



# DICHROMATIC NUMBER AND FRACTIONAL CHROMATIC NUMBER

BOJAN MOHAR<sup>1</sup> and HEHUI WU<sup>2,3</sup>

<sup>1</sup> Department of Mathematics, Simon Fraser University, Burnaby, BC V5A 1S6, Canada;  
email: mohar@sfu.ca

<sup>2</sup> 2202 East Guanghua Tower, Fudan University, 220 Handan Road, Shanghai, China 200433;  
email: hhwu@fudan.edu.cn

<sup>3</sup> Department of Mathematics, University of Mississippi, Oxford, MS 38677, USA

Received 5 February 2016; accepted 12 October 2016

## Abstract

The dichromatic number of a graph  $G$  is the maximum integer  $k$  such that there exists an orientation of the edges of  $G$  such that for every partition of the vertices into fewer than  $k$  parts, at least one of the parts must contain a directed cycle under this orientation. In 1979, Erdős and Neumann-Lara conjectured that if the dichromatic number of a graph is bounded, so is its chromatic number. We make the first significant progress on this conjecture by proving a fractional version of the conjecture. While our result uses a stronger assumption about the fractional chromatic number, it also gives a much stronger conclusion: if the fractional chromatic number of a graph is at least  $t$ , then the fractional version of the dichromatic number of the graph is at least  $\frac{1}{4}t/\log_2(2et^2)$ . This bound is best possible up to a small constant factor. Several related results of independent interest are given.

2010 Mathematics Subject Classification: 05C15

## 1. Introduction

For an undirected graph  $G$ , the *chromatic number*  $\chi(G)$  is the minimum number of independent sets whose union is  $V(G)$ . For a directed graph (digraph)  $D$ , the analogue of independent sets are acyclic vertex sets, where we call a vertex set *acyclic* if it does not contain a directed cycle. Then, the *chromatic number*  $\chi(D)$  of  $D$  is the minimum number of acyclic vertex sets whose union is  $V(G)$  (see [2, 10]). The *dichromatic number* of an undirected graph  $G$ , denoted by  $\bar{\chi}(G)$ , is the maximum chromatic number over all its orientations [4, 10].

© The Author(s) 2016. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted re-use, distribution, and reproduction in any medium, provided the original work is properly cited.

In the late 1970s, Erdős and Neumann-Lara [4] posed the following conjecture:

CONJECTURE 1.1 (Erdős and Neumann-Lara [4]). *For every integer  $k$ , there exists an integer  $f_k$ , such that if  $\chi(G) \geq f_k$ , then  $\bar{\chi}(G) \geq k$ .*

Clearly,  $f_1 = 1$  and  $f_2 = 3$ . But it is still an open question whether  $f_3$  exists.

Let  $\mathcal{I}(G)$  be the family of all independent vertex sets of  $G$ . For  $v \in V(G)$ , let  $\mathcal{I}(G, v)$  be the subfamily of all those independent sets that contain  $v$ . For each independent set  $I$ , consider a nonnegative real variable  $x_I$ . Let  $\chi_f(G)$  be the minimum value of  $\sum_{I \in \mathcal{I}(G)} x_I$  subject to

$$\begin{aligned} \sum_{I \in \mathcal{I}(G, v)} x_I &\geq 1 && \text{for each vertex } v, \\ x_I &\geq 0 && \text{for every } I \in \mathcal{I}(G). \end{aligned} \tag{1}$$

It is easy to see that in any optimal solution of this linear program, we have  $x_I \leq 1$  for every  $I \in \mathcal{I}(G)$ . If we request that the values  $x_I$  are integers 0 or 1, we obtain an integer program, whose optimal solution is the chromatic number  $\chi(G)$ . By this correspondence, we call  $\chi_f(G)$  the *fractional chromatic number* of the graph  $G$ .

The dual of this linear program computes the *fractional clique number*  $\omega_f(G)$ , a relaxation to the rationals of the integer concept of the clique number. That is, a weighting  $w : V \rightarrow \mathbb{R}^+$  of the vertex set  $V = V(G)$  into the set of nonnegative real numbers such that the total weight assigned to any independent set is at most 1, and the value  $w(V) = \sum_{v \in V} w(v)$  is maximum possible under these conditions.

We can also define the relaxation to the rationals of the integer concept of the independence number. For every weighting  $w : V \rightarrow \mathbb{R}^+$  of the vertices of  $G$ , for which the total weight is  $w(V) = \sum_{v \in V} w(v) = |V|$ , we consider the maximum weight  $w(I)$  over all independent sets  $I \in \mathcal{I}(G)$ . The minimum of this quantity, taken over all weightings  $w$ , is denoted by  $\alpha_f(G)$  and called the *fractional independence number* of  $G$ .

By linear programming duality, we have the following lemma (see, for example [6]).

LEMMA 1.2.  $\chi_f(G) = \omega_f(G)$ .

Analogously, we define the *fractional chromatic number*  $\chi_f(D)$  of a digraph  $D$  by using acyclic vertex sets playing the role of  $\mathcal{I}(G)$ . The maximum of  $\chi_f(D)$  taken over all orientations  $D$  of an undirected graph  $G$  is the *fractional dichromatic number*  $\bar{\chi}_f(G)$  of  $G$ .

The main result of this paper is the following fractional version of the Erdős and Neumann-Lara conjecture.

THEOREM 1.3. *If  $\chi_f(G) \geq t$ , then*

$$\vec{\chi}_f(G) \geq \frac{t}{4 \log(2et^2)}.$$

The logarithms used in Theorem 1.3 and throughout the rest of the paper are always taken with respect to base 2. Note that, when compared with Conjecture 1.1, this result uses the stronger assumption that the fractional chromatic number is large, but it also gives a stronger conclusion.

