not yet fully understood. In this article I concentrate on recent developments: equivalent formulations, a new proof, and progress on some generalizations.

Brief History

The Four-Color Problem dates back to 1852 when Francis Guthrie, while trying to color the map of the counties of England, noticed that four colors sufficed. He asked his brother Frederick if it was true that *any* map can be colored using four colors in such a way that adjacent regions (i.e., those sharing a common boundary segment, not just a point) receive different colors. Frederick Guthrie then communicated the conjecture to DeMorgan. The first printed reference is by Cayley in 1878.

A year later the first "proof" by Kempe appeared; its incorrectness was pointed out by Heawood eleven years later. Another failed proof was published by Tait in 1880; a gap in the argument was pointed out by Petersen in 1891. Both failed proofs did have some value, though. Kempe proved the five-color theorem (Theorem 2 below) and discovered what became known as Kempe chains, and Tait found an equivalent formulation of the Four-Color Theorem in terms of edge 3-coloring, stated here as Theorem 3.

The next major contribution came in 1913 from G. D. Birkhoff, whose work allowed Franklin to prove in 1922 that the four-color conjecture is true for maps with at most twenty-five regions. The same method was used by other mathematicians to make progress on the four-color problem. Important here is the work by Heesch, who developed the two main ingredients needed for the ultimate proof—"reducibility" and "discharging". While the concept of reducibility was studied by other researchers as well, the idea of discharging, crucial for the unavoidability part of the proof, is due to Heesch, and he also conjectured that a suitable development of this method would solve the Four-Color Problem. This was confirmed by Appel and Haken (abbreviated A&H) when they published their proof of the Four-Color Theorem in two 1977 papers, the second one joint with Koch. An expanded version of the proof was later reprinted in [1].

Let me state the result precisely. Rather than trying to define maps, countries, and their boundaries, it is easier to restate Guthrie's 1852 conjecture using planar duality. For each country we select a capital (an arbitrary point inside that country) and join the capitals of every pair of neighboring countries. Thus we arrive at the notion of a plane graph, which is formally defined as follows.

A *graph* G consists of a finite set $V(G)$, the set of *vertices* of G, a finite set $E(G)$, the set of *edges*

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Perspective

Colorings of plane graphs: A survey

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Dedicated to A.V. Kostochka on the occasion of his 60th birthday

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ABSTRACT

After a brief historical account, a few simple structural theorems about plane graphs useful for coloring are stated, and two simple applications of discharging are given. Afterwards, the following types of proper colorings of plane graphs are discussed, both in their classical and choosability (list coloring) versions: simultaneous colorings of vertices, edges, and faces (in all possible combinations, including total coloring), edge-coloring, cyclic coloring (all vertices in any small face have different colors), 3-coloring, acyclic coloring (no 2-colored cycles), oriented coloring (homomorphism of directed graphs to small tournaments), a special case of circular coloring (the colors are points of a small cycle, and the colors of any two adjacent vertices must be nearly opposite on this cycle), 2-distance coloring (no 2-colored paths on three vertices), and star coloring (no 2-colored paths on four vertices). The only improper coloring discussed is injective coloring (any two vertices having a common neighbor should have distinct colors).

1. Introduction and preliminaries

Coloring in a broad sense is a decomposition of a discrete object into simpler sub-objects. Due to its generality, this notion arises in various branches of discrete mathematics and has important applications. For example, one of the most natural models in the frequency assignment problem in mobile phoning is *$L(p, q)$-labeling*. The vertices of a planar graph (sources) should be colored (get frequencies assigned) so that the colors (integer frequencies) of vertices at distance 1 differ by at least p, while those at distance 2 differ by at least q. Sometimes, the set of available frequencies can vary from one source to another; this corresponds to "list $L(p, q)$-labeling".

The theory of plane graph coloring has a long history, extending back to the middle of the 19th century, inspired by the famous Four Color Problem (4CP), which asked if every plane map is 4-colorable. Now it is a broad area of research, with hundreds of contributors and thousands of contributions, and so is covered in this survey only partially.

The development of this area goes hand in hand with the study of the structure of plane graphs. Sometimes a new structural fact about plane graphs that is useful for coloring takes its place also in the structural theory; more often, it is not of independent interest and just serves as a tool for solving a specific coloring problem. Until several decades ago, the only coloring problem of broad interest was the 4CP, solved in 1976 by Appel and Haken [12] (see Theorem 1.1 below). Accordingly, the study of plane graphs from a structural viewpoint was for a long time almost exclusively concerned with plane triangulations of minimum degree 5. Since the 1960s, a rapidly growing number of interesting graph coloring problems on the plane have appeared (see Jensen and Toft's monograph [118]), and this advanced the study of the structure of plane graphs in general.

