IMPROVED BOUNDS FOR CENTERED COLORINGS

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Abstract. A vertex coloring ϕ of a graph G is *p-centered* if for every connected subgraph H of G either ϕ uses more than p colors on H or there is a color that appears exactly once on H. Centered colorings form one of the families of parameters that allow to capture notions of sparsity of graphs: A class of graphs has bounded expansion if and only if there is a function f such that for every $p \geqslant 1$, every graph in the class admits a p-centered coloring using at most $f(p)$ colors.

In this paper, we give upper bounds for the maximum number of colors needed in a p-centered coloring of graphs from several widely studied graph classes. We show that: (1) planar graphs admit p-centered colorings with $\mathcal{O}(p^3 \log p)$ colors where the previous bound was $\mathcal{O}(p^{19})$; (2) bounded degree graphs admit p-centered colorings with $\mathcal{O}(p)$ colors while it was conjectured that they may require exponential number of colors in p; (3) graphs avoiding a fixed graph as a topological minor admit p-centered colorings with a polynomial in p number of colors. All these upper bounds imply polynomial algorithms for computing the colorings. Prior to this work there were no non-trivial lower bounds known. We show that: (4) there are graphs of treewidth t that require $\binom{p+t}{t}$ colors in any p-centered coloring. This bound matches the upper bound; (5) there are planar graphs that require $\Omega(p^2 \log p)$ colors in any p-centered coloring. We also give asymptotically tight bounds for outerplanar graphs and planar graphs of treewidth 3. We prove our results with various proof techniques. The upper bound for planar graphs involves an application of a recent structure theorem while the upper bound for bounded degree graphs comes from the entropy compression method. We lift the result for bounded degree graphs to graphs avoiding a fixed topological minor using the Grohe-Marx structure theorem.

1. Introduction

Structural graph theory has expanded beyond the study of classes of graphs that exclude a fixed minor. One of the driving forces was, and is, to develop efficient algorithms for computationally hard problems for graphs that are 'structurally sparse'. Nešetřil and Ossona de Mendez introduced concepts of classes of graphs with *bounded expansion* [15]

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and classes of graphs that are *nowhere dense* [16]. These are very robust properties that include every class excluding a fixed graph as a minor but also graphs of bounded book-thickness or graphs that allow drawings with bounded number of crossings per edge, see [18]. At first sight, bounded expansion might seem to be a weak property for a graph class. Yet, this notion captures enough structure to allow solving a wide range of algorithmic problems efficiently: Dvořák, Král' and Thomas [4] devised an FPT algorithm for testing first-order definable properties in classes of bounded expansion.

One reason that these new notions attracted much attention is the realization that they can be characterized in several, seemingly different ways. Instead of providing the original definition we define bounded expansion in terms of centered chromatic numbers. Let p be an integer and $p \geqslant 1$. A vertex coloring ϕ of a graph G is *p-centered* if for every connected subgraph H of G either ϕ uses more than p colors on H or there is a color that appears exactly once on H. The *p-centered chromatic number* $\chi_p(G)$ of G is the minimum integer k such that there is a p-centered coloring of G using k colors.

A vertex coloring of a graph is 1-centered if and only if it is a *proper* vertex coloring, i.e., adjacent vertices receive distinct colors. A vertex coloring is 2-centered if and only if it is a *star coloring*, i.e., it is proper and every path on four vertices receives at least 3 distinct colors. A class $\mathcal{C}$ of graphs is of *bounded expansion* if and only if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for every integer $p \geqslant 1$ and every $G \in \mathcal{C}$ we have $\chi_p(G) \leqslant f(p)$. Nešetřil and Ossona de Mendez, who originally defined bounded expansion in terms of maximum densities of shallow minors, showed in [15] that the definitions are equivalent.

The *treedepth* of a graph G is the minimum height of a rooted forest F such that $G \subseteq \operatorname{clos}(F)$ where (1) a *rooted forest* is a disjoint union of rooted trees; (2) the *height* of a rooted forest F is maximum number of vertices on a path from a root to a leaf in F; (3) the closure $\operatorname{clos}(F)$ of F is the graph with vertex set $V(F)$ and edge set $\{\{v, w\} \mid v \text{ is an ancestor of } w \text{ in } F\}$. As observed in [17], every p-centered coloring of a graph G is also a *treedepth-p coloring* of G, i.e., for every $i \leqslant p$ the union of any i color classes induce a subgraph of G of treedepth at most i.

Low treedepth colorings are a central tool for designing parametrized algorithms in classes of bounded expansion. For example, Pilipczuk and Siebertz [20] showed that when $\mathcal{C}$ is a class of graphs avoiding a fixed graph as a minor, then given graphs H and G on p and n vertices, respectively, where G is in $\mathcal{C}$, it can be decided whether H is a subgraph of G in time $2^{\mathcal{O}(p \log p)} \cdot n^{\mathcal{O}(1)}$ and space $n^{\mathcal{O}(1)}$. The algorithm witnessing this statement starts with a computation of a p-centered coloring of G with $p^{\mathcal{O}(1)}$ colors, and for each p-tuple of colors it applies a procedure to solve this problem for graphs G of treedepth at most p. The results of our paper imply a corresponding algorithm for the case where $\mathcal{C}$ is a class of graphs avoiding a fixed graph as a *topological* minor.

The polynomial space complexity mentioned above is remarkable. Typically dynamic programming algorithms on tree decompositions use space exponential in the width of the decomposition and there are complexity-theoretical reasons to believe that without significant loss on time complexity, this cannot be avoided. On the other hand treedepth decompositions, sometimes called elimination trees, allow to devise algorithms using only polynomial space in the height of the decomposition, see a thorough study of this phenomenon by Pilipczuk and Wrochna in [21].

Clearly, the running times of algorithms based on p-centered colorings heavily depend on the number of colors used. A recent experimental work by O'Brien and Sullivan [19] points to the lack of efficient coloring procedures as the major bottleneck for applicability of these algorithms in practice.

In this paper, we improve the bounds for the maximum $\chi_p(G)$ for graphs G in several important sparse classes of graphs. Most importantly, we reduce the upper bound on the number of colors needed for a p-centered coloring of planar graphs from $\mathcal{O}(p^{19})$ to $\mathcal{O}(p^3 \log p)$ and we show how to color graphs of bounded degree with $\mathcal{O}(p)$ colors while it was believed that there is an exponential lower bound for this class. All our bounds are supported with polynomial time algorithms computing the coloring.

We proceed with a presentation of our results.

Theorem 1. *Planar graphs admit p-centered colorings with $\mathcal{O}(p^3 \log p)$ colors.*

The previously best known bound was $\mathcal{O}(p^{19})$ which was given by Pilipczuk and Siebertz [20]. A key tool responsible for the improvement of the exponent is a brand new structure theorem for planar graphs due to Dujmović et al. [3] which has its roots in [20]. In Section 2, we give a precise statement of the theorem and we show how to use it to color a planar graph in a p-centered way with $\mathcal{O}(p) \cdot f(p)$ colors where $f(p)$ is the maximum number of colors we need in a p-centered coloring of planar graphs of treewidth at most 3. The $\mathcal{O}(p^3 \log p)$ bound then follows from Theorem 6.(iii).

Next we conclude an improved bound for graphs drawn on surfaces with bounded genus.

Theorem 2. *Graphs with Euler genus g admit p-centered colorings with $\mathcal{O}(gp+p^3 \log p)$ colors.*

The previous best known bound was $\mathcal{O}(g^2p^3+p^{19})$, see [20]. Our result follows from the bound for planar graphs combined with a lemma from [3], which relates BFS-layerings of graphs with bounded genus with BFS-layerings of their large planar subgraphs.

Graphs with bounded maximum degree are sparse but somehow much less structured than planar graphs, e.g. they allow any graph as a minor. It was conjectured in [20], that the number of colors needed for p-centered colorings in the class of graphs of maximum degree 3 is exponential in p. This was supported by similar bounds for weak coloring numbers which is another family of parameters allowing to capture the notion of bounded expansion. We disprove the conjecture by providing an upper bound which is linear in p.

Theorem 3. *Graphs with bounded degree admit p-centered colorings with $\mathcal{O}(p)$ colors. More specifically, for every $p \geqslant 1$ and every graph G with maximum degree at most Δ we have*

$$\chi_p(G) \leqslant \left\lceil 2^{10} \cdot \Delta^{2-1/p} \cdot p \right\rceil .$$

A very new mansuscript of Dubois et al. [2] proves an almost matching lower bound. By analyzing an appropriate random graph they show that there are graphs of maximum degree Δ that require $\Omega(\Delta^{2-1/p} \cdot p \cdot \ln^{-1/p} \Delta)$ colors in any p-centered coloring.

We prove Theorem 3 with an entropy compression type argument. The idea is to run a naïve randomized algorithm for coloring a graph in the p-centered way. Assuming that this algorithm fails on a long run, for every possible evaluation of random experiments, we can compress a random string of bits below the entropy bound. The contradiction shows that there must be a sequence of bits which leads to a successful run. The entropy compression method is inspired by the algorithmic proof of the Lovász local lemma given by Moser and Tardos [14]. An instructive overview of the method can be found at Tao's blog [23] or in a paper by Grytczuk, Kozik, and Micek [7] where nonrepetitive sequences over a finite alphabet are constructed. In the case $p = 2$, i.e. for star chromatic number, our argument matches the upper bound $\mathcal{O}(\Delta^{1.5})$ by Fertin, Raspaud, and Reed [5].

One of the strong results from [20] is that the graphs excluding a fixed graph as a minor admit p-centered colorings with a polynomial in p number of colors. The proof goes through the structural graph minor theorem for graphs by Robertson and Seymour [22]. Grohe and Marx [6] extended the structure theorem to graphs avoiding a fixed topological minor. In some sense, they showed that incorporating the bounded-degree graphs into the structural decomposition given in [22] is enough to obtain approximate characterization for classes excluding a fixed graph as a topological minor. For these reasons, we have been able to lift our result for bounded degree graphs and obtain the following general statement.

Theorem 4. *For every graph H there is a polynomial f such that the graphs excluding H as a topological minor admit p-centered colorings with at most $f(p)$ colors.*

Prior to our work there were no non-trivial lower bounds for the maximum number of colors required in a p-centered coloring. The lower bounds for planar graphs, graphs with bounded treewidth and even graphs excluding a fixed graph as a minor were only linear in p. We present constructions forcing $\chi_p(G)$ to be superlinear in p.

Theorem 5. *For every $p \geqslant 0$ and $t \geqslant 0$, there is a graph G of treewidth at most t with*

$$\chi_p(G) \geqslant \binom{p+t}{t}.$$

This lower bound is sharp as Pilipczuk and Siebertz [20] showed that for every $p \geqslant 1$, every $t \geqslant 1$, every graph G of treewidth at most t has $\chi_p(G) \leqslant \binom{p+t}{t}$. In fact, we show a slightly stronger statement than Theorem 5 concerning a relaxation of p-centered colorings, namely the p-linear colorings introduced in [11]. See the precise statement in Section 6.

Finally, we present asymptotically tight lower and upper bounds for the maximum p-centered chromatic numbers of outerplanar graphs and planar graphs of treewidth at most 3, i.e. subgraphs of stacked triangulations (see [10] for more details). In particular Theorem 6.(iv) implies the best known lower bound $\Omega(p^2 \log p)$ for the maximum p-centered chromatic number of planar graphs.

Theorem 6.

(i) *Outerplanar graphs admit p-centered colorings with $\mathcal{O}(p \log p)$ colors.*
(ii) *There is a family of outerplanar graphs that require $\Omega(p \log p)$ colors in any p-centered coloring.*
(iii) *Planar graphs of treewidth at most 3, i.e. subgraphs of stacked triangulations, admit p-centered colorings with $\mathcal{O}(p^2 \log p)$ colors.*
(iv) *There is a family of planar graphs of treewidth 3 that require $\Omega(p^2 \log p)$ colors in any p-centered coloring.*

It turns out that the lower bounds (ii) and (iv) and the upper bounds (i) and (iii) of Theorem 6 are initial steps of a more general result stated in Theorem 7. The statement of the theorem involves the notion of *simple treewidth* which will be formally introduced in Subsection 5.2, for the moment we just remark that graphs of simple treewidth at most 2 are outerplanar graphs and graphs of simple treewidth at most 3 are subgraphs of stacked triangulations.

Theorem 7.

(i) *Graphs of simple treewidth at most k admit p-centered colorings with $\mathcal{O}(p^{k-1}\log p)$ colors.*
(ii) *There is a family of graphs of simple treewidth k that require $\Omega(p^{k-1}\log p)$ colors in any p-centered coloring.*

The rest of the paper is organized as follows. In Section 2, we present a new structure theorem for planar graphs from [3] and show how to apply it to get good bounds for $\chi_p(G)$ when G is planar or when G has bounded genus. In Section 3, we set up an entropy compression argument to prove an $\mathcal{O}(p)$ upper bound for graphs with bounded degree. In Section 4, we lift the previous result and get a polynomial bound for graphs avoiding a fixed graph as a topological minor. In Sections 5 and 7, we give the upper bounds and lower bounds, respectively, for graphs of bounded simple treewidth. Section 6 is devoted to the lower bound for graphs of bounded treewidth.