The proof of Theorem 1.3 occupies the whole of Section 2.

Erdős and Neumann-Lara proved [4] that  $c_1(n/\log n) \leq \vec{\chi}(K_n) \leq c_2(n/\log n)$  for some constants  $0 < c_1 < c_2$ . For large  $n$ ,  $c_1 \sim \frac{1}{2}$  and  $c_2 \sim 1$ . This implies that the bound of Theorem 1.3 is best possible up to the multiplicative factor of 8.

Theorem 1.3 implies the validity of Conjecture 1.1 for graphs whose chromatic number is bounded in terms of their fractional chromatic number. It is therefore natural to ask if some graphs with bounded fractional chromatic number might provide counterexamples to the conjecture of Erdős and Neumann-Lara. In Section 3, we treat such examples, in particular Kneser graphs with bounded fractional chromatic number. Let us observe that every graph with fractional chromatic number  $q$  has a homomorphism to a Kneser graph whose fractional chromatic number is also  $q$ . In view of our main Theorem 1.3, Kneser graphs with bounded fractional chromatic number are really the critical examples, for which the Erdős and Neumann-Lara conjecture should be tested first. We show that Kneser graphs have large dichromatic number as long as their chromatic number is large enough; see Theorem 3.5. The lower bound given by the theorem is surprisingly large and thus gives a firm support toward Conjecture 1.1.

**Previous work.** Dichromatic number was formally introduced by Neumann-Lara in 1982 [10], but its first appearance can be found in a paper by Erdős [4] in 1979, where he discusses several results and conjectures proposed by himself and Neumann-Lara. The notion of digraph coloring (for the chromatic and circular chromatic number) by using acyclic sets as color classes was rediscovered by Mohar [9] and studied in a follow-up paper by Bokal *et al.* [2]. A few years later, it became clear that this notion of the chromatic number, when restricted to tournaments, is closely related to the Erdős–Hajnal conjecture [3]; see, for example [1].

As pointed out by Erdős in [4], ‘It is surprisingly difficult to determine  $\vec{\chi}(G)$  even for the simplest graphs,’ it is thus not surprising that no real progress concerning the problems of him and Neumann-Lara in [4] has been made until today.

## 2. Principal and sparse vertex sets

This section is devoted to the proof of Theorem 1.3. We let  $V = V(G)$  and  $n = |V|$ . We can assume that  $\chi_f(G) = t$  and that  $t > 4 \log(2et^2)$ , which implies that  $t > 56$ . Since  $\omega_f(G) = \chi_f(G)$ , there exists a weight function  $w : V \rightarrow \mathbb{R}^+$ , such that  $w(V) = t$ , and for any  $I \in \mathcal{I}(G)$ ,  $w(I) \leq 1$ . Here and in the sequel we write  $w(A) = \sum_{v \in A} w(v)$  for any vertex set  $A \subseteq V$ , and call this value the *weight* of  $A$ .

From now on we fix  $w$  and assume that the vertices of  $V(G)$  are listed as  $v_1, \dots, v_n$  in the nonincreasing order of their weights, that is  $w(v_{i+1}) \leq w(v_i)$  for  $i = 1, \dots, n-1$ . For any subset  $X$  of  $V$ , we also rank the elements in  $X$  according to the ordering of  $V$ , and we denote by  $X_k$  the subset of the first  $k$  elements in  $X$ . In particular,  $V_k = \{v_1, \dots, v_k\}$ . We extend this notion to all positive real numbers by setting  $X_s := X_{\lfloor s \rfloor}$ .

For  $s \in \mathbb{R}^+$ , a nonempty subset  $X$  of a vertex set  $Y$  is said to be *s-principal* in  $Y$  if  $X \subseteq Y_{\lfloor s|X|}$ . That is, if  $X$  has size  $m$ , then all elements of  $X$  are within the first  $\lfloor sm \rfloor$  vertices in  $Y$ . On the other hand, a subset  $X$  of  $Y$  is *s-sparse* in  $Y$  if  $X$  contains no  $s$ -principal subset in  $Y$ . When the hosting set  $Y$  for  $s$ -principal or  $s$ -sparse is not specified, by default it is  $V$ .

The following property is clear from the definition of sparse sets.

CLAIM 1. *Any subset of an s-sparse set in Y is also s-sparse in Y.*

CLAIM 2. *X is an s-sparse subset in Y if and only if  $|Y_k \cap X| < k/s$  for  $1 \leq k \leq |Y|$ . In particular, if X is s-sparse in Y, then  $|X| < |Y|/s$ .*

*Proof.* It is clear that  $X$  is  $s$ -sparse if and only if for each  $r = 1, \dots, |X|$ ,  $|Y_{sr} \cap X| < r$ . If  $|Y_k \cap X| < k/s$  for every  $k$ , then this holds also for  $k = \lfloor sr \rfloor$ , implying that  $|Y_{sr} \cap X| < \lfloor sr \rfloor / s \leq r$ . Conversely, if  $|Y_{sr} \cap X| < r$ , then  $|Y_{sr} \cap X| \leq r - 1$ . Let  $s(r - 1) < k \leq sr$ . Then  $|Y_k \cap X| \leq |Y_{sr} \cap X| \leq r - 1 < k/s$ .  $\square$

The next claim about the total weight of an  $s$ -sparse set will be essential for us.

