Discrete Mathematics 313 (2013) 517–539

Contents lists available at SciVerse ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Perspective

Colorings of plane graphs: A survey

O.V. Borodin

Institute of Mathematics and Novosibirsk State University, Novosibirsk, 630090, Russia

ARTICLE INFO

Article history:
Received 13 April 2011
Received in revised form 6 November 2012
Accepted 7 November 2012
Available online 20 December 2012

Dedicated to A.V. Kostochka on the occasion of his 60th birthday

Keywords:
Plane graph
Planar graph
Coloring
List coloring

ABSTRACT

After a brief historical account, a few simple structural theorems about plane graphs useful for coloring are stated, and two simple applications of discharging are given. Afterwards, the following types of proper colorings of plane graphs are discussed, both in their classical and choosability (list coloring) versions: simultaneous colorings of vertices, edges, and faces (in all possible combinations, including total coloring), edge-coloring, cyclic coloring (all vertices in any small face have different colors), 3-coloring, acyclic coloring (no 2-colored cycles), oriented coloring (homomorphism of directed graphs to small tournaments), a special case of circular coloring (the colors are points of a small cycle, and the colors of any two adjacent vertices must be nearly opposite on this cycle), 2-distance coloring (no 2-colored paths on three vertices), and star coloring (no 2-colored paths on four vertices). The only improper coloring discussed is injective coloring (any two vertices having a common neighbor should have distinct colors).

1. Introduction and preliminaries

Coloring in a broad sense is a decomposition of a discrete object into simpler sub-objects. Due to its generality, this notion arises in various branches of discrete mathematics and has important applications. For example, one of the most natural models in the frequency assignment problem in mobile phoning is $L(p, q)$*-labeling*. The vertices of a planar graph (sources) should be colored (get frequencies assigned) so that the colors (integer frequencies) of vertices at distance 1 differ by at least p, while those at distance 2 differ by at least q. Sometimes, the set of available frequencies can vary from one source to another; this corresponds to "list $L(p, q)$-labeling".

The theory of plane graph coloring has a long history, extending back to the middle of the 19th century, inspired by the famous Four Color Problem (4CP), which asked if every plane map is 4-colorable. Now it is a broad area of research, with hundreds of contributors and thousands of contributions, and so is covered in this survey only partially.

The development of this area goes hand in hand with the study of the structure of plane graphs. Sometimes a new structural fact about plane graphs that is useful for coloring takes its place also in the structural theory; more often, it is not of independent interest and just serves as a tool for solving a specific coloring problem. Until several decades ago, the only coloring problem of broad interest was the 4CP, solved in 1976 by Appel and Haken [12] (see Theorem 1.1 below). Accordingly, the study of plane graphs from a structural viewpoint was for a long time almost exclusively concerned with plane triangulations of minimum degree 5. Since the 1960s, a rapidly growing number of interesting graph coloring problems on the plane have appeared (see Jensen and Toft's monograph [118]), and this advanced the study of the structure of plane graphs in general.

1.1. Reducible configurations, discharging, and the 4CP

The basic elements of a *plane map* are its vertices, edges, and faces. An *edge* is a closed Jordan curve; its end-points are *vertices*. A *loop* joins a vertex to itself; two vertices may be joined by several *multiple edges*. No edge can have an internal

E-mail address: brdnoleg@math.nsc.ru.

0012-365X/$ – see front matter © 2012 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2012.11.011

point in common with itself or with another edge. *Faces* are connected components of the complement of the map. A *plane graph* is a plane map with neither loops nor multiple edges. All maps considered in this survey are finite.

A set **S** of (usually small) plane graphs, called *configurations*, is *unavoidable* for a class **M** of plane maps if every map M in **M** has a configuration from **S** as a subgraph. In Section 2, we see a few examples of *unavoidable sets of configurations* (USC in the sequel) for various classes of plane maps. Among USCs, there are both trivial (vertices of degree at most 5) and very complicated (sets of up to fifteen hundred configurations, most of which consist of more than a dozen vertices; such a set was used by Appel and Haken [12]). A statement that **S** is a USC may comprise a nice theorem on the structure of plane graphs, having its own value (if it is formulated in basic terms and has precise numerical parameters). More often, it is a tool for solving a particular coloring problem. Rather rarely a USC combines both these virtues.

A link between USCs and coloring problems has been known for a long time, beginning with Heawood's Five Color Theorem, which is based on the trivial fact that every planar graph has a vertex of degree at most 5. Almost all previously known theorems on planar graph colorings are based on certain USCs (with very few exceptions, like Thomassen's celebrated Five Choosability Theorem [177]). The idea of the *method of reducible configurations* (MRC) is as follows. Given a type of coloring, a *reducible configuration* for that type of coloring is a graph that cannot lie in a minimal plane graph that cannot be colored as required. Accordingly, solving a plane graph coloring problem by the MRC is equivalent to finding an *unavoidable set of reducible configurations* (USRC) for it. To prove that a configuration C is reducible, a typical approach is to attempt to show that every coloring of the boundary of C can be extended to the whole C.