Following each argument for an upper bound, we give a short justification why the proof gives a polynomial time algorithm computing the actual coloring.

2. Upper bounds for planar graphs: An application of the product structure theorem

This section is devoted to the proof of Theorem 1 and Theorem 2. Assuming the statement of Theorem 6.(iii) these theorems directly follow from Theorem 9 and Corollary 11 respectively.

A *vertex-partition*, or simply *partition* of a graph G is a set $\mathcal{P}$ of non-empty sets of vertices in G such that each vertex of G is in exactly one element of $\mathcal{P}$. The *quotient* of $\mathcal{P}$ is the graph, denoted by $G/\mathcal{P}$, with vertex set $\mathcal{P}$ where distinct parts $A, B \in \mathcal{P}$ are adjacent in $G/\mathcal{P}$ if and only if some vertex in A is adjacent to some vertex in B.

A *layering* of a graph G is an ordered partition $(V_0, V_1, \ldots)$ of $V(G)$ such that for every edge $vw \in E(G)$, if $v \in V_i$ and $w \in V_j$, then $|i-j| \leqslant 1$. If r is a vertex in a connected graph G and $V_i = \{v \in V(G) \mid \operatorname{dist}_G(r,v) = i\}$ for all $i \geqslant 0$, then $(V_0, V_1, \ldots)$ is called a *BFS-layering* of G. This notion extends to disconnected graphs: If $G_1, \ldots, G_c$ are the components of G, and r_j is a vertex in G_j for $j \in \{1, \ldots, c\}$, and $V_i = \bigcup_{j=1}^{c}\{v \in V(G_j) \mid \operatorname{dist}_{G_j}(r_j, v) = i\}$ for all $i \geqslant 0$, then $(V_0, V_1, \ldots)$ is called a *BFS-layering* of G.

The *layered width* of a partition $\mathcal{P}$ of a graph G is the minimum integer ℓ such that for some layering $(V_0, V_1, \ldots)$ of G, we have $|X \cap V_i| \leqslant \ell$ for every $X \in \mathcal{P}$ and every integer $i \geqslant 0$.

The following lemma was shown by Dujmović et al. [3] as a key ingredient for their bound on the queue number of planar graphs, it is frequently refered to as the 'Product Structure Theorem'.

Lemma 8 ([3])**.** *Every planar G has a partition $\mathcal{P}$ with layered width 3 such that $G/\mathcal{P}$ is planar and $\operatorname{tw}(G/\mathcal{P}) \leqslant 3$. Moreover, there is such a partition for every BFS-layering of G.*

The statement of the lemma has its roots in a lemma by Pilipczuk and Siebertz [20]: Every planar graph G has a partition $\mathcal{P}$ into geodesic paths such that $\operatorname{tw}(G/\mathcal{P}) \leqslant 8$. (Recall that a path is *geodesic* if it is a shortest path between its endpoints.)

Below we use Lemma 8 to reduce the problem of bounding χ_p for planar graphs to the problem of bounding χ_p for planar graphs of treewidth at most 3. The reduction only introduces a single additional $\mathcal{O}(p)$ factor.

Theorem 9. *Let $f(p)$ be the maximum $\chi_p(H)$ over all graphs H such that H is planar and $\operatorname{tw}(H) \leqslant 3$. Then for every planar graph G we have*

$$\chi_p(G) \leqslant 3(p+1)f(p).$$

Proof. Let G be a planar graph. Let $(V_0, V_1, \ldots)$ be any BFS-layering of G. Let $\mathcal{P}$ be a partition of G of layered width 3 with respect to $(V_0, V_1, \ldots)$ given by Lemma 8. Therefore, $G/\mathcal{P}$ is planar, $\operatorname{tw}(G/\mathcal{P}) \leqslant 3$, and

$$|X \cap V_i| \leqslant 3 \quad \text{for every } X \in \mathcal{P} \text{ and } i \geqslant 0.$$

Let ψ be a p-centered coloring of $G/\mathcal{P}$ using at most $f(p)$ colors.

We define a vertex coloring ϕ of G to be $\phi(v) = (\alpha(v), \beta(v), \gamma(v))$ for every $v \in V(G)$ where the three coordinates of a color are defined below.

Let $v \in V(G)$ with $v \in V_i$ and $v \in X$ for some class X of $\mathcal{P}$. Then

$$\alpha(v) = i \mod (p+1),$$
$$\beta(v) = \psi(X).$$

Define $\gamma(v) \in \{1, 2, 3\}$ such that for every $i \geqslant 0$ and $X \in \mathcal{P}$ the colors of vertices in $V_i \cap X$ are distinct. This is possible as $|V_i \cap X| \leqslant 3$.

Clearly, the number of colors used by ϕ is bounded by $(p+1) \cdot f(p) \cdot 3$. It remains to show that ϕ is a p-centered coloring of G.

Consider a connected subgraph H of G. We want to show that either ϕ uses more than p colors on H or there is a color that appears exactly once on H. Let H' be a subgraph of $G/\mathcal{P}$ spanned by $\{X \in \mathcal{P} \mid V(H) \cap X \neq \emptyset\}$. Since H is connected, H' must be connected as well. Recall that ψ is a p-centered coloring of $G/\mathcal{P}$, therefore either ψ uses more than p colors on H' or there is X in H' with a unique color in H' under ψ. In the first case, $\beta(v)$ takes more than p values among $v \in V(H)$. Thus ϕ uses more than p colors on H, as desired. In the second case, fix X in H' with a unique color in H' under ψ. Since $V(H) \cap X \neq \emptyset$ we find $v \in V(H) \cap X$. If v has a unique color in H under ϕ then we are done. Otherwise, let $v' \neq v$ with $v' \in V(H)$ and $\phi(v) = \phi(v')$. Since X has a unique ψ-color in H' and $\beta(v') = \beta(v) = \psi(X)$ we conclude that $v' \in X$. Let $v \in V_i$ and $v' \in V_j$. If $i = j$ then v, v' are two distinct vertices in $X \cap V_i$. This, however, implies $\gamma(v) \neq \gamma(v')$. Therefore, we know that $i \neq j$. Since $\alpha(v) = \alpha(v')$ we know that $i \equiv_{p+1} j$. Combining it with $i \neq j$ we conclude that $|i-j| \geqslant p+1$. Since v, v' are two vertices in a connected graph H we know that H must intersect every layer V_k with $i \leqslant k \leqslant j$. This means that α takes all possible $p+1$ values on the vertices of H. Therefore, ϕ uses at least $p+1$ colors on H. □

The following lemma relates BFS-layerings of graphs with bounded Euler genus with their large planar subgraphs. With this lemma we can easily adapt the proof of Theorem 9 so that it works for graphs with bounded genus.

Lemma 10 ([3]). *Let G be a graph with Euler genus g. For every BFS-layering $(V_0, V_1, \ldots)$ there exists $Z \subseteq V(G)$ with $|V_i \cap Z| \leqslant 2g$ for all $i \geqslant 0$, such that $G - Z$ is planar. Moreover, there is a planar graph G^+ containing $G - Z$ as a subgraph, and there is a BFS-layering $(W_0, W_1, \ldots)$ of G^+ such that $W_i \cap V(G) = V_i - Z$, for all $i \geqslant 0$.*

Corollary 11. *Let $f(p)$ be the maximum $\chi_p(H)$ over all graphs H such that H is planar and $\mathrm{tw}(H) \leqslant 3$. Then for every graph G with Euler genus g we have*

$$\chi_p(G) \leqslant 2g(p+1) + 3(p+1)f(p).$$

Proof. Let G be a graph of Euler genus g. Fix a BFS-layering $(V_0, V_1, \ldots)$ of G. We apply Lemma 10 and get a set $Z \subseteq V(G)$, a graph G^+ containing $G - Z$, and a BFS-layering $(W_0, W_1, \ldots)$ of G^+ such that

(1) $|V_i \cap Z| \leqslant 2g$ for all $i \geqslant 0$;
(2) G^+ is planar;
(3) $W_i \cap V(G) = V_i - Z$ for all $i \geqslant 0$.

We apply Theorem 9 to the graph G^+, fixing at the beginning of its proof the layering to be $(W_0, W_1, \ldots)$, and get a p-centered coloring ϕ^+ of G^+ that uses $3(p+1)f(p)$ colors. In fact, we need to recall that $\phi^+(v) = (\alpha^+(v), \beta^+(v), \gamma^+(v))$ for $v \in V(G^+)$ where the respective functions are defined as before.

For a vertex $v \in V(G)$ with $v \in V_i$, we define $\alpha(v) = i \bmod (p+1)$. We pause here to note that for a vertex $v \in V(G) \setminus Z$ the values $\alpha(v)$ and $\alpha^+(v)$ are defined with respect to two different layerings $(V_0, V_1, \ldots)$ and $(W_0, W_1, \ldots)$. Nevertheless, the property 3 guarantees that $\alpha(v) = \alpha^+(v)$. For a vertex $v \in Z$, we define $z(v) \in \{1, \ldots, 2g\}$ in such a way that for every integer $i \geqslant 0$ the values of $z(v)$ are all distinct for $z \in V_i \cap Z$. It is possible as $|V_i \cap Z| \leqslant 2g$. Finally, we define

$$\phi(v) = \begin{cases} (0, \alpha(v), z(v)) & \text{for } v \in Z, \\ (1, \alpha^+(v), \beta^+(v), \gamma^+(v)) & \text{for } v \in V(G) - Z. \end{cases}$$

Clearly, the number of colors used by ϕ is bounded by $(p+1) \cdot 2g + (p+1) \cdot f(p) \cdot 3$. It remains to show that ϕ is a p-centered coloring of G.

Consider a connected subgraph H of G. We want to show that either ϕ uses more than p colors on H or there is a color that appears exactly once on H.

First assume that $V(H) \cap Z \neq \emptyset$ and let $v \in V(H) \cap Z$. If the color of v is unique in H under ϕ, then we are done. Otherwise, we fix a vertex v' in H with $v' \neq v$ such that $\phi(v) = \phi(v')$. In particular, $v' \in Z$. Let $v \in V_i$ and $v' \in V_j$. If $i = j$, then $z(v) \neq z(v')$.

Therefore, $i \neq j$. Since $\alpha(v) = \alpha(v')$ we know that $i = j \bmod (p+1)$. We conclude that $|i-j| \geqslant p+1$. Since v, v' are two vertices in a connected graph H we know that H must intersect every layer V_k with $i \leqslant k \leqslant j$. This means that α takes all possible $p+1$ values on the vertices of H. Therefore, ϕ uses at least $p+1$ colors on H.

If $V(H)$ and Z are disjoint, then H is a subgraph of G^+. Since $(\alpha^+, \beta^+, \gamma^+)$ is a p-centered coloring of G^+ and they are all coordinates of the ϕ-coloring. This completes the proof. □

As explained in [3], for every planar graph G on n vertices one can find a partition $\mathcal{P}$ stated in Lemma 8 and a witnessing tree-decomposition of $G/\mathcal{P}$ of width 3 in $\mathcal{O}(n^2)$ time. Given this tree-decomposition, the coloring described in the proof of Theorem 9 can be done in $\mathcal{O}(n)$ time plus the time required to color a planar graph of treewidth 3. As we will see in the proof of Theorem 6.(iii) this takes only $\mathcal{O}(n)$ time. Overall, we can color planar graphs with $\mathcal{O}(p^3 \log p)$ colors in quadratic time. Similarly, given a graph of Euler genus at most g we can embed it on a respective surface using the linear-time algorithm by Mohar [13]. Then we can process the BFS layerings from Lemma 10 in linear time and apply the algorithm from the planar case. This also results in a quadratic time coloring algorithm.

3. Upper bound for bounded degree: The entropy compression method

In this section we prove Theorem 3.

Let p be a positive integer and let G be a graph with maximum degree at most Δ. Fix an arbitrary ordering $\Pi = (v_1, \ldots, v_n)$ of $V(G)$. For every vertex $v \in V(G)$, fix an arbitrary ordering of the edges adjacent to v. Let $c = \lceil 2^{10} \cdot \Delta^{2-1/p} \cdot p \rceil$ and let M be a sufficiently large integer. For convenience we assume also that M is divisible by $2p$.

A *partial coloring* of G is a function $f : V(G) \to C \cup \{\diamondsuit\}$ where C is a set of colors used by f and $\diamondsuit$ is an extra value indicating that no color was assigned. A vertex v with $f(v) \in C$ is said to be *colored* by f while a vertex v with $f(v) = \diamondsuit$ is *uncolored* by f. A connected subgraph H of G is a *violator* with respect to a partial coloring f if

(1) all vertices in H are colored by f;
(2) f uses at most p colors on H; and
(3) no color is unique in H under f.