CLAIM 3. *If X is an s-sparse subset of Y, then  $w(X) \leq (1/s) w(Y)$ .*

*Proof.* Let  $y_1, \dots, y_r$  be the nonincreasing order of the elements of  $Y$  with  $r = |Y|$ , and let  $x_1, \dots, x_m$  be the ordering of  $X$  with  $m = |X|$ . Since  $X$  is an  $s$ -sparse subset of  $Y$ , we have  $x_i \notin Y_{si}$ . Hence for  $1 \leq i \leq m$ ,  $w(x_i) \leq w(y_j)$  if  $1 \leq j \leq si$ . Moreover, since  $x_i \in Y \setminus Y_{si}$ , we also have  $w(x_i) \leq w(y_j)$  for  $j = \lceil si \rceil$ .

For a real parameter  $z \in (0, |Y|]$ , define  $f(z) = w(y_{\lceil z \rceil})$ . Then  $f(z) \geq w(x_1)$  for  $0 < z \leq s$ ,  $f(z) \geq w(x_2)$  for  $s < z \leq 2s, \dots, f(z) \geq w(x_m)$  for  $(m - 1)s < z \leq ms$ . Therefore,

$$s w(X) = s \sum_{i=1}^m w(x_i) \leq \int_0^{sm} f(z) dz \leq \sum_{j=1}^{\lceil sm \rceil} w(y_j) \leq w(Y),$$

as required. □

The *average degree* of a graph  $G$ , denoted by  $\bar{d}(G)$ , equals  $(1/n) \sum_{v \in V(G)} d(v) = 2e(G)/n$ , where  $e(G)$  denotes the number of edges in  $G$ . When  $A$  is a vertex set of  $G$ , the *average degree* of  $A$ , denoted by  $\bar{d}(A)$ , means the average degree  $\bar{d}(G[A])$  of the corresponding induced subgraph of  $G$ .

The following lemmas will be instrumental in proving Theorem 1.3.

LEMMA 2.1. *Suppose that  $\chi_f(G) \geq t$ . Let  $d = 2 \log(et^2)$  and suppose that  $t \geq 2(d + 1)$ . Then there exists an orientation of the edges of  $G$  such that every  $t$ -principal set of  $G$  with average degree at least  $d$  contains a directed cycle.*

LEMMA 2.2. *Let  $d = 2 \log(et^2)$  and suppose that  $t \geq 2(d + 1)$ . If  $A$  is a vertex set with weight more than  $2d + 4$  such that each independent subset of  $A$  has weight at most 1, then  $A$  contains a  $t$ -principal subset whose average degree is at least  $d$ .*

Our main result, Theorem 1.3, easily follows from Lemmas 2.1 and 2.2.

*Proof of Theorem 1.3.* Let  $d$  be as in Lemmas 2.1 and 2.2. We may assume that  $t > 4 \log(2et^2) = 2d + 4$ . By our assumption on the weight function of  $G$  being the one that gives the fractional independence number of  $G$ , there is no independent set with weight more than 1. Hence by Lemma 2.2, every vertex set with weight more than  $2d + 4$  contains a  $t$ -principal set with average degree at least  $d$ . By Lemma 2.1, there exists an orientation  $D$ , such that every  $t$ -principal set with average degree at least  $d$  is cyclic. Hence every vertex set with weight more than  $2d + 4$  is cyclic. Since the total weight for  $V(G)$  is  $t$ , and every acyclic set in  $D$  has weight at most  $2d + 4$ , we must have  $\chi_f(D) \geq t/(2d + 4)$ . Therefore  $\bar{\chi}_f(G) \geq t/(2d + 4)$ , which is the same as the bound in the theorem. □

First we give a proof of Lemma 2.2.

*Proof of Lemma 2.2.* Suppose  $A$  is a vertex set whose weight is more than  $2d + 4$  and suppose that  $A$  contains no  $t$ -principal subsets with average degree at least  $d$ . For a vertex  $v \in A$ , let  $d^>(v)$  be the number of neighbors of  $v$  in  $G[A]$  that

appear before  $v$  in the ordering  $v_1, \dots, v_n$ . We have  $e(A_i) = \sum_{v \in A_i} d^>(v)$  for each  $i = 1, \dots, |A|$ . Let  $L = \{v \in A : d^>(v) \geq d\}$ , and assume  $L = \{v_{i_1}, \dots, v_{i_l}\}$ , where  $l = |L|$ . Then  $e(V_{i_j} \cap A) \geq dj$  for  $1 \leq j \leq l$ . In particular, if  $|V_{i_j} \cap A| \leq 2j$ , then the average degree  $\bar{d}(V_{i_j} \cap A)$  will be at least  $d$ . Since  $A$  does not contain any  $t$ -principal set with average degree at least  $d$ , the set  $V_{i_j} \cap A$  would not be  $t$ -principal in this case. Thus, we have one of the following:

- (1)  $|V_{i_j} \cap A| > 2j$ ; or
- (2)  $|V_{i_j}| > t|V_{i_j} \cap A|$ .

Let  $L_1 = \{v_{i_j} \in L : |V_{i_j} \cap A| > 2j\}$ , and let  $L_2 = \{v_{i_j} \in L : |V_{i_j}| > t|V_{i_j} \cap A|\}$ . Then  $L = L_1 \cup L_2$ . If  $v_{i_j} \in L_1$ , then it is not among the first  $2j$  elements of  $A$ . As the  $j$ th element in  $L_1$  does not appear before  $v_{i_j}$ , the  $j$ th element of  $L_1$  is not among the first  $2j$  elements of  $A$ . Therefore,  $L_1$  is a 2-sparse subset of  $A$ . By Claim 3,  $w(L_1) \leq \frac{1}{2}w(A)$ .