1.1. Reducible configurations, discharging, and the 4CP

The basic elements of a *plane map* are its vertices, edges, and faces. An *edge* is a closed Jordan curve; its end-points are *vertices*. A *loop* joins a vertex to itself; two vertices may be joined by several *multiple edges*. No edge can have an internal

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Planar 4-critical graphs with four triangles

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Abstract

By the Grünbaum-Aksenov Theorem (extending Grötzsch's Theorem) every planar graph with at most three triangles is 3-colorable. However, there are infinitely many planar 4-critical graphs with exactly four triangles. We describe all such graphs. This answers a question of Erdős from 1990.

1 Introduction

The classical Grötzsch's Theorem [14] says that every planar triangle-free graph is 3-colorable. The following refinement of it is known as the Grünbaum-Aksenov Theorem (the original proof of Grünbaum [15] was incorrect, and Aksenov [1] fixed the proof).

Theorem 1 ([1, 7, 15]). *Let G be a planar graph containing at most three triangles. Then G is 3-colorable.*

The example of the complete 4-vertex graph K_4 shows that "three" in Theorem 1 cannot be replaced by "four". But maybe there are not many plane 4-critical graphs with exactly four triangles (Pl_4-*graphs*, for short)?

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On two generalizations of the Alon–Tarsi polynomial method

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ABSTRACT

In a seminal paper (Alon and Tarsi, 1992 [6]), Alon and Tarsi have introduced an algebraic technique for proving upper bounds on the choice number of graphs (and thus, in particular, upper bounds on their chromatic number). The upper bound on the choice number of G obtained via their method, was later coined the *Alon–Tarsi number of* G and was denoted by $AT(G)$ (see e.g. Jensen and Toft (1995) [20]). They have provided a combinatorial interpretation of this parameter in terms of the eulerian subdigraphs of an appropriate orientation of G. Their characterization can be restated as follows. Let D be an orientation of G. Assign a weight $\omega_D(H)$ to every subgraph H of D: if $H \subseteq D$ is eulerian, then $\omega_D(H) = (-1)^{e(H)}$, otherwise $\omega_D(H) = 0$. Alon and Tarsi proved that $AT(G) \leqslant k$ if and only if there exists an orientation D of G in which the out-degree of every vertex is strictly less than k, and moreover $\sum_{H \subseteq D} \omega_D(H) \neq 0$. Shortly afterwards (Alon, 1993 [3]), for the special case of line graphs of d-regular d-edge-colorable graphs, Alon gave another interpretation of $AT(G)$, this time in terms of the signed d-colorings of the line graph. In this paper we generalize both results. The first characterization is generalized by showing that there is an infinite family of weight functions (which includes the one considered by Alon and Tarsi), each of which can be used to characterize $AT(G)$. The second characterization is generalized to all graphs (in fact the result is even more general—in particular it applies to hypergraphs). We then use the second generalization to prove that $\chi(G) = ch(G) = AT(G)$ holds for certain families of graphs G. Some of these results generalize certain known choosability results.

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Topological lower bounds for the chromatic number: A hierarchy

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Abstract

This paper is a study of "topological" lower bounds for the chromatic number of a graph. Such a lower bound was first introduced by Lovász in 1978, in his famous proof of the *Kneser conjecture* via Algebraic Topology. This conjecture stated that the *Kneser graph* $\mathrm{KG}_{m,n}$, the graph with all k-element subsets of $\{1, 2, \ldots, n\}$ as vertices and all pairs of disjoint sets as edges, has chromatic number $n-2k+2$. Several other proofs have since been published (by Bárány, Schrijver, Doľnikov, Sarkaria, Kříž, Greene, and others), all of them based on some version of the Borsuk–Ulam theorem, but otherwise quite different. Each can be extended to yield some lower bound on the chromatic number of an arbitrary graph. (Indeed, we observe that *every* finite graph may be represented as a generalized Kneser graph, to which the above bounds apply.)

We show that these bounds are almost linearly ordered by strength, the strongest one being essentially Lovász' original bound in terms of a neighborhood complex. We also present and compare various definitions of a *box complex* of a graph (developing ideas of Alon, Frankl, and Lovász and of Kříž). A suitable box complex is equivalent to Lovász' complex, but the construction is simpler and functorial, mapping graphs with homomorphisms to $\mathbb{Z}_2$-spaces with $\mathbb{Z}_2$-maps.