We propose a naïve randomized algorithm which tries to color G in a p-centered fashion with c colors, see Algorithm 1. The algorithm starts with all vertices being uncolored. In every step, the first uncolored vertex from Π is colored with a random color from $\{1, \ldots, c\}$. If this new assignment creates a violator H, then $\min(|V(H)|, 2p)$ vertices in H are uncolored, including the vertex colored in this step. (One could erase the colors of all the vertices in H and get a $\mathcal{O}(\Delta^2 p)$ bound; we present this slightly more involved version to get $\mathcal{O}(\Delta^{2-1/p} p)$.)

To trade probabilistic arguments against counting we replace random sampling by a prescribed sequence $X = (x_1, \ldots, x_M)$ of colors, where $x_i \in [c]$ for every $i \in [M]$, and take X as part of the input of the algorithm.

Algorithm 1 Input: $X = (x_1, \ldots, x_M)$ with $x_i \in \{1, \ldots, c\}$

1: $f(v) \leftarrow \diamondsuit$, for every v in G
2: **for** $i = 1, 2, \ldots, M$ **do**
3:     **if** all vertices of G are colored under f **then**
4:         **break**
5:     $w_i \leftarrow v_j$, where v_j is the first uncolored vertex in Π
6:     $f(w_i) \leftarrow x_i$
7:     **if** there exists a violator of f **then**
        $H_i \leftarrow$ a violator of f
        $T_i \leftarrow$ a spanning tree of H_i rooted at w_i
        $W_i \leftarrow$ a traversal of T_i that starts and ends in w_i, and traverses each edge of T_i twice
        $f(v) \leftarrow \diamondsuit$, for every v among first $\min(|V(H_i)|, 2p)$ distinct vertices along W_i
8: **return** f

If the algorithm breaks the loop at line 4 and outputs a coloring f, then f is a p-centered coloring of G using at most c colors. Indeed, we keep an invariant that after each iteration there is no violator with respect to f. This is obviously true before the first iteration. Now within step i, the newly colored vertex w_i could create violators, but each of them must contain w_i. When the algorithm uncolors some vertices in a single violator, it always uncolors w_i itself. This way there are no violators left.

We define a structure called $\textsc{Log}(X)$ which accumulates data during the run of the algorithm. We will show that unless input X leads to a break we can reconstruct X from $\textsc{Log}(X)$, i.e., the map $X \mapsto \textsc{Log}(X)$ is injective. On the other hand, $\textsc{Log}(X)$ takes strictly less than c^M values. This is a contradiction as there is no injective map from a set of size c^M into a smaller one. Thus there has to be an input which leads to a break (line 4), i.e. to a valid coloring.

We continue to define $\textsc{Log}$ and to show how to reconstruct X from $\textsc{Log}(X)$ when the algorithm with input X does not break at line 4.

The data in $\textsc{Log}(X)$ is a tuple $\textsc{Log}(X) = (Z, \Sigma, \Gamma, F)$. We define one by one the coordinates of the tuple.

In $Z = (z_0, z_1, \ldots, z_M)$ we record the number of colored vertices over the process. Specifically, z_i is the number of colored vertices after finishing the i-th step of the loop. Note that $z_0 = 0$, $z_1 = 1$ and for $i > 1$ either $z_i = z_{i-1} + 1$ or $z_i = z_{i-1} + 1 - \min(h_i, 2p)$, where $h_i = |V(H_i)|$ is the number of vertices of the violator H_i.

In Σ we collect information about the violator subgraphs H which get partially uncolored at some iteration of the loop.

Suppose that in the i-th step of the algorithm a subgraph H_i is recognized as a violator. Let T_i and W_i be the structures fixed by the algorithm after line 7, so T_i is a spanning tree of H_i rooted at w_i and $W_i = (u_0, u_1, \ldots, u_{2h_i-2})$ is a walk in G traversing T_i such that the walk starts and ends at w_i, i.e., $u_0 = u_{2h_i-2} = w_i$, and each edge of T_i is traversed twice. Let W_i' be the walk in G going along W_i until W_i reaches $m_i = \min(|V(H_i)|, 2p)$ distinct vertices and after that W_i' simply traverses back in T_i to the root w_i, and ends there. Clearly, W_i' has length $2m_i - 2$ (number of edges) and we denote its vertices by $W_i' = (u_0', u_1', \dots, u_{2m_i-2}')$. For $j \in \{0, \dots, 2m_i - 3\}$, $u_j'u_{j+1}'$ is a *forward step* if u_j' is the parent of u_{j+1}' in T_i. Otherwise, u_{j+1}' is the parent of u_j' in T_i and $u_j'u_{j+1}'$ is a *backward step*.

We encode W_i' in two lists. The binary list $B_i = (b_0, b_1, b_2, \dots, b_{2m_i-3})$ records which steps of W_i' are forward and which are backward, i.e., $b_j = 1$ if $u_j'u_{j+1}'$ is a forward step and $b_j = 0$ if $u_j'u_{j+1}'$ is backward. Clearly, $m_i - 1$ steps are forward and $m_i - 1$ steps are backward. The list $L_i = (\ell_1, \dots, \ell_{m_i-1})$ with $\ell_k \in \{1, \dots, \Delta\}$ collects data about the forward steps of W_i'. If $u_j'u_{j+1}'$ is the k-th forward step in W_i', then ℓ_k is the unique number such that $u_j'u_{j+1}'$ is the ℓ_k-th edge in the fixed ordering of edges adjacent to u_j'.

Initially $\Sigma = (B, L)$ consists of two empty lists, and whenever it comes to uncoloring a violator H_i in the course of the algorithm then B_i is appended to B and L_i is appended to L.

In Γ we collect information about the colors of the vertices of violator subgraphs at the moment they are uncolored within the iteration process.

Again, suppose that in the i-th step of the algorithm a subgraph H_i with $h_i = |V(H_i)|$ is partially uncolored, and let $m_i = \min(h_i, 2p)$. Let W_i' be the traversal of a subgraph of H_i fixed in the definitions of B_i and L_i. Let $V_i = (v_1', \dots, v_{m_i}')$ be the vertices of W_i' sorted by their first-appearance in W_i'. Let $B_i' = (b_1', \dots, b_{m_i}')$ be a binary sequence such that $b_i' = 1$ if $f(v_i') \notin \{f(v_1'), \dots, f(v_{i-1}')\}$ and $b_i' = 0$ otherwise. Let $c_i = \sum b_j'$ that is c_i is the number of colors used by f at W_i' before it is uncolored. Since $V(W_i') \subseteq V(H_i)$ and H_i is a violator we have $c_i \leqslant p$. Let $d_i = m_i - c_i$. We claim that $d_i \geqslant c_i$. Indeed, if $h_i > 2p$ then $d_i = m_i - c_i = 2p - c_i \geqslant p$. Otherwise, $m_i = |V(H_i)|$ and since H_i is a violator every color in H_i is used at least twice so $d_i = h_i - c_i \geqslant c_i$ as well.

Let $V_i^1 = (v_1^1, \dots, v_{c_i}^1)$ be the restriction of V_i to those indices j with $b_j' = 1$. Let $V_i^0 = (v_1^0, \dots, v_{d_i}^0)$ be the restriction of V_i to those indices j with $b_j' = 0$. Finally, we put $A_i = (f(v_1^1), \dots, f(v_{c_i}^1))$ and $P_i = (p_1, \dots, p_{d_i})$ where p_j is chosen so $f(v_j^0) = f(v_{p_j}^1)$.

Initially (B', A, P) consists of three empty lists, and whenever it comes to uncoloring a violator H_i in the course of the algorithm then B_i' is appended to B', A_i is appended to A, and P_i is appended to P. Finally, we merge A and P into a single list $^*AP$ by reversing A and appending P (only for technical reasons to simplify the counting). This will yield $\Gamma = (B', {}^*AP)$.

The last component F of $\textsc{Log}(X)$ consists of the partial coloring f of G when the algorithm stops, i.e., after the M iterations of the loop. More explicitly, $F = (f(v_1), f(v_2), \dots, f(v_n))$ where the coloring f is taken from the output.

For the counting it will be convenient to have lists of length independent of X. Let γ be the number of colored vertices in F, i.e., the number of v_i with $f(v_i) \neq \diamondsuit$. Note that $M - \sum m_i = \gamma$. We append γ entries to the lists B', $^*AP$ from Γ and $2\gamma + 2v$ entries to B from Σ, where v is the total number of violators. Now $|B| = 2M$, $|B'| = M$, and

$|{}^*AP| = M$. The length of L is $\sum(m_i - 1) \leqslant (M-\gamma)\left(1-\frac{1}{2p}\right) \leqslant \left(1-\frac{1}{2p}\right)M$ and we append elements to L to make the length of L equal to this value.

Claim. *The mapping $X \to \textsc{Log}(X)$ is injective.*

Proof. We will show how to reconstruct the original input X from the value of $\textsc{Log}(X)$. The reconstruction is done in two phases. In the forward phase we reconstruct:

(i) $(w_1, w_2, \ldots, w_M)$ where w_i is the vertex colored with x_i at line 6 in the i-th iteration;
(ii) $(U_0, U_1, \ldots, U_M)$ where U_i is the set of uncolored vertices after the i-th iteration.
(iii) W_i' for every i such that the algorithm encounters a violator in the i-th step, where W_i' is the traversal of the subgraph of H_i with all uncolored vertices, fixed in the definition of $\textsc{Log}(X)$.

Note first that $U_0 = V(G)$. For the two lists in $\Sigma \in \textsc{Log}(X)$ we initialize $\mathrm{pos}(B) = 0$ and $\mathrm{pos}(L) = 0$. The invariant is that when U_{i-1} has been computed $\mathrm{pos}(B)$ and $\mathrm{pos}(L)$ mark the tail of the lists constructed up to the end of the $(i-1)$-st iteration of the loop.

Let $i \geqslant 1$ and suppose that U_{i-1} is determined. Then the vertex w_i is identified as the first element of U_{i-1} in Π. Now, if $z_i = z_{i-1} + 1$ then there was no violator in the i-th iteration step and we simply have $U_i = U_{i-1} - \{w_i\}$.

If $z_i \leqslant z_{i-1}$, then the coloring of w_i led to a violator H_i whose vertices were partially uncolored. The number of uncolored vertices of H_i can be identified as $m_i = z_{i-1}+1-z_i$. Knowing m_i we can separate the blocks of the lists B and L corresponding to the subgraph H_i. Let $B_i = (b_0, b_1, b_2, \ldots, b_{2m_i-3})$ be the block of $2m_i - 2$ entries of B starting from the current position $\mathrm{pos}(B)$ and advance $\mathrm{pos}(B) \leftarrow \mathrm{pos}(B) + 2m_i - 2$. Let $L_i = (\ell_1, \ldots, \ell_{m_i-1})$ be the block of $m_i - 1$ entries of L starting from the current position $\mathrm{pos}(L)$ and advance $\mathrm{pos}(L) \leftarrow \mathrm{pos}(L)+m_i-1$. Having identified w_i, B_i, and L_i we can reconstruct the traversal $W_i' = (u_0', u_1', u_2', \ldots, u_{2m_i-2}')$ that was fixed computing the value of $\textsc{Log}(X)$. Indeed, $u_0' = w_i$. We keep track of the number k of the next forward step to come, so we start with $k = 1$. When we have identified the vertex u_j', for $j \geqslant 0$, then the value b_j indicates whether the next step is forward or backward. If the step is forward, then we know that the $u_j'u_{j+1}'$ edge is the ℓ_k-th edge in the fixed ordering of edges adjacent to u_j'. Therefore, we identified u_{j+1}' and we increment k. If the step is backward, then we identify u_{j+1}' from the discovered so far subtree of T_i. When the sequence $W_i' = (u_0', u_1', u_2', \ldots, u_{2m_i-2}')$ is reconstructed, in particular, we have identified all the uncolored vertices of H_i. Finally, $U_i = U_{i-1} \cup \{u_0', u_1', u_2', \ldots, u_{2m_i-2}'\}$. This completes the description of the forward phase.

In the backward phase we reconstruct:

(i) $(f_M, f_{M-1}, \ldots, f_1)$ where f_i is the partial coloring f after the i-th iteration; and
(ii) $(x_M, x_{M-1}, \ldots, x_1)$.

Note that $f_M = F$ is a part of the evaluation of $\textsc{Log}(X)$. In particular, the number γ of vertices colored after the whole process is known. For the two lists in $\Gamma \in \textsc{Log}(X)$ we initialize $\mathrm{pos}(B') = M - \gamma$, $\mathrm{pos}(A) = 0$, and $\mathrm{pos}(P) = M - \gamma$. The invariant is that when f_i has been computed these functions mark the tail of the lists after the $(i-1)$-st iteration of the loop. In order to mark the tail of list A, we mark the head of the respective part of $^*\!AP$, because A is reversed there. The tail of P and $^*\!AP$ coincide except for the γ added elements.