For each  $v_{i_j} \in L_2$ , we have  $|V_{i_j} \cap A| < (1/t)|V_{i_j}|$ . As  $|V_{i_j} \cap L_2| \leq |V_{i_j} \cap A| < (1/t)|V_{i_j}|$ , we see by Claim 2 that  $L_2$  is a  $t$ -sparse subset of  $V$ . By Claim 3,  $w(L_2) \leq (1/t)w(V) = 1$ .

Let  $S = A \setminus L = \{v \in A : d^>(v) < d\}$ . Then we have  $w(S) \geq w(A) - w(L_1) - w(L_2) \geq w(A)/2 - 1$ . Also,  $G[S]$  is a  $\lfloor d \rfloor$ -degenerate graph, hence  $S$  is  $\lfloor d + 1 \rfloor$ -colorable. Therefore, there is at least one independent set with weight at least

$$\frac{w(S)}{\lfloor d + 1 \rfloor} \geq \frac{w(A)/2 - 1}{d + 1} > 1.$$

In the last inequality we used the fact that  $w(A) > 2d + 4$ . This completes the proof. □

In the rest of this section, we are going to use probabilistic method to prove Lemma 2.1, that there exists an orientation such that no  $t$ -principal set of  $G$  with average degree at least  $d$  is acyclic. We will need the following lemma about acyclic orientations.

LEMMA 2.3. *The number of acyclic orientations of  $G$  is at most*

$$\prod_{v \in V(G)} (d_G(v) + 1).$$

*Proof.* Let  $v_1, \dots, v_n$  be the vertices of  $G$ . For any orientation  $D$  of  $G$ , we consider its out-degree sequence  $(d_D^+(v_1), d_D^+(v_2), \dots, d_D^+(v_n))$ . Since  $d_D^+(v) \leq d_G(v)$ , there are at most  $\prod_{v \in V(G)} (d_G(v) + 1)$  different out-degree sequences. On the other hand, for any out-degree sequence corresponding to

some acyclic orientation, we can determine the orientation inductively, starting from that there is a vertex with out-degree 0, and all the edges incident to it are oriented toward it. So given different acyclic orientations, it is easy to see that their out-degree sequence cannot be the same. Therefore, there are at most  $\prod_{v \in V(G)} (d_G(v) + 1)$  acyclic orientations.  $\square$

We pick an orientation of the graph  $G$  among all the  $2^{e(G)}$  orientations uniformly at random. Then for each edge, the probability for each direction we pick is  $1/2$ , and it is mutually independent from the orientation of other edges. We have the following corollary of Lemma 2.3.

**COROLLARY 2.4.** *The probability that a random orientation of  $G$  is acyclic is less than  $2^{-e(G)(1-2\alpha)}$ , where  $\alpha = \log(\bar{d}(G) + 1)/\bar{d}(G)$ .*

*Proof.* Let  $f(x) = \ln(x + 1)$ . Since  $f'(x) = 1/(x + 1)$  and  $f''(x) = -1/(x + 1)^2 < 0$ , we see that  $f(x)$  is concave for  $x > -1$ . Therefore,  $\sum_{v \in V(G)} \ln(d_G(v) + 1) \leq n \ln(\bar{d}(G) + 1)$ . That is

$$\prod_{v \in V(G)} (d_G(v) + 1) \leq (\bar{d}(G) + 1)^n = 2^{n\bar{d}(G)} = 2^{2\alpha e(G)}.$$

Since there are  $2^{e(G)}$  different orientations of  $G$ , among which at most  $(\bar{d}(G) + 1)^n$  are acyclic, the probability that a random orientation of  $G$  is acyclic is at most  $2^{2\alpha e(G)} 2^{-e(G)} = 2^{-e(G)(1-2\alpha)}$ .  $\square$

We are going to use the following fact.

**FACT 1.**  $\binom{[tk]}{k} < (et)^k$ .

*Proof.* By Stirling's formula,  $k! \geq \sqrt{2\pi k} (k/e)^k > (k/e)^k$ . Therefore, we have

$$\binom{[tk]}{k} \leq \frac{tk(tk - 1) \cdots (tk - k + 1)}{k!} < \frac{(tk)^k}{\left(\frac{k}{e}\right)^k} = (et)^k. \quad \square$$

Now we are ready to prove Lemma 2.1.

*Proof of Lemma 2.1.* Let  $A$  be a  $t$ -principal vertex set of cardinality  $k$  and with average degree at least  $d$ . By Corollary 2.4 applied to the graph  $G[A]$ , the probability that a random orientation of  $G$  is acyclic on  $A$  is small:

$$\Pr(A \text{ is acyclic}) \leq 2^{-e(G[A])(1-2\alpha)} = 2^{-k\bar{d}(A)(1-2\alpha)/2}, \quad (2)$$

where  $\alpha = \log(\bar{d}(A) + 1)/\bar{d}(A)$ . Since  $\bar{d}(A) \geq d$ , we also have the following inequality:

$$\frac{1}{2} k \bar{d}(A) (1 - 2\alpha) \geq \frac{1}{2} kd - k \log(d + 1). \tag{3}$$

As a  $t$ -principal set,  $A$  is a  $k$ -set contained in  $V_{kt}$ . Thus, there are at most  $\binom{\lfloor tk \rfloor}{k}$   $t$ -principal sets with  $k$  elements. By (2) and (3), the probability that some  $t$ -principal  $k$ -set with average degree at least  $d$  is acyclic is at most

$$\begin{aligned} 2^{-kd/2+k \log(d+1)} \binom{\lfloor tk \rfloor}{k} &< 2^{-kd/2+k \log(d+1)} (et)^k \\ &= (2^{-d/2+\log(d+1)} et)^k \\ &= \left(\frac{d+1}{t}\right)^k \leq 2^{-k}. \end{aligned}$$