1 Introduction

Graph coloring is a classical combinatorial topic: For a given (finite) graph G, determine how to distribute a minimal number of colors to the vertices in such a way that adjacent vertices get different colors. The minimum number of colors is $\chi(G)$, the *chromatic number* of the graph. The graph coloring problem has numerous important practical motivations; among the more recent ones, we mention that it appears as a (simplified) model for the frequency assignment problem in mobile communication (cf. Borndörfer et. al. [6] and Eisenblätter et al. [13]).

The most famous graph coloring problem is, of course, the Four Color Problem, asking whether every planar graph can be colored by four colors, which was answered positively by Haken and Appell 1977 and re-solved by Robertson, Sanders, Seymour & Thomas [30]. Even for planar graphs, though, determining 3-colorability is already algorithmically difficult (NP-hard), and beyond the range of planar graphs, the gaps between the upper and the lower

09

Duality, Nowhere-Zero Flows, Colorings and Cycle Covers *

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Abstract

Nowhere-Zero Flows Problems, Coloring problems, Cycle cover problems lie in the core of graph theory. The strong relationship between them is the duality. Several important (and beautiful) conjectures are still open and this a very active field of study. We presente this subject with the enlightening notion of duality.

1 Introduction

This paper was written for the traditional Spring School of the Combinatorial Seminar at Charles University which was held in April 1999 in Borová Lada and in Finsterau. In 1999 this school was organized jointly with Humboldt Universitaet Berlin, Universitaet Bonn and Université Bordeaux I. Teachers from these schools took part in the meeting. The text tries to provide a study text for (undergraduate) students and it should serve as a background for the discussions and lectures at Spring School.

Some additional material and some complementary information can be found in the following articles:

[19] F. Jaeger. *Flows and Generalized Coloring Theorems in Graphs.* J.Combin. Theory Ser. B 26 (1979),pp. 205-216.

[22] F. Jaeger. *Nowhere zero-flow Problems.* Selected topics in Graph Theory 3 Academic Press, London 1988, 71 95.

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Tournaments and colouring

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DICHROMATIC NUMBER AND FRACTIONAL CHROMATIC NUMBER

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Abstract

The dichromatic number of a graph G is the maximum integer k such that there exists an orientation of the edges of G such that for every partition of the vertices into fewer than k parts, at least one of the parts must contain a directed cycle under this orientation. In 1979, Erdős and Neumann-Lara conjectured that if the dichromatic number of a graph is bounded, so is its chromatic number. We make the first significant progress on this conjecture by proving a fractional version of the conjecture. While our result uses a stronger assumption about the fractional chromatic number, it also gives a much stronger conclusion: if the fractional chromatic number of a graph is at least t, then the fractional version of the dichromatic number of the graph is at least $\frac{1}{4}t/\log_2(2et^2)$. This bound is best possible up to a small constant factor. Several related results of independent interest are given.

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1. Introduction

For an undirected graph G, the *chromatic number* $\chi(G)$ is the minimum number of independent sets whose union is $V(G)$. For a directed graph (digraph) D, the analogue of independent sets are acyclic vertex sets, where we call a vertex set *acyclic* if it does not contain a directed cycle. Then, the *chromatic number* $\chi(D)$ of D is the minimum number of acyclic vertex sets whose union is $V(G)$ (see [2, 10]). The *dichromatic number* of an undirected graph G, denoted by $\vec{\chi}(G)$, is the maximum chromatic number over all its orientations [4, 10].

Circle graphs are quadratically χ-bounded

James Davies* and Rose McCarty†

Abstract

We prove that the chromatic number of a circle graph with clique number ω is at most $7\omega^2$.

1 Introduction

We prove the following result.

Theorem 1. *The chromatic number of a circle graph with clique number ω is at most $7\omega^2$.*

We prove Theorem 1 as a consequence of the following.

Theorem 2. *The vertex set of a circle graph with clique number ω can be partitioned into 7ω sets, each of which induces a permutation graph.*

Theorem 2 implies Theorem 1 since permutation graphs are perfect (see [3]).

A class of graphs is *χ-bounded* if the chromatic number of each graph in the class is bounded above by some fixed function of its clique number. Such a function is called a *χ-bounding function* for the class. If a polynomial χ-bounding function exists, then the class is *polynomially χ-bounded*. (See Scott and Seymour [10] for a survey of χ-boundedness.)

Gyárfás [4] proved that the class of circle graphs is χ-bounded. Kostochka and Kratochvíl [8] showed that there exists a χ-bounding function of order 2^ω for the class of polygon circle graphs, which includes all circle graphs. (A *polygon circle graph* is the intersection graph of a finite set of polygons inscribed in the unit circle.) We further improve on these results by showing that the class of circle graphs is polynomially χ-bounded. Using a result of Krawczyk and Walczak [9], we obtain the following corollary.