Let $1 < i \leqslant M$ and suppose that f_i has been identified. Now, if $z_i = z_{i-1} + 1$ then there was no violator in the i-th iteration step and we simply have $x_i = f_i(w_i)$ and f_{i-1} is obtained from f_i by uncoloring w_i.

If $z_i \leqslant z_{i-1}$, then the coloring of w_i led to a violator H_i. From the forward phase we know the traversal $W_i' = (u_0', u_1', u_2', \dots, u_{2m_i-2}')$ with $u_0' = w_i$ of a spanning tree of the subgraph of H_i uncolored in the ith step. Let $V_i = (v_1', u_2', \dots, v_{m_i}')$ be the uncolored vertices of H_i sorted by their first-appearance in W_i'. From B' take the block $B_i' = (b_1', \dots, b_{m_i}')$ of length m_i preceding the current position. The sequence B_i' guides us how to identify the split of V_i into $V_i^1 = (v_1^1, \dots, v_{c_i}^1)$ and $V_i^0 = (v_1^0, \dots, v_{d_i}^0)$. They are simply the restrictions of V_i to those indices j with $b_j' = 1$ and $b_j' = 0$, respectively. As we see above, from the lengths of the identified sequences we can reconstruct the values of c_i and d_i, i.e., $c_i = |V_i^1|$ and $d_i = |V_i^0|$. Now we can identify the next block $A_i = (\alpha_1, \dots, \alpha_{c_i})$ of length c_i in $^*\!AP$ by reversing the next c_i elements after $\mathrm{pos}(A)$ and the block $P_i = (p_1, \dots, p_{d_i})$ of size d_i preceding the position $\mathrm{pos}(P)$ in $^*\!AP$. The list A_i contains the colors of vertices in V_i^1, namely, $f_{i-1}(v_j^1) = \alpha_j$. For vertices in V_i^0 we use the list P_i and have $f_{i-1}(v_j^0) = \alpha_{p_j}$. Hence, we finally get $x_i = \alpha_1$ and after resetting $f_{i-1}(w_i) = \diamondsuit$ we have reconstructed the coloring f_{i-1}. It only remains to update $\mathrm{pos}(B') \leftarrow \mathrm{pos}(B') - m_i$, $\mathrm{pos}(A) \leftarrow \mathrm{pos}(A) + c_i$, and $\mathrm{pos}(P) \leftarrow \mathrm{pos}(P) - d_i$. This completes the description of the backward phase and the proof of the claim. $\square$

We now come to the final task in the proof, we estimate the size of the image of the LOG-function. Recall that $\textsc{Log}(X) = (Z, \Sigma, \Gamma, F)$, we will independently estimate the number of values each of the four components can take.

Recall that $Z = (z_0, z_1, \dots, z_M)$. Based on Z, we build an auxiliary sequence $S_Z = (s_1, \dots, s_{2M})$ where each s_i is either $+1$ or -1. The sequence is obtained from left to right. Set $s_1 = z_1 - z_0 = 1$. Assume that $z_0, \dots z_j$ have been processed. If $z_{j+1} - z_j = 1$, then append $+1$ to S_Z and if $z_{j+1} - z_j = -r \leqslant 0$ append a single $+1$ and a sequence of $r + 1$ entries of -1 to S_Z, finally add a sequence of γ entries of -1. Note that replacing Z by S_Z in $\textsc{Log}(X)$ would not change the encoded information. Indeed knowing γ from $F \in \textsc{Log}(X)$ the final sequence can be removed and z_i is obtained as the sum of the i-th $+1$ entry and the block of -1 entries following it in the truncated S_Z.

In S_Z the number of $+1$ entries and the number of -1 entries are exactly M and every prefix of S_Z has a non-negative sum. These sequences are known as Dyck-paths and counted by the Catalan number C_M. It is well-known that C_M is in $o(4^M)$.

For Σ and Γ we consider the data accumulated with a violator H_i and its m_i uncolored vertices. In $\Sigma = (B, L)$ we add a binary list B_i of length $2m_i - 2$ and a list L_i of length $m_i - 1$ with entries in $\{1, \dots, \Delta\}$. In $\Gamma = (B', {}^*\!AP)$ we add a binary list B_i' of length m_i, a list A_i of length c_i with entries in $\{1, \dots, c\}$, and a list P_i of length d_i with entries in $\{1, \dots, p\}$. From the properties of a violator we have $c_i + d_i = m_i$ and $d_i \geqslant c_i$.

Since we know the lengths of these lists, we can upper bound the possibilities for B, L, and B' respectively by 2^{2M}, $\Delta^{\left(1-\frac{1}{2p}\right)M}$, and 2^M. Lists A and P together have M entries and for each i we have $d_i \geqslant c_i$. Since $c > p$ we only overcount when assuming that A

and P are of the same length, therefore, the number of possibilities for $^*AP$ is bounded by $c^{M/2}p^{M/2}$.

For the final coloring F there are $(c+1)^n$ options.

This way we obtain an upper bound on the size of the image of the Log-function

$$o(4^M)\cdot 2^{2M}\Delta^{\left(1-\frac{1}{2p}\right)M}2^M c^{M/2}p^{M/2}(c+1)^n.$$

Since Log is injective we get

$$\begin{aligned} o(4^M)\cdot 2^{2M}\Delta^{\left(1-\frac{1}{2p}\right)M}2^M c^{M/2}p^{M/2}\cdot(c+1)^n &\geqslant c^M,\\ o(32^M)\cdot \Delta^{\left(1-\frac{1}{2p}\right)M}p^{M/2}\cdot(c+1)^n &\geqslant c^{M/2},\\ o\left(\left(2^{10}\cdot\Delta^{2-1/p}\cdot p\right)^M\right)\cdot(c+1)^{2n} &\geqslant c^M. \end{aligned}$$

Since we have chosen $c=\left\lceil 2^{10}\Delta^{2-1/p}p\right\rceil$, we see that for large enough M the inequality above cannot hold. Therefore, for some input X the algorithm outputs a p-centered coloring of G using c colors. This completes the proof.

Let $c=\left\lceil 2^{10}\Delta^{2-1/p}p\right\rceil$. If the naïve randomized coloring algorithm is allowed to use $2c$ colors it succeeds in finding a valid coloring in expected $\leqslant 2n\log(2c+1)+4$ iterations. Indeed, let $M=2n\log(2c+1)$ and consider the probability that after $M+t$ iterations the algorithm did not break. This probability is upper bounded by $\frac{c^{(M+t)/2}(2c+1)^n}{(2c)^{(M+t)/2}}\leqslant\frac{1}{2^{t/2}}$, whence the expected number of iterations is upper bounded by $M+\sum\frac{1}{2^{t/2}}<M+4$.

A single iteration step can be done in $\mathcal{O}(n)$ time. Indeed, the most time consuming thing to do is to check whether after coloring w_i there is a violator. This can be done by iterating over all subsets $C\subseteq\{1,\ldots,c\}$ of size at most p. For each such C, we can determine in $\mathcal{O}(n)$ time if the component of the subgraph spanned by vertices with colors from C that contains w_i is a violator. Since the number of such subsets is a function of c and p only, i.e. of Δ and p, we conclude that a single iteration can be done in $\mathcal{O}(n)$ time.

Therefore, the expected runtime of the randomized algorithm is $\mathcal{O}(n^2)$.

4. Upper bound for graphs excluding a fixed topological minor: Lifting through the structure theorem

In this section we prove Theorem 4. We start with an informal statement of the structural theorem by Grohe and Marx [6]. For every graph H, every graph excluding H as a topological minor has a tree-decomposition with bounded adhesion such that every torso either has bounded degree with the exception of a bounded number of vertices, or excludes a fixed graph as a minor. Furthermore, such a decomposition for a graph G can be computed in time $f(H)\cdot|V(G)|^{\mathcal{O}(1)}$ for some computable function f.

We proceed with all the necessary definitions.

A graph H is a *minor* of a graph G if H can be obtained from G by deleting vertices, deleting edges, and contracting edges. A graph H is a *topological minor* of a graph G

if H can be obtained from G by deleting vertices, deleting edges, and contracting edges with at least one endpoint of degree 2.

A *tree-decomposition* of a graph G is a pair $(T, \{B_t\}_{t\in V(T)})$ where T is a tree and the sets B_t $(t \in V(T))$ are subsets of $V(G)$ called *bags* satisfying

(1) for each edge $uv \in E(G)$ there is a bag containing both u and v, and
(2) for each vertex $v \in V(G)$ the set of vertices $t \in V(T)$ with $v \in B_t$ induces a non-empty subtree of T.

The *treewidth* of a graph G is the minimum k such that G has a tree-decomposition with all bags of size at most $k+1$. An *adhesion set* in the tree-decomposition $(T, \{B_t\}_{t\in V(T)})$ of G is a set $B_t \cap B_{t'}$ where $tt' \in E(T)$. The *adhesion* of a tree-decomposition is the maximum size of an adhesion set. The *torso* of a bag B_t is the graph obtained from $G[B_t]$ by adding all the edges between vertices in every adhesion set $B_t \cap B_{t'}$ where $tt' \in E(T)$.

We are ready to present the precise statement of the structure theorem for graphs avoiding a fixed graph as a topological minor.

Theorem 12 ([6])**.** *For every integer $k \geqslant 1$, there exist $a(k)$, $c(k)$, $d(k)$, and $e(k)$ such that the following holds: If H is a graph on k vertices and G avoids H as a topological minor, then G has a tree-decomposition $(T, \{B_t\}_{t\in V(T)})$ with adhesion at most $a(k)$ such that for every $t \in V(T)$ we have*

(i) *the torso of B_t has at most $c(k)$ vertices of degree larger than $d(k)$, or*
(ii) *the torso of B_t avoids $K_{e(k)}$ as a minor.*

To lift the statement through the tree-decompositions of bounded adhesion we use the following lemma from [20]. We say that a tree-decomposition $(T, \{B_t\}_{t\in V(T)})$ of a graph G is *over a class of graphs $\mathcal{C}$* if for every $t \in V(T)$ the torso of B_t lies in $\mathcal{C}$.

Lemma 13 ([20])**.** *Let $\mathcal{C}$ be a class of graphs such that for a fixed d and every $p \geqslant 1$, every graph in $\mathcal{C}$ admits a p-centered coloring with $\mathcal{O}(p^d)$ colors. If $\mathcal{D}$ is a class of graphs such that every graph $G \in \mathcal{D}$ has a tree-decomposition over $\mathcal{C}$ with adhesion at most k, then for every $p \geqslant 1$, every graph in $\mathcal{D}$ admits a p-centered coloring with $\mathcal{O}(p^{d+k})$ colors.*

We also need the following theorem from [20].

Theorem 14 ([20])**.** *For every graph H there is a polynomial f such that the graphs excluding H as a minor admit p-centered colorings with at most $f(p)$ colors.*

With these tools in hand we can give a proof of a polynomial upper bound for graphs avoiding a fixed topological minor (Theorem 4).

Let H be a graph on k vertices and let G be a graph avoiding H as a topological minor. From Theorem 12 we obtain a tree-decomposition $(T, \{B_t\}_{t\in V(T)})$ with adhesion at most $a(k)$ such that every torso of a bag in the tree-decomposition satisfies one of the two properties mentioned there.

For fixed $p \geqslant 1$, we construct a p-centered coloring of G. Let $t \in V(T)$ and consider the torso $\tau(t)$ of the bag B_t. Suppose that $\tau(t)$ has at most $c(k)$ vertices of degree larger than $d(k)$. We color the vertices of $\tau(t)$ as follows:

(1) use at most $c(k)$ distinct colors on vertices of degree larger than $d(k)$;

(2) color the remaining vertices using at most $(32 \cdot d(k))^2 \cdot p$ colors applying Theorem 3.

Clearly, we obtain a p-centered coloring of $\tau(t)$ using at most $c(k)+(32 \cdot d(k))^2 \cdot p$ colors.

Now suppose that $\tau(t)$ avoids $K_{e(k)}$ as a minor. By Theorem 14, there is a constant $f(k)$ such that $\tau(t)$ admits a p-centered coloring using $\mathcal{O}(p^{f(k)})$ colors.

Therefore, by Lemma 13, the graph G admits a p-centered coloring using $\mathcal{O}(p^{1+f(k)+a(k)})$ colors. This completes the proof.

As mentioned, Theorem 12 comes with an algorithm that produces a desired tree-decomposition for a graph G on n vertices in $f(H) \cdot n^{\mathcal{O}(1)}$ in time for some computable function f. Both results from [20], Lemma 13 and Theorem 14, are also supported by algorithms with running times $n^{\mathcal{O}(1)}$. Finally, within the proof of our theorem the only non-trivial thing to comment is when we call the algorithm to compute a coloring in the bounded degree case. As we have seen in the previous section, the expected running time is $\mathcal{O}(n^2)$. All in all, there is a randomized polynomial time algorithm to color a graph without H as a topological minor in a p-centered way with polynomial in p number of colors.