Here we used the facts that  $d = 2 \log(et^2)$  and  $t \geq 2(d + 1)$ . Therefore, the probability that some  $t$ -principal set with average degree at least  $d$  is acyclic is at most

$$\sum_{k=1}^n \Pr(\exists \text{ an acyclic } t\text{-principal } k\text{-set } A \text{ with } \bar{d}(A) \geq d) < \sum_{k=1}^n 2^{-k} < 1.$$

This implies that there exists an orientation such that no  $t$ -principal sets of  $G$  with average degree at least  $d$  is acyclic. □

In the last summation in the above proof, we could also use the fact that  $A$  needs to have average degree at least  $d$ , hence  $k = |A| \geq d + 1$ . Then the probability is less than  $2^{-d}(1/2 + 1/4 + \dots) = 2^{-d} = e^{-2}t^{-4}$ . This shows that most orientations give the conclusion of the lemma.

### 3. Blow-ups and Kneser graphs

Theorem 1.3 raises the question of whether the real condition in Conjecture 1.1 should be about the fractional chromatic number instead of the usual chromatic number. It also shows that any possible infinite family of graphs providing a counterexample to the conjecture would have bounded fractional chromatic number (and arbitrarily large chromatic number). Any graph  $G$  with  $\chi_f(G) \leq q$  has a homomorphism into a Kneser graph  $K(n, k)$  with  $n/k \leq q$  (see, for example, [6]), and thus it is natural to check the validity of the Erdős and Neumann-Lara conjecture for Kneser graphs. We do this in this section. The results may be of independent interest because of the tools involved in the proofs.



Given a graph  $H$ , the *blow-up* of  $H$  with *power*  $m$ , denoted by  $H^{(m)}$ , is the graph obtained from  $H$  by replacing each vertex by an independent set of size  $m$  (called the *blow-up* of the vertex), and for each edge  $xy$  in  $H$ , the two blow-ups of  $x$  and  $y$  form a complete bipartite graph  $K_{m,m}$ . The subgraph of  $H^{(m)}$  replacing an edge of  $H$  is isomorphic to  $K_{m,m}$  and will be referred to as the *blow-up* of that edge.

LEMMA 3.1. *Let  $k$  be a positive integer and let  $H$  be a graph with  $\chi(H) > k$ . If there is an orientation  $D$  of  $H^{(m)}$ , such that in the blow-up of any edge, no subgraph isomorphic to  $K_{\lceil m/k \rceil, \lceil m/k \rceil}$  is acyclic, then  $\chi(D) > k$ .*

*Proof.* Suppose that  $D$  is an orientation of  $H^{(m)}$  with the stated property, and that  $D$  is  $k$ -colorable. We are going to define a  $k$ -coloring of  $H$  as follows. For each vertex  $x$  of  $H$ , let  $X$  be the blow-up of  $x$ . There is a color  $c = c(x)$  that is used for at least  $\lceil m/k \rceil$  vertices of  $X$  in the  $k$ -coloring of  $D$ . Consider an edge  $xy \in E(H)$ . If  $c(x) = c(y)$ , let  $A$  be the set of vertices in the blow-up of  $x$  having color  $c(x)$ , and let  $B$  be the set of vertices in the blow-up of  $y$  colored  $c(y)$ . Then  $|A|, |B| \geq \lceil m/k \rceil$ . By assumption,  $A \cup B$  is not acyclic in  $D$ . This contradicts the fact that all vertices in  $A \cup B$  receive the same color. We conclude that  $c(x) \neq c(y)$ . This shows that the function  $c : V(H) \rightarrow [k]$  is a  $k$ -coloring of  $H$ . This contradiction completes the proof. □

LEMMA 3.2. *If  $2 + 2 \log m \leq \lceil m/k \rceil$ , and  $\chi(H) > k$ , then  $\bar{\chi}(H^{(m)}) > k$ .*

*Proof.* Let  $r = \lceil m/k \rceil$ . By Lemma 3.1, it suffices to prove that for  $m \geq 4k \log(ke)$ , there is an orientation of  $H^{(m)}$ , such that no copy of  $K_{r,r}$  in the blow-up of any edge is acyclic. It suffices to show that the blow-up of each edge of  $H$  has such an orientation, and we will prove that a random orientation will have this property with positive probability.

Let us now consider a random orientation of the blow-up  $K \cong K_{m,m}$  of a fixed edge of  $H$ . As there are  $(2r)!$  permutations of the vertices of  $K_{r,r}$ , and each acyclic orientation corresponds to at least one such permutation, the probability that a copy of  $K_{r,r}$  in  $K$  is acyclic is at most  $(2r)!/2^{r^2}$ . As there are  $\binom{m}{r}^2$  copies of  $K_{r,r}$  in  $K$ , the probability that some copy of  $K_{r,r}$  is acyclic is at most

$$(2r)! 2^{-r^2} \binom{m}{r}^2 = \binom{2r}{r} \left( \frac{m!}{(m-r)!} \right)^2 2^{-r^2} \leq \binom{2r}{r} m^{2r} 2^{-r^2} < 2^{-r^2+2r} m^{2r}.$$

The value on the right-hand side of this inequality is at most 1 if  $2 + 2 \log m \leq r$ . Thus, there is at least one orientation for  $K$ , such that no copy of  $K_{r,r}$  in  $K$  is acyclic. This completes the proof. □

A *Kneser graph*, denoted by  $KG(n, k)$ , is a graph whose vertices correspond to the  $k$ -element subsets of a set with  $n$  elements, and two vertices are adjacent if their corresponding sets are disjoint. In 1978, Lovász [8] proved that the chromatic number of  $KG(n, k)$  is  $n - 2k + 2$ , which was conjectured by Kneser in 1955. It is also known that the fractional chromatic number of  $KG(n, k)$  is  $n/k$ . Thus, Kneser graphs provide examples of graphs, whose fractional chromatic number can be bounded, while their chromatic number would be arbitrarily large.