The configurations in an unavoidable set may be reducible for a certain coloring problem or may describe some interesting structural property of plane graphs. A common method of proving unavoidability of a USC, called *discharging*, works as follows.

Assuming the existence of a map M that avoids all configurations from **S**, we associate a real number with each vertex v, which we call the *initial charge* of v. Sometimes we give an initial charge to every face of M as well. The initial charges are defined in such a way that their sum is negative: see Section 3 for examples of how this can be done. Therefore, some elements of M have a negative initial charge (are *deficient*). Using the absence in M of configurations that belong to **S**, we try to redistribute charges (preserving their sum) in favor of deficient elements, taking some charge from elements having positive initial charge. If we succeed in making the *new charges* of all elements nonnegative, then an obvious contradiction arises, which completes the proof.

This simple scheme is in fact an example of the usual proof by contradiction, and it is widely known among graph theorists. It is a dominant tool used in studying the structural and colorability properties of plane graphs. However, being acquainted with it does not guarantee success; many known problems in the area remained open for many years. Moreover, we should understand that in fact *every* discharging, no matter how artificial, still produces a USC; namely, the USC consisting of all those configurations that are deficient under this discharging. (Roughly speaking, a deficient configuration is a subgraph such that the sum of the charges of its elements is negative.) Another question is how valuable such a USC is. So, we should try to construct "smart" or at least sensible dischargings.

In solving the 4CP, some of the milestones were Wernicke's Theorem [202], dated 1904, on the existence in a plane graph with minimum degree 5 of a vertex of degree 5 adjacent to a vertex of degree at most 6, and the similar result due to Franklin [97], which guarantees existence of a vertex of degree 5 with two neighbors of degree at most 6. We note that these theorems are formulated in terms of USCs.

In 1940, Lebesgue [134] suggested to distribute the charges of vertices and/or faces uniformly among the neighboring vertices and/or edges and/or faces (in any combination); this provides some approximate structural information about various subclasses of plane graphs. In particular, Lebesgue [134] described the structure of the neighborhoods of vertices of degree 5 in plane triangulations with minimum degree 5. This description was not closely enough tied to 4-coloring: many of the listed neighborhoods could not be shown to be reducible by the standard means described above.

In 1969, Heesch [108] introduced a concept of *likely to be reducible configurations*, all lying within distance 2 from a vertex of degree 5. He believed that in fact these configurations could all be proved to be reducible with respect to 4-coloring by a certain specific technique, computed their cardinality to be 8900, and proved that they constitute a USC. This program of solving the 4CP became a general framework for Appel and Haken's [12] glorious proof of their Four Color Theorem (4CT), obtained as a result of massive manual and computerized work.

Theorem 1.1 (*Appel–Haken [12]*)**.** *Every planar graph is* 4*-colorable.*

The USRC in [12] comprises around 1400 items. However, this complicated computer-based proof for such a broadly known problem formulated in an elementary way led to some dissatisfaction in a more conservative part of the mathematical community.

A subsequent new proof of the 4CT, dated 1997, by Robertson et al. [164] improves on some features of the first proof; in particular, it uses fewer than 700 reducible configurations. However, finding a traditional humanly checkable proof of the 4CT remains a very attractive open problem.

The *dual map* M^* of a plane map M is the map of adjacency of the faces of M, where two faces are *adjacent* if they have a common boundary edge. By duality, a 4-coloring of M^* yields a 4-face-coloring of M, and vice versa. Therefore, Theorem 1.1 also says that every plane map without self-adjacent faces is 4-face-colorable.

1.2. Basic definitions, notation, and conventions

The sets of vertices, edges, and faces of a plane map M are denoted by $V(M)$, $E(M)$, and $F(M)$, respectively. (We will drop the argument whenever this does not lead to confusion.) A *planar graph* is one that can be expressed as a plane graph.

The *degree* $d(x)$ of $x \in V \cup F$ is the number of edges incident with x (in particular, each cut-edge incident with a face f and vertex v contributes 1 to $d(v)$ and 2 to $d(f)$). A *d-vertex* or *d-face* is one of degree d; a *d^+-vertex* is a vertex of degree at least d, a *d^--face* is one of degree at most d, etc. Let Δ denote the maximum vertex degree of a graph or map, and let δ be its minimum vertex degree. Let ρ be the minimum face degree of a plane map. A plane map is *normal* if $\delta \geq 3$ and $\rho \geq 3$. In particular, every plane graph with $\delta \geq 3$ is a normal map.

We say that two cycles in a plane map are *adjacent* or *intersecting* if they have an edge or a vertex in common, respectively. "Triangle" is another name for a 3-cycle.

A *proper k-coloring* of a graph G is a function $\varphi: V(G) \to \{1, \ldots, k\}$ such that $\varphi(u) \neq \varphi(v)$ whenever $uv \in E(G)$. A graph is *k-colorable* if it has a proper k-coloring. The minimum k such that G is k-colorable is its *chromatic number*, denoted by $\chi(G)$.