5. Upper bounds for graphs of bounded simple treewidth

In order to deduce Theorem 1 from Theorem 9, we need a good upper bound for planar graphs of treewidth at most 3, i.e., subgraphs of stacked triangulations, see [10]. We proceed to prove Theorem 6.(i) and Theorem 7.(i) which also gives Theorem 6.(iii) To establish these results, we need a certain structural property of chordal graphs, sometimes called the *shadow completeness*, that was first observed by Kündgen and Pelsmajer in [12].

Lemma 15 (shadow completeness)**.** *Let G be a chordal graph and let $(V_0, V_1, \ldots)$ be a BFS-layering of G. Let $i > 0$ and let $C \subseteq G[\bigcup_{j>i} V_j]$ be connected. Let the* shadow *$S_i(C)$ of C in V_i be the set of $v \in V_i$ such that there is a path P from v to C with $P \cap V_i = \{v\}$. Then $S_i(C)$ is a clique. Moreover, $S_i(C)$ separates C from $G[\bigcup_{j<i} V_j]$.*

Proof. Let C be a connected subgraph of $G[\bigcup_{j>i} V_j]$. Let v and v' be two vertices in $S_i(C)$. Let P and P' be the certifying paths of v and v'. The ends of P and P' can be connected in C yielding a path connecting v and v' which has all inner vertices in $G[\bigcup_{j>i} V_j]$. Let P_1 be a shortest path from v to v' with all inner vertices in $G[\bigcup_{j>i} V_j]$. Let P_2 be a shortest path from v to v' with all inner vertices in $\bigcup_{j<i} V_j$ (there is one as $G\left[\bigcup_{j<i} V_j\right]$ is connected in the BFS-tree). Now $P_1 \cup P_2$ is a cycle in G of length at least 4 and the only possible chord is an edge vv'. Since G is chordal, we have $vv' \in E(G)$, as desired.

Since every path from a vertex in $\bigcup_{j<i} V_j$ to a vertex in C contains a vertex in V_i the shadow $S_i(C)$ is a separator for these two sets. □

Corollary 16. *Let G be a chordal graph and let $(V_0, V_1, \ldots)$ be a BFS-layering of G. Let $i \geqslant 0$ and let $H \subseteq G[\bigcup_{j\geqslant i} V_j]$ be connected. Then $H \cap G[V_i]$ is connected.*

Proof. Consider two vertices u, v in $H \cap G[V_i]$. Take a path P from u to v in H. If all vertices of P are in $H \cap G[V_i]$, then we are done. Otherwise, consider a vertex x of P with $x \in V_i$ such that the vertex after x in P is not in V_i. Let y be the first vertex of P after x with $y \in V_i$. Lemma 15 implies that x and y are adjacent. Thus P is not a shortest path between u and v in H, a contradiction. $\square$

5.1. **Proof of the upper bound for outerplanar graphs.**

Let G be an outerplanar graph. Then G is a subgraph of a maximal outerplanar graph G^+. Since any p-centered coloring of G^+ can be restricted to G, we may assume $G = G^+$. It is well-known that maximal outerplanar graphs are 2-trees. In particular, G is chordal. Let $(V_0, V_1, \ldots)$ be a BFS-layering of G.

Claim. *The subgraph $G[V_i]$ of G induced by V_i is a linear forest, i.e., each connected component of it is a path.*

Proof. Contracting $G[\bigcup_{j<i} V_j]$ and deleting $G[\bigcup_{j>i} V_j]$ yields a minor M of G that consists of $G[V_i]$ and an additional apex a adjacent to every $v \in V_i$. Since adding an apex increases the treewidth by exactly one, we have $\operatorname{tw}(G[V_i]) \leqslant 1$, so $G[V_i]$ is a forest. Assume $v \in G[V_i]$ has (at least) 3 neighbors $x, y, z \in G[V_i]$. Deleting the edge av and all vertices except x, y, z, v, a from M yields a $K_{2,3}$-minor of G, which is a contradiction with G being outerplanar. Thus $G[V_i]$ is a forest with all vertices of degree at most 2, as desired. $\square$

We will refer to the connected components of $G[V_i]$ as the *layer paths* of $G[V_i]$.

In the following, we construct a coloring $\phi : V(G) \to C$ with C being a set of colors of size $p\lceil \log(p+1) \rceil + 2p + 1$.

We construct ϕ layer by layer. In each layer we color the layer paths independently. When it comes to coloring a layer path P we have a set $\mathcal{F}(P)$ of colors which are forbidden for P. We initialize $\mathcal{F}(P) = \emptyset$ for every layer path P in G. The set $\mathcal{F}(P)$ will be of size at most $p\lceil \log(p+1) \rceil + p$. The path P will be colored with a set of $p+1$ colors from $C - \mathcal{F}(P)$.

The layer V_0 contains only a single vertex v. Set $\phi(v) = \alpha$ for some color $\alpha \in C$. To make sure that α does not appear on the next p layers we add α into $\mathcal{F}(P)$ for every layer path P of $G[V_i]$ with $0 < i \leqslant p$.

If all layers V_j with $0 \leqslant j < i$ are colored and P is a layer path of $G[V_i]$, then we extend ϕ to P as follows: Choose a set $C_P \subseteq C - \mathcal{F}(P)$ of $p+1$ colors, say $C_P = \{0_P, 1_P, \ldots, p_P\}$. Color $P = (v_0, v_1, \ldots, v_m)$ periodically, i.e., let $\phi(v_k) = j_P$ if $k = j \bmod (p+1)$.

Now consider a layer path P' in $G[V_j]$ with $i < j$, and note that the shadow $S_i(P')$ of P' in V_i is the set

$$\left\{ v \in V_i \mid v \text{ is adjacent to the component of } G\left[\bigcup\nolimits_{j'>i} V_{j'}\right] \text{ containing } P' \right\}.$$

By Lemma 15 the set $S_i(P')$ induces a clique in $G[V_i]$. Therefore, $S_i(P')$ is contained in one of the layer paths of $G[V_i]$ and $|S_i(P')| \leqslant 2$. Moreover $S_i(P')$ is a separator. For $i < k < j$ let P_k be the layer path of V_k containing $S_k(P')$. The separator property implies the following transitivity of shadows: $S_i(P') = S_i(P_k)$. Indeed, a path from V_i to P' has to pass through $S_k(P') \subseteq P_k$ and every path from V_i to P_k can be extended within $G[\bigcup_{j' \geqslant k} V_{j'}]$ to a path from V_i to P'.

For each layer path P' in $G[V_k]$ with $i < k \leqslant i + p$ such that $S_i(P') \subseteq V(P)$ we extend $\mathcal{F}(P')$ by some colors used on P. To determine these forbidden colors we use an auxiliary structure. Let T be a binary tree on p nodes with root r_T and height $\lceil \log(p + 1)\rceil$ where height of a rooted tree is the maximum number of nodes on a path from the root to a leaf. We label the nodes of T with the totally ordered set of colors $1_P < 2_P \ldots < p_P$ using an *in-order* traversal, i.e., the label of each node is larger than all labels of nodes in its left subtree and smaller than all labels of nodes in its right subtree. For $\ell \in \{1_P, \ldots, p_P\}$ let $F(\ell)$ denote the set of labels of ancestors of the node u labeled ℓ, i.e., $F(\ell)$ is the set of labels seen on the path from u to r_T in T. We extend this definition by $F(0_P) = \emptyset$.

Let $i < j \leqslant i + p$ and consider a layer path P' of $G[V_j]$ with $S_i(P') \subseteq V(P)$. We extend the set of forbidden colors of P' putting

$$\mathcal{F}(P') \leftarrow \mathcal{F}(P') \cup \{0_P\} \cup \textstyle\bigcup_{v \in S_i(P')} F(\phi(v)).$$

Claim. *If ℓ, ℓ' are consecutive in the cyclic order on $\{0_P, 1_P, \ldots, p_P\}$, then*

$$F(\ell) \subseteq F(\ell') \quad \textit{or} \quad F(\ell') \subseteq F(\ell).$$

Proof. If either ℓ or ℓ' is 0, the statement is obvious. Otherwise $\ell' = \ell + 1$ and there are nodes u and u' with these labels in T. It is easy to see that every two consecutive nodes in an in-order traversal must be in the ancestor-descendant relation in the tree. This immediately gives $F(\ell) \subseteq F(\ell')$ or $F(\ell') \subseteq F(\ell)$. □

Note that if $S_i(P') = \{v, v'\}$, then v and v' are consecutive on P. By the claim proven above we have

$$\left|\textstyle\bigcup_{v \in S_i(P')} F(\phi(v))\right| \leqslant \lceil \log(p + 1)\rceil.$$

Hence at most $\lceil \log(p + 1)\rceil + 1$ colors are added to $\mathcal{F}(P')$ after coloring the layer paths in $G[V_i]$. Since $\mathcal{F}(P')$ is extended at most p times we get an upper bound

$$|\mathcal{F}(P')| \leqslant p\lceil \log(p + 1)\rceil + p.$$

Claim. *ϕ is a p-centered coloring of G.*

Proof. Let H be a connected subgraph of G. We want to show that ϕ uses more than p colors on H or there is a color used only once on H. Let $i := \min\{j \geqslant 0 \mid V_j \cap V(H) \neq \emptyset\}$. We distinguish two cases:

Case 1. H contains a vertex from V_{i+p}.

Let Q be a layer path in V_{i+p} containing a vertex u_{i+p} from H. For all j with $i \leqslant j < i + p$ the connectivity of H together with the separator property of $S_j(Q)$ imply that $S_j(Q) \cap V(H) \neq \emptyset$. Therefore we can choose $u_j \in S_j(Q) \cap V(H)$ for $i \leqslant j < i + p$.

We claim that $\phi(u_j) \neq \phi(u_k)$ for all $i \leqslant j < k \leqslant i + p$. Let Q_k be the layer path containing u_k. From $S_k(Q) \subseteq Q_k$ it follows from the transitivity property of shadows that $S_j(Q_k) = S_j(Q)$, whence $u_j \in S_j(Q_k)$. Now $\phi(u_j) \in F(\phi(u_j)) \subseteq \mathcal{F}(Q_k)$, Therefore, the color $\phi(u_j)$ is forbidden for the path Q_k and $\phi(u_k) \neq \phi(u_j)$.

We conclude that the vertices $u_i, \dots, u_{i+p}$ are colored with $p+1$ distinct colors. Therefore ϕ uses on H more than p colors.

Case 2. *H contains no vertex from V_{i+p}.*

Corollary 16 implies that $Q = G[V(H) \cap V_i]$ is connected, so it is a subpath of some layer path P of $G[V_i]$. Let $C_P = \{0_P, ..., p_P\}$ be the set of colors used to color P. Let $C_Q \subseteq C_P$ be the set of colors used on Q. If two vertices of Q have the same color, then due to the periodicity of ϕ on P all the $p+1$ colors from $\{0_P, \dots, p_P\}$ are used on Q, whence ϕ uses on H more than p colors. From now on we assume that all colors on Q are unique.

If $0_P \in C_Q$, then recall that $0_P \in \mathcal{F}(P')$ for any layer path P' of $G[V_k]$ with $i < k \leqslant i+p$ such that $S_i(P') \subseteq V(P)$. Therefore 0_P is contained in $\mathcal{F}(P')$ for all P' intersected by $V(H)$. Hence 0_P is a unique color on H. From now on we assume that $0_P \notin C_Q$.

Consider the rooted binary tree T on p nodes devised during the coloring procedure of P. Since C_Q is an interval of $\{1, \dots, p\}$ there is a unique node r_Q in T with a label from C_Q such that r_Q is a predecessor of all nodes with labels from C_Q in T. Let β be the label of r_Q. It follows that $\beta \in F(\ell)$ for all $\ell \in C_Q$, whence $\beta \in F(\phi(v))$ for all $v \in Q$. Therefore $\beta \in \mathcal{F}(P')$ for any layer path P' of $G[V_k]$ with $i < k \leqslant i+p$ such that $S_i(P') \subseteq V(P)$. This means that β is unique on H, as desired. $\square$

In summary, any outerplanar graph has a p-centered coloring with $p\lceil \log(p+1)\rceil + 2p+1 \in \mathcal{O}(p \log p)$. This completes the proof of Theorem 6.(i). The described algorithm coloring outerplanar graphs has a straightforward $\mathcal{O}(n)$ time implementation.

5.2. **Proof of the upper bound for graphs of bounded simple treewidth.** For $p \geqslant 1$, let $f(p)$ be the maximum $\chi_p(G)$ when G is outerplanar. We have just seen that $f(p) \in \mathcal{O}(p \log p)$. Now we are going to show that every planar graph of treewidth at most 3 admits a p-centered coloring with at most $(p+1)f(p)$ colors. This proof can be generalized though, which is what we will do here. The essence of the generalization is that maximal outerplanar graphs are simple 2-trees. We continue with a discussion of simple treewidth.