**THEOREM 3.3.** *Let  $n, k, t$ , and  $x$  be nonnegative integers such that  $0 < k < n$  and  $x < kt$ . The Kneser graph  $KG(nt, kt - x)$  contains the blow-up of  $KG(n, k)$  with power  $\binom{k(t-1)}{x}$  as a subgraph. Furthermore, when  $x < t$ , it contains the blow-up of  $KG(n, k)$  with power  $\binom{kt}{x}$ , and when  $x = t$ , it contains the blow-up of  $KG(n, k)$  with power  $\binom{kt}{x} - k$ .*

*Proof.* Let  $G = KG(nt, kt - x)$  and  $H = KG(n, k)$ . We can consider the vertices of  $G$  as the  $(kt - x)$ -subsets of  $[n] \times [t]$ , and the vertices of  $H$  as the  $k$ -subsets of  $[n]$ . Given a vertex  $A \in V(G)$ , let  $f(A) = \{a \in [n] : (a, b) \in A \text{ for some } b \in [t]\}$ ; that is,  $f$  is the projection from  $[n] \times [t]$  to  $[n]$ .

Notice that for two vertices  $A$  and  $B$  of  $G$ , if  $f(A) \cap f(B) = \emptyset$ , then  $A \cap B = \emptyset$ . In particular, if  $|f(A)| = |f(B)| = k$  and  $f(A)$  and  $f(B)$  are adjacent in  $H$ , then  $A$  and  $B$  are adjacent in  $G$ . For every vertex  $X \in V(H)$ ,  $X$  is a  $k$ -set. If  $f(A) = X$  for some vertex in  $G$ , then  $A$  is a  $(kt - x)$ -subset of  $X \times [t]$ .

If  $x < t$ , then for every  $(kt - x)$ -subset  $A$  of  $X \times [t]$ , we have  $f(A) = X$ . So there are  $\binom{kt}{x}$  vertices in  $G$  that map to  $X$  by  $f$ . This shows that  $G$  contains a blow-up of  $H$  with power  $\binom{kt}{x}$ .

If  $x = t$ , among the  $(kt - x)$ -subsets of  $X \times [t]$ , only  $k$  of them have  $f(A) \neq X$ , and each misses one element of  $X$ . Therefore,  $G$  contains a blow-up of  $H$  with power  $\binom{kt}{x} - k$ .

In general, we can consider those  $(kt - x)$ -subsets of  $X \times [t]$ , which contain all the elements in  $X \times \{1\}$ . There are  $\binom{k(t-1)}{x}$  such subsets, and each of them is mapped to  $X$  by  $f$ . Therefore,  $G$  contains a blow-up of  $H$  with power  $\binom{k(t-1)}{x}$ .  $\square$

Erdős and Neumann-Lara (see [4]) proved that

$$\bar{\chi}(K_n) \geq \frac{n}{2 \log(n)}. \quad (4)$$

This result can be extended to blow-ups of complete graphs as follows.

**THEOREM 3.4.**  $\vec{\chi}(K_n^{(k)}) > \min\{nk/4 \log(nk), n/2\}$ .

*Proof.* Let  $G = K_n^{(k)}$ , and let  $t = \max\{\lceil 4 \log(nk) \rceil, 2k\}$ . It suffices to show that there is an orientation of  $G$  such that every vertex set of size  $t$  has a directed cycle.

Let  $A \subseteq V(G)$  be a set of size  $t$ . As  $G$  is a balanced complete  $n$ -partite graph and each part has size  $k$ , each vertex in  $A$  has at least  $t - k$  neighbors in  $A$ . Therefore,  $G[A]$  has at least  $t(t - k)/2$  edges. Among all orientations of  $G[A]$ , at most  $t!$  of them are acyclic. Thus,  $\Pr(A \text{ is acyclic}) \leq t! 2^{-t(t-k)/2}$ . There are  $\binom{nk}{t}$  such set. It follows that with a random orientation of  $G$  we have the following:

$$\begin{aligned} \Pr(\exists A : |A| = t \text{ and } A \text{ is acyclic}) &\leq \binom{nk}{t} t! 2^{-t(t-k)/2} \\ &< (nk 2^{-(t-k)/2})^t. \end{aligned} \tag{5}$$

Since  $t = \max\{\lceil 4 \log(nk) \rceil, 2k\}$ , we have  $t - k \geq 2 \log(nk)$ . Therefore,  $nk 2^{-(t-k)/2} \leq 1$ . This implies that the value on the right-hand side of (5) is at most 1. Consequently, there is at least one orientation  $D$  of  $G$  such that any set of size  $t$  has a directed cycle. Therefore, in a proper coloring of  $D$ , each color class has size at most  $t - 1$ . Hence  $\vec{\chi}(K_n^{(k)}) \geq \chi(D) \geq nk/(t - 1) > \min\{(nk/4 \log(nk)), n/2\}$ .  $\square$