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Graph Isomorphism, Color Refinement, and Compactness

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Abstract

Color refinement is a classical technique to show that two given graphs G and H are non-isomorphic; it is very efficient, even if incomplete in general. We call a graph G *amenable* to color refinement if this algorithm succeeds in distinguishing G from any non-isomorphic graph H. Babai, Erdős, and Selkow (1982) proved that almost all graphs G are amenable. We here determine the exact range of applicability of color refinement by showing that the class of all amenable graphs is recognizable in time $O((n+m)\log n)$, where n and m denote the number of vertices and the number of edges in the input graph.

Furthermore, we prove that amenable graphs are compact in the sense of Tinhofer (1991), that is, their polytopes of fractional automorphisms are integral. The concept of compactness was introduced in order to identify the class of graphs G for which isomorphism $G \cong H$ can be decided by computing an extreme point of the polytope of fractional isomorphisms from G to H and checking if this point is integral. Our result implies that this linear programming approach to isomorphism testing has the applicability range at least as large as the combinatorial approach based on color refinement.

1 Introduction

The well-known *color refinement* (or *naive vertex classification*) algorithm begins with a uniform coloring of the vertices of two graphs G and H and refines it step by step so that, if two vertices have equal colors but differently colored neighborhoods (with the multiplicities of colors counted), then these vertices get new different colors in the next refinement step. The algorithm terminates as soon as no further refinement is possible and concludes that G and H are non-isomorphic if the multisets of colors occurring in these graphs are different. If this happens, the conclusion is correct. However, color refinement cannot sometimes distinguish non-isomorphic graphs. The simplest example is given by any two non-isomorphic regular graphs of the same degree with the same number of vertices. We say that color refinement *applies* to a graph G if it succeeds in distinguishing G from any non-isomorphic H. The most obvious class of graphs to which color refinement is applicable is formed by *unigraphs*. Those are the graphs which are determined up to isomorphism by their degree sequences; see, e.g., [4, 24]. Another class where color refinement works

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COLOURINGS OF (m, n)-COLOURED MIXED GRAPHS

GARY MACGILLIVRAY, SHAHLA NASSERASR, FEIRAN YANG

Abstract. A mixed graph is, informally, an object obtained from a simple undirected graph by choosing an orientation for a subset of its edges. A mixed graph is (m, n)-coloured if each edge is assigned one of $m \geq 0$ colours, and each arc is assigned one of $n \geq 0$ colours. Oriented graphs are $(0, 1)$-coloured mixed graphs, and 2-edge-coloured graphs are $(2, 0)$-coloured mixed graphs. We show that results of Sopena for vertex colourings of oriented graphs, and of Kostochka, Sopena and Zhu for vertex colourings oriented graphs and 2-edge-coloured graphs, are special cases of results about vertex colourings of (m, n)-coloured mixed graphs. Both of these can be regarded as a version of Brooks' Theorem.

1. Introduction

There are parallels between vertex colourings of oriented graphs and vertex colourings of 2-edge-coloured graphs: statements that hold for one family often also hold for the other with more or less the same proof. For examples see [1, 2, 6, 7, 8, 11, 12]. On the other hand, Sen [12] gives examples where results that hold for oriented graphs do not hold for 2-edge-coloured graphs. For example, the maximum value of the oriented chromatic number of an orientation of P_5 is 3, while there are 2-colourings of P_5 that have chromatic number 4. Thus it is unlikely that there is a direct translation of results for graphs in one of these families to graphs in the other one.

A connection between oriented graphs and 2-edge-coloured graphs arises through the (m, n)-*coloured mixed graphs* introduced by Nešetřil and Raspaud [10], of which both are special cases. Theorems that hold for (m, n)-coloured mixed graphs hold for subfamilies, and methods which prove such results can be applied to subfamilies. Conversely, if a statement holds for both 2-edge-coloured graphs and oriented graphs with essentially the same proof, then there is some evidence that these may be special cases of a general theorem for (m, n)-coloured mixed graphs. For example, results in [1, 11] are shown to hold for (m, n)-mixed graphs in [10]. An ongoing project is to find generalizations to (m, n)-coloured of theorems common to 2-edge-coloured graphs and oriented graphs.