For $k \geqslant 1$, a tree-decomposition $(T, \mathcal{B})$ of G is *k-simple* if $(T, \mathcal{B})$ is of width at most k and for every $X \subseteq V(G)$ with $|X| = k$, we have that $X \subseteq B_t$ for at most two distinct $t \in V(T)$. The *simple treewidth* of G, denoted by $\mathrm{stw}(G)$, is the least integer k such that G has a k-simple tree-decomposition.

The notion of simple treewidth was proposed by Knauer and Ueckerdt [9], where graphs of $\mathrm{stw}(G)$ are defined as subgraphs of simple k-trees. Since we do not want to discuss k-trees and their construction order we stick to our equivalent definition, see [24] for further discussion. It is easy to see that $\mathrm{stw}(G) \leqslant \mathrm{tw}(G) \leqslant \mathrm{stw}(G) + 1$, for every graph G. Graphs of simple treewidth at most 1 are disjoint unions of paths. Graphs of simple treewidth at most 2 are outerplanar graphs. Graphs of simple treewidth at most 3 are planar graphs of treewidth at most 3, i.e. subgraphs of stacked triangulations, see [10].

The first thing to learn about simple treewidth is that it is monotone under taking minors, i.e., if H is a minor of G, then $\mathrm{stw}(H) \leqslant \mathrm{stw}(G)$, see [8]. The next key fact is that the simple treewidth goes down when we restrict the graph to a single BFS layer.

Lemma 17 ([8])**.** *Let G be a connected graph with $\mathrm{stw}(G) = k \geqslant 1$ and let $(V_0, V_1, \ldots)$ be a BFS-layering of G. Then $\mathrm{stw}(G[V_i]) \leqslant k-1$ for every $i \geqslant 0$.*

For $k \geqslant 2$ and $p \geqslant 1$, let $f_k(p)$ be the maximum p-centered chromatic number of graphs of simple treewidth at most k. We will show that $f_{k+1}(p) \leqslant (p+1)f_k(p)$. Given that $f_2(p) \in \mathcal{O}(p \log p)$ this yields $f_k(p) \in \mathcal{O}(p^{k-1} \log p)$ for all $k \geqslant 2$. Below we proceed with the induction step for $k \geqslant 3$.

Let G be a graph and $\mathrm{stw}(G) \leqslant k$. Fix a tree-decomposition $(T, \mathcal{B})$ of G witnessing the simple treewidth and let G^+ be a supergraph of G so that each $B \in \mathcal{B}$ induces a clique in G^+. Clearly, $\mathrm{stw}(G^+) \leqslant k$ and G^+ is chordal. Since every p-centered coloring of G^+ is a p-centered coloring of G we work with G^+. Let $(V_0, V_1, \ldots)$ be a BFS layering of G^+.

Let v be a vertex in G^+ with $v \in V_i$. We define $\alpha(v) = i \bmod (p+1)$. By Lemma 17 $\mathrm{stw}(G^+[V_i]) \leqslant k-1$. Let β_i be a p-centered coloring of $G^+[V_i]$ using at most $f_{k-1}(p)$ colors. We define $\beta(v) = \beta_i(v)$. Finally, for a vertex $v \in V(G^+)$ we define $\phi(v) = (\alpha(v), \beta(v))$. Clearly, ϕ uses at most $(p+1) \cdot f_{k-1}(p)$ colors. We claim that ϕ is a p-centered coloring of G^+.

Let H be a connected subgraph of G^+. We want to show that either ϕ uses more than p colors on H or there is a color that appears exactly once on H. Let ℓ be minimal such that $V(H) \cap V_\ell \neq \emptyset$. The set $X = V(H) \cap V_\ell$ induces a connected subgraph of G^+ by Corollary 16 (here we are using that G^+ is chordal). Since β_ℓ is a p-centered coloring of $G^+[V_\ell]$ we have that either $|\beta(X)| > p$ or there is a vertex in X of unique color under β. In the first case, ϕ takes more than p values on $V(H)$. In the second case, fix the vertex $x \in X$ of the unique color under β. If x has a unique color under ϕ in H, then we are done. Otherwise, let $x' \in V(H)$ be a vertex with $x \neq x'$ and $\phi(x) = \phi(x')$. Let $x' \in V_{\ell'}$. Since x has a unique color in $V(H) \cap V_\ell$, we get that $\ell \neq \ell'$. Since $\alpha(x) = \alpha(x')$ we conclude that $\ell' - \ell \geqslant p+1$. Since x and x' are two vertices in a connected graph H we know that H must intersect every layer V_k with $\ell \leqslant k \leqslant \ell'$. This means that α takes all possible $p+1$ values on the vertices of H. Therefore, ϕ uses at least $p+1$ colors on H, as desired.

The proof implies an algorithm for coloring simple k-trees. This algorithm may have many calls for the corresponding algorithm for $(k-1)$-trees but, since the graphs of these calls are disjoint, the overall time complexity remains $\mathcal{O}(n)$ for fixed k.

6. Lower bound for graphs of bounded treewidth

Let p be an integer and $p \geqslant 1$. A vertex coloring ϕ of a graph G is *p-linear* if for every path P in G either ϕ uses more than ϕ colors on P or there is a color that appears exactly once on P. The *p-linear chromatic number* $\mathrm{lin}_p(G)$ of G is the minimum integer k such that there is a p-linear coloring of G using k colors.

Clearly, every p-centered coloring is a p-linear coloring. Thus, $\chi_p(G) \leqslant \mathrm{lin}_p(G)$ for every graph G. We are going to show that for every $p \geqslant 0$ and $t \geqslant 0$, there is a graph

G of treewidth at most t with

$$\operatorname{lin}_p(G) \geqslant \binom{p+t}{t}.$$

We start with the key definition for our inductive construction. For an integer x with $x \geqslant 1$, a rooted tree R is *x-ary*, if every non-leaf vertex of R has degree at least x. The *depth* of a vertex v in a rooted tree R is the number of vertices on the path between v and the root in R. A rooted tree R has *depth d* if every leaf in R has depth d. A subgraph R of a graph G is *(p, d, x)-brushed* if

(i) R is an x-ary tree of depth d; and
(ii) for every vertex v in R, if $i \in [d]$ is the depth of v in R, then there exist indices $j_1, \ldots, j_k$ with $1 \leqslant k \leqslant p$ and $i = j_1 < \cdots < j_k = d$ such that for every leaf w of R that lies in the subtree of v, if $u_i \cdots u_d$ is a path in R connecting $u = u_i$ with $w = u_d$, then the sequence $u_{j_1} \cdots u_{j_k}$ induce a path in G.

We construct a family of graphs $\{G_{(p,t,x,N)} \mid p \geqslant 0,\ t \geqslant 0,\ x \geqslant 2,\ N \geqslant 1\}$ such that the following invariants hold

(1) $\operatorname{tw}(G_{(p,t,x,N)}) \leqslant t$; and
(2) for every integer p' with $p' \geqslant p$ and every p'-linear coloring ϕ of $G_{(p,t,x,N)}$ either
   (a) ϕ uses at least N colors on $G_{(p,t,x,N)}$, or
   (b) G has a $\left(p, \binom{p+t}{t}, x\right)$-brushed subgraph R and there is a sequence of distinct colors $\left(\alpha_1, \ldots, \alpha_{\binom{p+t}{t}}\right)$ so that for every $i \in \left\{1, \ldots, \binom{p+t}{t}\right\}$ and every vertex v of depth i in R, we have $\phi(v) = \alpha_i$.

This family, in particular $G_{\left(p,t,2,\binom{p+t}{t}\right)}$, will witness the statement of the theorem.

We start the construction with the base cases. Let $x \geqslant 2$ and $N \geqslant 1$. For every $p \geqslant 0$ and every $t \geqslant 0$ the graphs $G_{(0,t,x,N)}$ and $G_{(p,0,x,N)}$ contain just a single vertex. Since these graphs have no edges, their treewidth is $0 \leqslant t$, so invariant (1) holds. A single vertex is also an x-ary tree of depth 1, so invariant (2b) holds.

For the inductive step let $p \geqslant 1$, $t \geqslant 1$, $x \geqslant 2$, and $N \geqslant 1$. Let $M = \binom{p+t-1}{t-1}$ and $X = (x-1)N^M + 1$. The graph $G_{(p,t,x,N)}$ is obtained from a copy G^0 of $G_{(p-1,t,X,N)}$ by adding for every vertex v of G^0 disjoint copies $G^{v,1}, \ldots, G^{v,X}$ of $G_{(p,t-1,x,N)}$ in such a way that each vertex of $G^{v,i}$ with $i \in [X]$ is adjacent to v and to no other vertices outside $G^{v,i}$.

First, we argue that $\operatorname{tw}(G_{(p,t,x,N)}) \leqslant t$. Take a tree-decomposition $(T^0, \mathcal{B}^0)$ of G^0 of width $\leqslant t$ and for each $v \in V(G^0)$ and each $i \in [X]$ take a tree-decomposition $(T^{v,i}, \mathcal{B}^{v,i})$ of $G^{v,i}$ of width $\leqslant t-1$. We construct a tree-decomposition $(T, \mathcal{B})$ of $G_{(p,t,x,N)}$ as follows. Let $V(T) = V(T^0) \cup \bigcup V(T^{v,i})$. The edges of T include all the edges of T^0 and all the edges of $T^{v,i}$ for every $v \in V(G^0)$ and $i \in [X]$. Additionally, for every $v \in V(G^0)$, fix some $t_v \in V(T)$ such that $v \in B^0_{t_v} \in \mathcal{B}^0$ and for each i fix some vertex $t^{v,i}$ of $T^{v,i}$. In T the tree T is connected to $T^{v,i}$ with the extra edge $t_v t^{v,i}$. In simple words, we make a tree out of the forest formed by previous trees. The set $\mathcal{B}$ of bags for T are defined as follows: for every $t \in V(T^0)$ let $B_t = B^0_t$ where $B^0_t \in \mathcal{B}^0$, and for every $t \in V(T^{v,i})$

let $B_t = B_t^{v,i} \cup \{v\}$. It is elementary to verify that $(T, \mathcal{B})$ is a tree-decomposition of $G_{(p,t,x,N)}$ and the width of $(T, \mathcal{B})$ is at most t.

Let $p' \geqslant p$ and let ϕ be a p'-linear coloring of $G_{(p,t,x,N)}$. We want to show that (2) is satisfied for the coloring ϕ. First of all, if ϕ uses at least N colors, then (2a) holds. Therefore from now on we assume that ϕ uses less than N colors.

Since G^0 is isomorphic to $G_{(p-1,t,X,N)}$ we have that G^0 has a $\big(p, \binom{p-1+t}{t}, X\big)$-brushed subgraph that satisifes (2b). We fix a witnessing X-ary tree R^0 of depth $\binom{p-1+t}{t}$ and the respective sequence of colors $C^0 = \Big(\alpha_1, \dots, \alpha_{\binom{p-1+t}{t}}\Big)$ such that $\phi(u) = \alpha_i$, for every vertex u of depth i in R^0.

Let u be a leaf of R^0 and $j \in [X]$. We claim that no vertex in $G^{u,j}$ is colored by ϕ with a color from C^0. In order to get a contradiction, suppose that $\phi(v) = \alpha_i$ for some v in $G^{u,j}$ and $i \in \big[\binom{p-1+t}{t}\big]$. If $i = \binom{p-1+t}{t}$, then $\phi(u) = \alpha_i = \phi(v)$, while u and v are adjacent in $G_{(p,t,x,N)}$. Since $p' \geqslant p \geqslant 1$, the connected subgraph of $G_{(p,t,x,N)}$ induced by $\{u, v\}$ contradicts the fact that ϕ is p'-linear. Now assume that $1 \leqslant i < \binom{p-1+t}{t}$. Let u_i and u_{i+1} be the ancestors of u in R^0 of depth i and $i+1$, respectively; ($u_{i+1} = u$ is possible). Since $X \geqslant 2$, u_i has some child u' distinct from u_{i+1}. Let u'' be a leaf (vertex of depth $\binom{p-1+t}{t}$) of R^0 in the subtree of u'. Consider the paths $u_i \cdots u_{\binom{p-1+t}{t}}$ and $w_i \cdots w_{\binom{p-1+t}{t}}$ in R where $u_{\binom{p-1+t}{t}} = u$, $w_i = u_i$, $w_{i+1} = u'$, and $w_{\binom{p-1+t}{t}} = u''$. Since R^0 is $\big(p, \binom{p-1+t}{t}, X\big)$-brushed in G^0 there exist indices $j_1, \dots, j_k$ with $1 \leqslant k \leqslant p-1$ and $i = j_1 < j_2 < \cdots < j_k = \binom{p-1+t}{t}$ such that $u_{j_1} \cdots u_{j_k}$ and $w_{j_1} \cdots w_{j_k}$ are paths in G^0 connecting $u_i = u_{j_1}$ with $u = u_{j_k}$ and $u_i = w_{j_1}$ with $u'' = w_{j_k}$, respectively. Consider the path $P = u_{j_k} \cdots u_{j_1} w_{j_2} \cdots w_{j_k}$ connecting $u = u_{j_k}$ and $u'' = w_{j_k}$ in G^0 and recall that $\phi(u_{j_\ell}) = \alpha_{j_\ell} = \phi(w_{j_\ell})$, for all $\ell \in [k]$. Thus, ϕ uses exactly k colors on P and the only unique color on P is $\phi(u_i) = \alpha_i$. Now since v and $u_{j_k} = u$ are adjacent in $G_{(p,t,x,N)}$, the following sequence is a path in $G_{(p,t,x,N)}$:

$$v u_{j_k} \cdots u_{j_1} w_{j_2} \cdots w_{j_k}.$$

Since $\phi(v) = \alpha_i = \phi(u_i)$, there is no color used by ϕ exactly once on that path. Furthermore, ϕ uses exactly $k \leqslant p-1 \leqslant p'$ colors on that path. This contradicts the fact that ϕ is p'-linear coloring of $G_{(p,t,x,N)}$. Thus, no vertex in $G^{u,j}$ is colored by ϕ with a color from C^0

Let u be a leaf of R^0 and $j \in [X]$. Since $G^{u,j}$ is isomorphic to $G_{(p,t-1,x,N)}$, we have that $G^{u,j}$ has a (p, M, x)-brushed subgraph $R^{u,j}$ that satisifes (2b). We fix a witnessing x-ary tree $R^{u,j}$ of depth M and the respective sequence of colors $C^{u,j} = \big(\beta_1^{u,j}, \dots, \beta_M^{u,j}\big)$ such that $\phi(v) = \beta_k^{u,j}$, for every $k \in [M]$ and every vertex v of depth k in $R^{u,j}$. It follows from the previous paragraph that $\alpha_i \neq \beta_k^{u,j}$, for all $i \in \big[\binom{p-1+t}{t}\big]$ and $k \in [M]$.