**THEOREM 3.5.** *For any positive integers  $n, k$  with  $n \geq 2k$ , we have*

$$\vec{\chi}(KG(n, k)) \geq \left\lfloor \frac{n - 2k + 2}{8 \log(n/k)} \right\rfloor.$$

*Proof.* Let  $G = KG(n, k)$ . Observe that  $\vec{\chi}(G) = 1$  only when the graph is a forest. In our case this happens only when  $n = 2k$ , and the inequality holds in this case. Let  $z = \lfloor \chi(G)/8 \log(\chi_f(G)) \rfloor = \lfloor (n - 2k + 2)/8 \log(n/k) \rfloor$ . We just need to consider the case when  $z \geq 2$ . Hence, we may assume  $\chi(G) = n - 2k + 2 \geq 16$ . For  $k \leq 3$ ,  $G$  contains the complete graph on  $\lfloor n/k \rfloor$  vertices. By using (4), it is easy to see that the theorem holds in this case. Thus, we may assume that  $k \geq 4$ .

Let  $H_1 = KG(\lfloor n/2k \rfloor, 1) = K_{\lfloor n/2k \rfloor}$ . Theorem 3.3 with  $t = 2k$  and  $x = k$  implies that  $KG(2k \lfloor n/2k \rfloor, 2k - k)$  contains the blow-up of  $H_1$  with power  $\binom{2k}{k}$ . Since  $n \geq 2k \lfloor n/2k \rfloor =: n'$ ,  $G$  contains  $KG(n', k)$ , and hence also the blow-up of  $H_1$  with power  $\binom{2k}{k}$ . By Theorem 3.4, we have

$$\vec{\chi}(G) > \min \left\{ \frac{1}{2} \left\lfloor \frac{n}{2k} \right\rfloor, \frac{\binom{2k}{k} \lfloor \frac{n}{2k} \rfloor}{4 \log\left(\binom{2k}{k} \lfloor \frac{n}{2k} \rfloor\right)} \right\}. \tag{6}$$

Suppose first that  $k \leq 2 \log(n/k)$ . If the minimum in (6) is attained with the first term, then we have  $\vec{\chi}(G) > \frac{1}{2} \lfloor n/2k \rfloor \geq (n - 2k + 1)/4k$ . Since  $\vec{\chi}(G)$  is an

integer, we conclude that  $\vec{\chi}(G) \geq (n - 2k + 2)/4k \geq (n - 2k + 2)/8 \log(n/k) \geq z$  as claimed. If the minimum is attained with the second term, then noting that  $\binom{2k}{k} \geq 4k$ , we see that

$$\begin{aligned} \vec{\chi}(G) &\geq \frac{4k \lfloor \frac{n}{2k} \rfloor}{4 \log(4k \lfloor \frac{n}{2k} \rfloor)} \geq \frac{n}{4 \log(2n)} = \frac{n}{4 \log(n/k) + 4 \log(k/2)} \\ &\geq \frac{n}{8 \log(n/k)} \geq z. \end{aligned}$$

Suppose now that  $k > 2 \log(n/k)$ , that is  $n < k2^{k/2}$ . To simplify notation, we also set  $r := \lfloor k/2 \rfloor$ . We will distinguish two cases.

Suppose first that  $n \geq 4k$ . The result is clear if  $z \leq 2$ . Otherwise,  $z \geq 3$  and we have  $n - 2k + 2 \geq 24 \log(n/k) \geq 48$ . This contradicts the assumption that  $n < k2^{k/2}$  if  $k \leq 6$ . If  $k = 7$ , then  $n \geq 60$  and hence  $n/k \geq 8$ . In particular,  $G$  contains  $K_7$  as a subgraph. Since  $\vec{\chi}(K_7) = 3$  [11], we may assume that  $z \geq 4$ ; otherwise the theorem is proved. However, if  $z \geq 4$  and  $n/k \geq 8$ , then  $n - 2k + 2 \geq 32 \log(n/k) \geq 96$ . Again, this contradicts the assumption that  $n < k2^{k/2}$ . We need to do one more special case, when  $k = 9$ . In this case, we conclude as above, first that  $n \geq 64$ , next that  $G$  contains  $K_7$  and that we may assume that  $z \geq 4$ . This implies that  $n \geq 16 + 32 \log(n/9)$ . The smallest  $n$  for which this inequality is satisfied is 128. Therefore,  $G$  contains the complete graph  $K_{14}$  as a subgraph. It is known [11] that  $\vec{\chi}(K_{11}) = 4$ , thus we may assume henceforth that  $z \geq 5$ . In this case, the inequality for  $z$  becomes  $n \geq 16 + 40 \log(n/9)$ . The smallest  $n$  for which this is satisfied is 193. Thus,  $193 \leq n \leq 203 = \lfloor k2^{k/2} \rfloor$ . In this case we may use (6) which shows that  $\vec{\chi}(G) > 5$ . The assumption that  $z \geq 6$  now gives a contradiction to the fact that  $n \leq 203$ .

From now on we may assume that either  $k = 8$  or  $k \geq 10$ . Let  $H_2 = KG(\lfloor n/3 \rfloor, r)$ . By Theorem 3.3 with  $t = 3$  and  $x = r$  (when  $k$  is even) or  $x = r - 1$  (when  $k$  is odd), we conclude that  $G$  contains a blow-up of  $H_2$  with power  $m = \binom{2r}{x}$ .