In this paper we consider vertex colourings of (m, n)-coloured mixed graphs and bounds for the (m, n)-coloured mixed chromatic number, $\chi(G, m, n)$. Definitions and terminology appear in the next section. In the subsequent sections we extend results which can be considered as versions of Brooks' Theorem to (m, n)-coloured mixed graphs. Sopena [13] proved constructively that the oriented chromatic number of an oriented graph with maximum degree $\Delta \geq 2$ satisfies $\chi_o \leq (2\Delta - 1)2^{2\Delta-2}$. In Section 3 we extend this statement to (m, n)-coloured mixed graphs by using similar methods to prove constructively that the (m, n)-coloured mixed chromatic number of an (m, n)-coloured mixed graph G with maximum degree $\Delta \geq 2$ satisfies $\chi(G, m, n) \leq (2\Delta - 1)(m + 2n)^{2\Delta-2}$. Kostochka, Sopena and Zhu [8] use the probabilistic method to give a better bound. They show that the oriented chromatic number of an oriented graph with maximum degree Δ satisfies $\chi_o \leq \Delta^2 2^{\Delta+1}$, and the corresponding statement holds for the number of colours needed for a vertex-colouring of a 2-edge-coloured graph. In Section 4 we extend this statement to (m, n)-coloured mixed graphs by using similar methods to show that $\chi(G, m, n) \leq \Delta^2(m + 2n)^{\Delta+1}$.

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IMPROVED BOUNDS FOR CENTERED COLORINGS

MICHAŁ DĘBSKI, STEFAN FELSNER, PIOTR MICEK, AND FELIX SCHRÖDER

Abstract. A vertex coloring ϕ of a graph G is *p-centered* if for every connected subgraph H of G either ϕ uses more than p colors on H or there is a color that appears exactly once on H. Centered colorings form one of the families of parameters that allow to capture notions of sparsity of graphs: A class of graphs has bounded expansion if and only if there is a function f such that for every $p \geqslant 1$, every graph in the class admits a p-centered coloring using at most $f(p)$ colors.

In this paper, we give upper bounds for the maximum number of colors needed in a p-centered coloring of graphs from several widely studied graph classes. We show that: (1) planar graphs admit p-centered colorings with $\mathcal{O}(p^3 \log p)$ colors where the previous bound was $\mathcal{O}(p^{19})$; (2) bounded degree graphs admit p-centered colorings with $\mathcal{O}(p)$ colors while it was conjectured that they may require exponential number of colors in p; (3) graphs avoiding a fixed graph as a topological minor admit p-centered colorings with a polynomial in p number of colors. All these upper bounds imply polynomial algorithms for computing the colorings. Prior to this work there were no non-trivial lower bounds known. We show that: (4) there are graphs of treewidth t that require $\binom{p+t}{t}$ colors in any p-centered coloring. This bound matches the upper bound; (5) there are planar graphs that require $\Omega(p^2 \log p)$ colors in any p-centered coloring. We also give asymptotically tight bounds for outerplanar graphs and planar graphs of treewidth 3. We prove our results with various proof techniques. The upper bound for planar graphs involves an application of a recent structure theorem while the upper bound for bounded degree graphs comes from the entropy compression method. We lift the result for bounded degree graphs to graphs avoiding a fixed topological minor using the Grohe-Marx structure theorem.

1. Introduction

Structural graph theory has expanded beyond the study of classes of graphs that exclude a fixed minor. One of the driving forces was, and is, to develop efficient algorithms for computationally hard problems for graphs that are 'structurally sparse'. Nešetřil and Ossona de Mendez introduced concepts of classes of graphs with *bounded expansion* [15]

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Complexity of Coloring Graphs without Forbidden Induced Subgraphs

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Abstract. We give a complete characterization of parameter graphs H for which the problem of coloring H–free graphs is polynomial and for which it is NP–complete. We further initiate a study of this problem for two forbidden subgraphs.

1 Preliminaries and Overview of Results

Graph coloring belongs to the most important and applied graph problems. It also belongs to the first identified NP–complete problems. Many classes of graphs were shown to allow polynomial-time solution (e.g., interval graphs, chordal graphs, etc.). In this paper we aim at classifying the computational complexity of this problem when restricted to graphs that do not contain certain forbidden induced subgraphs. Related results appear in [8], where 3-colorability is studied. We consider, on the other hand, the coloring problem with the number of colors being part of the input. For one forbidden subgraph we obtain a complete characterization of the complexity, which performs the polynomial-time/NP–complete dichotomy. First results in the direction of two forbidden subgraphs are gathered in the last section, but a complete characterization is not yet at hand.

We consider finite simple undirected graphs. We say that a graph G is H–*free*, where H is another graph, if G does not contain an induced subgraph

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