Since ϕ takes less than N values, there are less then N^M possibilities for the color sequence $C^{u,j}$. Since $X = (x-1)N^M + 1$, we find x values $j \in [X]$ such that the color sequences $C^{u,j}$ are identical, we let $C^u = (\beta_1^u, \dots, \beta_M^u)$ be this repeated color sequence.

We now perform a bottom-up traversal on R^0. We start by marking all the leaves as being *visited*. When reaching an internal vertex v of R^0 its children $u_1, \dots, u_X$ have been visited and each of them has an associated color sequence C^{u_j}. Again, since

$X = (x-1)N^M + 1$, we find x values $j \in [X]$ such that the color sequences C^{u_j} are identical, we define C^v to be this repeated color sequence and mark v as visited.

When all vertices have been maked visited every internal vertex v in R_0 has x children $u_1, \ldots, u_x$ such that $C^v = C^{u_i}$ for all $i \in [x]$. This way starting from the root of R^0 we can filter out a subtree R' of R^0 such that R' is an x-ary tree of depth $\binom{p-1+t}{t}$ rooted at the root of R_0 with the properties that

(i) for every vertex v in R', v has the same depth in R' as in R^0; and
(ii) there is a sequence $C = (\beta_1, \ldots, \beta_M)$ such that $C^v = C$ for all v in R'.

Finally, we define a tree R as a tree obtained from R' by attaching to each leaf u of R' exactly x trees among $R^{u,j}$ for $j \in [X]$ such that $C^{u,j} = C$. Therefore R is an x-ary tree. Note also that the depth of R is the sum of the depth of R' and $M = \binom{p+t-1}{t-1}$ (which is the depth of each $R^{u,j}$). Thus, the depth of R is

$$\binom{p-1+t}{t} + \binom{p+t-1}{t-1} = \binom{p+t}{t}.$$

We claim that R is $\left(p, \binom{p+t}{t}, x\right)$-brushed in $G_{(p,t,x,N)}$. To prove this, we need to verify the item (ii) of the definition. Let v be a vertex in R and i be the depth of v in R.

If $i > \binom{p-1+t}{t}$, then v lies in one of the trees attached to R' in the construction of R, say v is in $R^{u,j}$ for some leaf u of R^0 and $j \in [X]$. Clearly, the depth of v in $R^{u,j}$ is $i' = i - \binom{p-1+t}{t}$. Since $R^{u,j}$ is (p, M, x)-brushed in $G^{u,j}$, we get the indices $j_1, \ldots, j_k$ with $1 \leqslant k \leqslant p$ and $i' = j_1 < \cdots < j_k = M$ such that for every leaf w of $R^{u,j}$ that lies in the subtree of v, if $u_{i'} \cdots u_M$ is a path in $R^{u,j}$ connecting $v = u_{i'}$ with $w = u_M$, then the sequence $u_{j_1} \cdots u_{j_k}$ induce a path in $G^{u,j}$. Thus, in this case (ii) holds for v.

If $i \leqslant \binom{p-1+t}{t}$, then v lies in R'. Since R^0 is $\left(p-1, \binom{p-1+t}{t}, X\right)$-brushed in G^0, we get the indices $j_1, \ldots, j_k$ with $1 \leqslant k \leqslant p-1$ and $i = j_1 < \cdots < j_k = \binom{p-1+t}{t}$ such that for every leaf w of R^0 that lies in the subtree of v, if $u_i \cdots u_{\binom{p-1+t}{t}}$ is a path in R^0 connecting $u = u_i$ with $w = u_{\binom{p-1+t}{t}}$, then the sequence $u_{j_1} \cdots u_{j_k}$ induces a path in G^0. Consider a sequence $j_1, \ldots, j_k$ extended with $j_{k+1} = \binom{p+t}{t}$. Note that its length is $k+1 \leqslant p$. Now consider any leaf w' of R that lies in the subtree of v. Let w be the leaf of R^0 such that w' lies in $R^{w,j}$ for some $j \in [X]$. Let $u_i \cdots u_{\binom{p+t}{t}}$ be the path in R between $u_i = v$ and $u_{\binom{p+t}{t}} = w'$. We claim that $u_{j_1} \cdots u_{j_{k+1}}$ is a path in $G_{(p,t,x,N)}$. Indeed, $u_{j_1} \cdots u_{j_k}$ is a path in G^0 and $u_{j_k} = w$ is adjacent with $u_{j_{k+1}} = w'$ in $G^{u,j}$. This completes the proof that R is $\left(p, \binom{p+t}{t}, x\right)$-brushed in $G_{(p,t,x,N)}$.

It remains to argue that the item (2b) holds for the coloring ϕ and the subgraph R in $G_{(p,t,x,N)}$. Consider the concatenated color sequence

$$\left(\alpha_1, \ldots, \alpha_{\binom{p-1+t}{t}}, \beta_1, \ldots, \beta_M\right).$$

Let $i \in \left[\binom{p+t}{t}\right]$ and let v be a vertex of depth i in R. If $i \leqslant \binom{p-1+t}{t}$, then v is a vertex in R^0 and as such $\phi(v) = \alpha_i$. If $i > \binom{p-1+t}{t}$, then v is a vertex in $R^{u,j}$ for some u leaf of R^0 and $j \in [X]$. Since $R^{u,j}$ was chosen to attach to R' in the construction of R, we have $C^{u,j} = C$. Therefore, $\phi(v) = \beta_{i - \binom{p-1+t}{t}}$, as required.

Therefore, $G_{(p,t,x,N)}$ satisfies both invariants. This completes the proof.

7. Lower bounds for graphs of bounded simple treewidth

In this section we prove Theorem 7.(ii). We first deal with the case $k = 2$, i.e., with the case of outerplanar graphs, which was independently stated as Theorem 6.(ii). We then generalize the proof for the case of larger k. The case $k = 3$ yields Theorem 6.(iv).

7.1. **Proof of the lower bound for outerplanar graphs.**

1. Constructing the family of graphs.

The *tree of fans* $F(w, d)$ is obtained from a rooted complete w-ary tree of depth d by connecting the children of each inner vertex with a path. Note that $F(w, d)$ is outerplanar, the vertices are partitioned into $d + 1$ levels. Level 0 consists of the root vertex and level j has w^j vertices, hence there are $\sum_{j=0}^{d} w^j$ vertices in total.

Let $s = \lfloor p/2 \rfloor$ and $t = \lfloor s/2 \rfloor + 1$ and $f_2(p) = t\lfloor \log s \rfloor$. We let H be a tree of fans $F(sf_2(p), t)$ and ϕ be a p-centered coloring of H. We claim that ϕ is using at least $f_2(p)$ colors.

2. A clean subgraph of H.

We identify a *clean* subgraph H^* of H, this subgraph will be isomorphic to $F(s, t)$. The cleaning is done top-down. The root v_0 is *clean* by definition. If a vertex v is identified as clean we look at the coloring of the path P_v on its children. Let U_v be the set of unique colors, i.e., the set of colors which occur only once on P_v. If $|U_v| \geqslant f_2(p)$ we have a proof that ϕ uses at least $f_2(p)$ colors. Otherwise considering the length of P_v which is $sf_2(p)$ and the bound $|U_v| < f_2(p)$ we conclude that P_v contains a subpath of size s which contains no vertex with a unique color, i.e., every color that appears on the subpath is assigned to at least two vertices of P_v. Fix such a subpath Q_v and declare its vertices to be *clean*. Then H^* is simply the subgraph of H induced by the clean vertices.

3. The spine for the master caterpillar of H^*.

A *caterpillar* rooted at w_0 of *depth d* in H consists of a *spine*, this is a path $w_0, w_1, \ldots w_d$ such that w_{i+1} is a child of w_i, additionally there may be *hairs*, they are leaves attached to vertices of the spine. We require that hairs attached to a spine vertex w are children of w, moreover, w_d has no attached hairs, whence, we also count it as a hair attached to w_{d-1}. A caterpillar in H is *H^*-based* if all spine vertices belong to H^*.

Starting from the root v_0 of H^* we are going to identify a path $S = v_0, v_1, \ldots, v_t$ in H^* such that v_i is a vertex of level i. Later we will see that there is a caterpillar K_0 in H with spine S such that ϕ is using at least $f_2(p)$ colors on K_0.

Suppose the spine vertex $v_{\ell-1}$ has been identified. We now describe a procedure to find v_ℓ. Let Q be the path of clean children of $v_{\ell-1}$ in H^*, this path consists of s vertices. With iterated halving we identify a sequence of nested intervals $Q = Q_0 \supseteq Q_1 \supseteq \cdots$ which closes in at a single vertex v_ℓ. The iteration requires at least $\lfloor \log s \rfloor$ steps. When Q_j has been identified let U_j be the set of unique colors on Q_j. Since $|Q_j| \leqslant s \leqslant p$ and ϕ is p-centered $U_j \neq \emptyset$. Divide Q_j into its left and right half $Q_j^{(\ell)}$ and $Q_j^{(r)}$, such that $||Q_j^{(\ell)}| - |Q_j^{(r)}|| \leqslant 1$. Let $U_j^{(\ell)}$ and $U_j^{(r)}$ be the subsets of those colors of U_j which appear in $Q_j^{(\ell)}$ and $Q_j^{(r)}$, respectively. It may happen that one of $U_j^{(\ell)}$ and $U_j^{(r)}$ is empty.

In the following we let $\sigma \in \{\ell, r\}$ and $\bar{\sigma}$ be such that $\{\sigma, \bar{\sigma}\} = \{\ell, r\}$. Let $d^{(\sigma)}$ be the minimum depth of an H^*-based caterpillar rooted at a vertex of $Q_j^{(\sigma)}$ which contains all the colors of $U_j^{(\bar{\sigma})}$. If there is no such caterpillar we let $d^{(\sigma)} = p$.

Claim. $(d^{(\ell)} - 1) + (d^{(r)} - 1) > p - |Q_j|$.

Proof. For $\sigma \in \{\ell, r\}$ let $K^{(\sigma)}$ be an H^*-based caterpillar of depth $d^{(\sigma)}$ rooted at a vertex of $Q_j^{(\sigma)}$ which collects all colors of $U_j^{(\bar{\sigma})}$. We assume that every hair in $K^{(\sigma)}$ has a color from $U^{(\bar{\sigma})}$ as otherwise we simply remove it from $K^{(\sigma)}$. Let Γ_0 be the subgraph of H which is obtained as the union of Q_j, $K^{(\ell)}$, and $K^{(r)}$. Note that the number of colors used by ϕ on Γ_0 is bounded by $|Q_j| + (d^{(\ell)} - 1) + (d^{(r)} - 1)$ as hairs, including the last vertices of the spines, only reuse colors used on Q_j.

In order to get a contradiction, suppose that $(d^{(\ell)} - 1) + (d^{(r)} - 1) \leqslant p - |Q_j|$. Then we have that the number of colors used by ϕ on Γ_0 is bounded by

$$|Q_j| + (d^{(\ell)} - 1) + (d^{(r)} - 1) \leqslant |Q_j| + p - |Q_j| = p.$$

Next we are going to construct a connected supergraph Γ of Γ_0 which contains the same colors as Γ_0, i.e., at most p, and has no unique color. Such a Γ can not exist because ϕ is a p-centered coloring of H.