Our goal here is to apply Lemma 3.2, so we need to argue that  $2 + 2 \log m \leq \lceil m/z \rceil$ . We can estimate  $z$  as follows:

$$z \leq \frac{n}{8 \log(n/k)} < \frac{k2^{k/2}}{8 \log\left(\frac{2^{k/2}k}{k}\right)} = 2^{k/2-2}.$$

Thus, it suffices to see that

$$2 + 2 \log m \leq \lceil m/\lfloor 2^{k/2-2} \rfloor \rceil. \tag{7}$$

Inequality (7) can be proved by induction on  $k$  as follows. First of all, the inequality holds for  $k = 8$  and  $k = 11$  (base cases), which the reader can easily

verify. Next, we observe that when we increase  $k$  by 2, the left-hand side increases by at most 4. On the other hand, the right-hand side increases by more than 4 when  $k \geq 8$ .

This shows that we can apply Lemma 3.2 when  $k = 8$  or  $k \geq 10$ . We do this for every  $n$  such that  $4k \leq n < k2^{k/2}$ . Since  $\chi(H_2) = \lfloor n/3 \rfloor - 2\lfloor k/2 \rfloor + 2 \geq (n - 2k + 2)/16 \geq \chi(G)/8 \log(n/k) \geq z$ , we have by Lemma 3.2 that  $\bar{\chi}(G) \geq z$ .

For the remaining subcase, we assume that  $n < 4k$ . Let  $H_3 = KG(\lfloor n/2 \rfloor, r+2)$ . By Theorem 3.3 with  $t = 2$  (and  $x = k - 2r + 4 \geq 4$ ), we see that  $G$  contains a blow-up of  $H_3$  with power  $\binom{r+2}{4}$ . Observe that

$$z = \left\lfloor \frac{n - 2k + 2}{8 \log(n/k)} \right\rfloor \leq \left\lfloor \frac{4k - 2k + 2}{8 \log(4k/k)} \right\rfloor = \left\lfloor \frac{k + 1}{8} \right\rfloor.$$

By induction on  $k$ , we easily prove the following:

$$2 + 2 \log \binom{r + 2}{4} \leq \left\lceil \frac{\binom{r+2}{4}}{\lfloor \frac{k+1}{8} \rfloor} \right\rceil \leq \left\lceil \frac{\binom{r+2}{4}}{z} \right\rceil.$$

This shows that we can apply Lemma 3.2. Recall that  $n - 2k + 2 \geq 16$ . Then,  $\chi(H_3) = \lfloor n/2 \rfloor - 2r - 2 \geq \lfloor (n - 2k + 2)/8 \log(n/k) \rfloor = z$ . By Lemma 3.2, we conclude that  $\bar{\chi}(G) \geq z$ . □

Let  $n = 2k + z - 2$  with  $k \gg z$ , and let  $G = KG(n, k)$ . Then  $\chi(G) = z$  and  $\chi_f(G) = n/k < 2 + z/k$ . By Theorem 3.5,  $\bar{\chi}(G) \geq z/8 \log(n/k) > z/16$ . Thus, we have the following result.

**COROLLARY 3.6.** *For any  $\varepsilon > 0$  and any integer  $t$ , there is a graph  $G$  with  $\chi_f(G) < 2 + \varepsilon$  and  $\bar{\chi}(G) \geq t$ .*

So  $\bar{\chi}(G)$  is lower bounded by a function of  $\chi_f(G)$  but not upper bounded by a function of  $\chi_f(G)$ .

### Acknowledgements

B. Mohar was supported in part by an NSERC Discovery Grant (Canada), by the Canada Research Chair program, and by the Research Grant P1–0297 of ARRS (Slovenia). H. Wu was supported in part by the Thousands Talent (for Youth) Plan (China). This work was done while H. Wu was a PIMS Postdoctoral Fellow at the Department of Mathematics, Simon Fraser University, Burnaby, B.C.

## References

- [1] E. Berger, K. Choromanski, M. Chudnovsky, J. Fox, M. Loebli, A. Scott, P. Seymour and S. Thomassé, ‘Tournaments and coloring’, *J. Combin. Theory B* **103** (2013), 1–20.
- [2] D. Bokal, G. Fijavž, M. Juvan, P. M. Kayll and B. Mohar, ‘The circular chromatic number of a digraph’, *J. Graph Theory* **46** (2004), 227–240.
- [3] M. Chudnovsky, ‘The Erdős–Hajnal conjecture—A Survey’, *J. Graph Theory* **75** (2014), 178–190.
- [4] P. Erdős, ‘Problems and results in number theory and graph theory’, in *Proceedings of Ninth Manitoba Conference on Numerical Math. and Computing, 1979*, pp. 3–21 <https://www.renyi.hu/~p.erdos/1980-30.pdf>.
- [5] P. Erdős, J. Gimbel and D. Kratsch, ‘Some extremal results in cochromatic and dichromatic theory’, *J. Graph Theory* **15**(6) (1991), 579–585.
- [6] C. Godsil and G. Royle, *Algebraic Graph Theory*, Graduate Texts in Mathematics, 207 (Springer, New York, 2001).
- [7] A. Harutyunyan and B. Mohar, ‘Two results on the digraph chromatic number’, *Discrete Math.* **312**(10) (2012), 1823–1826.
- [8] L. Lovász, ‘Kneser’s conjecture, chromatic number, and homotopy’, *J. Combin. Theory A* **25** (1978), 319–324.
- [9] B. Mohar, ‘Circular colorings of edge-weighted graphs’, *J. Graph Theory* **43** (2003), 107–116.
- [10] V. Neumann-Lara, ‘The dichromatic number of a digraph’, *J. Combin. Theory B* **33** (1982), 265–270.
- [11] V. Neumann-Lara, ‘The 3 and 4-dichromatic tournaments of minimum order’, *Discrete Math.* **135** (1994), 233–243.