Consider a color α that is used exactly once in Γ_0. Since all the colors used exactly once at Q_j are repeated in $K^{(\ell)}$ or $K^{(r)}$, the color α must appear in one of the $K^{(\sigma)}$, say $K^{(\ell)}$. Since all the colors of the hairs are also used at Q_j, the color α must be used at a vertex v of the spine of $K^{(\ell)}$ but not within Q_j. Therefore the parent of v, say v', also belongs to $K^{(\ell)}$. Since $K^{(\ell)}$ is H^*-based, v and v' belong to H^*, i.e., they are clean. Therefore there is a child v'' of v' in H with $v'' \neq v$ and $\phi(v'') = \phi(v) = \alpha$. We add such a vertex v'' to Γ. We apply this procedure for every unique color α of Γ_0.

The resulting graph Γ is clearly connected and $|\phi(\Gamma)| = |\phi(\Gamma_0)| \leqslant p$ while Γ has no unique color under ϕ. This contradiction completes the proof of the claim. $\square$

From the claim we get a $\sigma \in \{\ell, r\}$ with $d^{(\sigma)} - 1 \geqslant \frac{1}{2}(p - |Q_j|) \geqslant \frac{1}{2}(p - s) \geqslant \frac{p}{4} \geqslant t - 1$. Use this σ to define $Q_{j+1} = Q_j^{(\sigma)}$ and $A_{j+1} = U_j^{(\bar{\sigma})}$.

The iterated halving ends with a vertex v_ℓ and a sequence $A_1, A_2, \ldots, A_{\lfloor \log s \rfloor}$ of sets of colors. From the construction it follows that A_{i+1} is the set of unique colors of Q_i which appear in $Q_i^{(\bar{\sigma})}$ while for $j > i + 1$ the colors of A_j appear in $Q_i^{(\sigma)}$. This shows that the sets $A_1, A_2, \ldots, A_{\lfloor \log s \rfloor}$ are pairwise disjoint. Define $B_\ell = \bigcup_{i=1}^{\lfloor \log s \rfloor} A_i$. From the definition of $d^{(\sigma)}$ and the inequality $d^{(\sigma)} \geqslant t$ we can deduce the following observation, which will be crucial:

> Every H^*-based caterpillar of depth at most $t - 1$ rooted at v_ℓ misses at least one color from each A_i, i.e., it misses at least $\lfloor \log s \rfloor$ colors from B_ℓ.

4. Color collecting subcaterpillars of the master.

Having defined the vertices $v_0, v_1, \dots, v_t$ we let K_ℓ be the caterpillar with spine $v_\ell, \dots, v_t$ which includes all the children of v_i in H^* for each $i = \ell, \dots, t-1$. With C_ℓ we denote the set of colors of K_ℓ. Since for $0 \leqslant j < \ell \leqslant t$ all vertices of K_ℓ are also vertices of K_j we get $C_0 \supseteq C_1 \supseteq \cdots \supseteq C_t$. Note that B_ℓ is a set of colors used by children of $v_{\ell-1}$, therefore, $B_\ell \subseteq C_{\ell-1}$. For $\ell > 0$, the caterpillar K_ℓ is an H^*-based caterpillar of depth $t-\ell \leqslant t-1$ rooted at v_ℓ, therefore, K_ℓ misses at least $\lfloor \log s \rfloor$ colors from B_ℓ. Together this shows that $|C_{\ell-1}| \geqslant |C_\ell| + \lfloor \log s \rfloor$, whence $|C_0| \geqslant t \lfloor \log s \rfloor$.

7.2. **Proof of the lower bound for simple treewidth k.**

The proof of the lower bound for graphs of simple treewidth k is based on the same four steps as the proof of the lower bound for outerplanar graphs.

1. Constructing the family of graphs.

The underlying graph for the lower bound is $G_k(w,d)$.

We give a recursive definition: $G_2(w,d) = F(w,d)$ where $F(w,d)$ is the tree of fans, the vertices of level d of this tree of fans constitute the *boundary* L_2 of $G_2(w,d)$. Having defined $G_{j-1}(w,d)$ and its boundary L_{j-1} we aim at defining $G_j(w,d)$. We introduce an additional parameter δ running from 1 to d and graphs $G_j(w,d,\delta)$. We then define $G_j(w,d) = G_j(w,d,d)$.

The graph $G_j(w,d,1)$ is obtained from $G_{j-1}(w,d)$ by making the root v an universal vertex, i.e., for every vertex u of the graph we ensure that the edge (u,v) belongs to the graph. The boundary of $G_j(w,d,1)$ is the boundary L_{j-1} of $G_{j-1}(w,d)$. From $G_j(w,d,\delta)$ we obtain $G_j(w,d,\delta+1)$ by taking a copy $G(u)$ of $G_j(w,d,1)$ for each u in the boundary and identifying the root of $G(u)$ with u. The boundary of $G_j(w,d,\delta+1)$ is the union of the boundaries of all the new subgraphs $G(u)$.

The following properties of $G_k(w,d)$ will be useful.

(1) $G_k(w,d)$ contains a spanning subgraph isomorphic to $F(w,d^{k-1})$. Indeed, if we modify the definition slightly by deleting the instruction *make the root an universal vertex*, then througout the construction we only glue new trees of fans to the boundary vertices of an already constructed tree of fans. The boundary vertices are always the vertices of the largest level. This shows that we obtain a tree of fans. For the parameters use induction and the fact that the depth of the tree of fans underlying $G_j(w,d)$ is d times the depth of the tree of fans underlying $G_{j-1}(w,d)$.

(2) The simple treewidth of $G_j(w,d)$ is at most one more than the simple treewidth of $G_{j-1}(w,d)$, whence by induction $\mathrm{stw}(G_k(w,d)) \leqslant k$.

First note that the following argument implies $\mathrm{stw}(G_j(w,d,1)) \leqslant \mathrm{stw}(G_{j-1}(w,d))+1$. If $\mathrm{stw}(G) = t$ and G^+ is obtained from G by making some vertex v of G an universal vertex, then $\mathrm{stw}(G-v) \leqslant t$ and $\mathrm{stw}(G^+) = \mathrm{stw}(G-v)+1$.

Since every 2-connected component of $G_j(w,d)$ is a copy of $G_j(w,d,1)$ we also have $\mathrm{stw}(G_j(w,d)) \leqslant \mathrm{stw}(G_j(w,d,1))$.

Let $s = \lfloor \frac{p}{k} \rfloor$, and $t = \frac{k-1}{k}\frac{p}{2}$ and $h = \lfloor \frac{t}{k-1} \rfloor + 1 = \lfloor \frac{p}{2k} \rfloor + 1$. We also define $f_k(p) = h^{k-1}\lfloor \log s \rfloor$.

Let H be a $G_k(sf_k(p), h)$ and ϕ be a p-centered coloring of H. We claim that ϕ is using at least $f_k(p)$ colors.

2. A clean subgraph of H.

In a first step we define a *clean* subgraph H^* of H. Let F^* be obtained by the cleaning procedure of the previous section applied to the tree of fans $F(sf_k(p), h^{k-1})$ contained in H. Note that F^* is a tree of fans $F(s, h^{k-1})$. Define H^* as the graph induced in H by the clean vertices, i.e., by $V(F^*)$.

3. The spine for the master caterpillar of H^*.

We adopt a notion of level in H from the tree of fans $F(sf_k(p), h^{k-1})$ contained in H. Let $\text{lev}(u)$ denote the level of vertex u, the level of the root is 0 and the level of vertices from the boundary of H is h^{k-1}. If vw is an edge of H and $\text{lev}(v) < \text{lev}(w)$ we say that w is a *child* of v.

With this child relation the definition of a caterpillar and an H^*-based caterpillar are exactly as in the previous subsection.

The next step is to define the spine of the master caterpillar. This path $S = v_0, v_1, \ldots, v_{h^{k-1}}$ starting from the root v_0 of H^* is computed within the clean spanning tree of fans F^* contained in H^*. The procedure is exactly as before, we only recall the main steps.

Suppose $v_{\ell-1}$ has been identified. The next vertex v_ℓ of S is determined by iterated halving of the path of clean children of $v_{\ell-1}$ in F^*. This produces $\lfloor \log s \rfloor$ sets of colors whose union is denoted B_ℓ.

With the same definitions as before we again have $(d^{(\ell)} - 1) + (d^{(r)} - 1) > p - |Q_j|$.

Based on the inequality $s + 2t \leqslant p$ we get[1]:

> Every H^*-based caterpillar of depth at most t rooted at v_ℓ misses at least $\lfloor \log s \rfloor$ colors from B_ℓ.

4. Color collecting subcaterpillars of the master.

Having defined the vertices of the path S we define a caterpillar K_i for each vertex v_i of S as follows. The spine $S_i = w_0, w_1, \ldots, w_{d_i}$ of K_i is determined on the basis of S. For the first vertex we take $w_0 = v_i$. When $w_0, \ldots, w_j$ have been determined and $w_j \neq v_{h^{k-1}}$, then let w_{j+1} be the last child of w_j in S, i.e, the last vertex w on S such that $w_j w$ is an edge of H^*. With a vertex w_j of the spine S_i the caterpillar K_i also includes all the children of w_j in H^* as hairs.

Claim. $V(K_{i+1}) \subseteq V(K_i)$.

[1]The difference to the corresponding statement in the previous subsection comes from the fact that there the definition of t had a $+1$ which is now shifted to the definition of h.

Proof. From the definition $\operatorname{lev}(v_i) = \operatorname{lev}(v_{i+1}) - 1$. First suppose that $\operatorname{lev}(v_i)$ is not divisible by h. Then in the construction of H the vertex v_i was never element of a boundary, i.e., it wasn't used as an universal vertex of a subgraph. Therefore, all the children of v_i in H and in particular v_{i+1} belong to level $\operatorname{lev}(v_i) + 1$. The spine S_i thus starts with v_i, v_{i+1} and by construction $S_i = v_i S_{i+1}$. This implies the claim in this case.

Now suppose that $\ell \geqslant 1$ is maximal with the property that h^ℓ divides $\operatorname{lev}(v_i)$. In this case v_i was made the universal vertex of some copy $G(v_i)$ of $G_{\ell+1}(sf_k(p), h, 1)$ used in the construction. If the spine S_i thus starts with v_i, w, then $\operatorname{lev}(w) = \operatorname{lev}(v_i) + h^\ell$, i.e., $w = v_{i+h^\ell}$. Moreover, all the elements v_j of the spine S which are between v_i and w and all their children also belong to $G(v_i)$ and hence as children of v_i also belong to K_i. The previous considerations also imply that w belongs to the spine of S_{i+1}. This proves the claim in this case. □

Let C_i be the set of colors of K_i. From the previous claim we get the nesting of the colorsets $C_0 \supseteq C_1 \supseteq \ldots \supseteq C_{h^{k-1}}$.

From the construction we know that for each vertex v_i in S the caterpillar K_i is an H^*-based caterpillar.

Claim. *For $i > 0$, the depth d_i of K_i is smaller than $(k-1)h$.*

Proof. Let $S_i = w_0, w_1, \ldots w_{d_i}$. Clearly $w_{d_i} = v_{h^{k-1}}$. Let j be the least index such that $\operatorname{lev}(w_j)$ is divisible by h^{k-2}. Then $\operatorname{lev}(w_\ell)$ is also divisible by h^{k-2} for all $j < \ell \leqslant d_i$. Therefore, there are at most h vertices w_ℓ with $\operatorname{lev}(w_\ell)$ divisible by h^{k-2} in S_i.

From the minimality of j it follows that $w_0, w_1, \ldots w_j$ are all taken from the same copy of $G_{k-1}(sf_k(p), h)$ used in the construction of H. By induction, the base case is a tree of fans, we can asume that $j < (k-2)h$. □

Since $(k-1)h \leqslant t+1$ we conclude that the depth of K_i is at most t for all $i > 0$.

The set B_i is a set of colors used by children of the predecessor v_{i-1} of v_i, therefore, $B_i \subseteq C_{i-1}$. Caterpillar K_i is H^*-based, rooted at v_i, and of depth at most t, therefore, K_i misses at least $\lfloor \log s \rfloor$ colors from B_i. Together this shows that $|C_{i-1}| \geqslant |C_i| + \lfloor \log s \rfloor$, whence $|C_0| \geqslant h^{k-1} \lfloor \log s \rfloor$. This completes the proof that ϕ uses at least $f_k(p)$ colors on H.

8. Further directions

We would like to finish the paper with our two favourite problems.

Conjecture 18. *Planar graphs admit p-centered colorings with $\mathcal{O}(p^2 \log p)$ colors.*

Question 19. *Do outerplanar graphs admit p-linear colorings with $\mathcal{O}(p)$ colors?*

The best we know for Question 19 is $\mathcal{O}(p \log p)$ by Theorem 6.(i).

Another line of thought is that our upper bound for bounded degree graphs implies the existence of p-centered colorings with $\mathcal{O}(p)$ colors of planar grids. Interestingly, the authors have not been able to provide a direct construction for such a coloring.

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