A *list assignment L* on G is a function that assigns to every vertex v a list $L(v)$ of colors available to be used on v (the colors are represented by positive integers). An *L-coloring* is a proper coloring of G such that the color on v is chosen from $L(v)$, for each $v \in V(G)$. A graph G is *k-choosable* (or *list k-colorable*) if G has an L-coloring whenever L is a list assignment such that $|L(v)| \geq k$ for all $v \in V(G)$. The minimum k such that G is k-choosable is the *list chromatic number* (or the *choosability*) of G, denoted by $\chi^{\text{list}}(G)$. (This notation for the choosability is used in this survey in order to make clear distinctions among many coloring parameters; in most research papers, it is denoted by χ_ℓ or ch.) We see that $\chi^{\text{list}}(G) \geq \chi(G)$ for every G, since when the lists happen to be identical their size must be at least $\chi(G)$ to permit a proper coloring to be chosen.

The other types of coloring considered in this paper are defined in the corresponding sections, along with the notation for their chromatic and list chromatic numbers. The same is true for other special notions and notation.

A graph is *k-degenerate* if its vertices can be deleted in some order such that every vertex has, at the moment of its deletion, degree at most k in the remaining subgraph. Every edgeless graph is 0-degenerate, and every tree is 1-degenerate. An easy consequence of Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$ for connected plane graphs G (see (1) in Section 3.1) is that every planar graph is 5-degenerate. A simple inductive argument shows that every k-degenerate graph satisfies $\chi^{\text{list}} \leq k+1$.

1.3. Topics covered

In Section 2, we state a few simple structural theorems about plane graphs and explain why some of them are useful for colorings. Section 3 shows two simple proofs by means of discharging.

Sections 4–6 are devoted to studying the three interrelated types of colorings introduced in 1965 by Ringel [163]: coloring of 1-plane graphs (a 1-plane graph is a graph having a drawing in the plane in which every edge has at most one internal point in common with other edges), simultaneous colorings of vertices, edges, and faces of plane graphs (it is required that any two neighboring elements to be colored in any of the four specific versions of simultaneous coloring should get distinct colors), and k-cyclic coloring (any two vertices lying on a common k^--face must have different colors). The 4CT tells us that every plane graph is 3-cyclically 4-colorable. More specifically, Ringel [163] proved that seven colors suffice for coloring 1-plane graphs, as well as for vertex-face and 4-cyclic colorings of plane graphs. He conjectured that in fact six colors suffice in these three cases, which was proved by Borodin [17] in 1984. Simultaneous vertex-edge-coloring, also called total, is discussed in Section 6, along with edge-coloring of planar graphs.

In 1959, Grötzsch [98] proved his fundamental Three Color Theorem, saying that every triangle-free planar graph is 3-colorable. In 1995, Voigt [186] constructed a triangle-free planar graph that is not 3-choosable. The present state of the art of 3-coloring and 3-choosability of planar graphs is described in Sections 7 and 8, respectively.

In 1973, Grünbaum [99] defined a proper coloring to be acyclic if it has no 2-colored cycles and conjectured that every planar graph is acyclically 5-colorable. Kostochka and Mel'nikov [127] constructed a 2-degenerate bipartite planar graph that is not acyclically 4-colorable. Borodin [18] confirmed Grünbaum's conjecture by finding a USRC of cardinality 450 for the family of plane triangulations with minimum degree 4. This proof occupies about 25 journal pages and does not use a computer. A crucial feature was that among the 450 reducible configurations used, some 400 formed a family growing from a single irreducible configuration as a root, and the reducibility proof was given for that whole family at once in just six pages. (This trick is instructive, and with luck it may possibly be used elsewhere.) Lately, acyclic coloring has found applications to some other plane graph coloring and partitioning problems. In particular, acyclic 5-colorability implies, by means of short nice arguments, the following best-known upper bounds for coloring parameters on planar graphs: 80 for the oriented chromatic number (Raspaud–Sopena [161]) and, combined with the 4CT, 20 for the star chromatic number (Albertson et al. [8]). The results on acyclic coloring are summarized in Section 9.

In Sections 10 and 11, we discuss oriented and circular colorings. Briefly, an oriented coloring is a homomorphism from an oriented graph to a tournament, and we seek the smallest such tournament. We discuss the special case of circular coloring of graphs that is equivalent to finding a homomorphism into a cycle, seeking the shortest such cycle.

The simplest but most important models of $L(p, q)$-labeling are $p = q = 1$ (2-distance coloring) and $p = 0, q = 1$ (which is the same as injective coloring for triangle-free graphs); these are considered in Sections 12 and 13. Note that ordinary proper coloring is precisely the case $p = 1, q = 0$ of $L(p, q)$-labeling. Injective coloring, which requires that every two vertices having a common neighbor get distinct colors, is the only improper coloring mentioned in this paper. Also observe that a total coloring of a graph G is a 2-distance coloring of the 1-subdivision of G (obtained from G by putting a vertex of degree 2 on each edge).

Finally, Section 14 is devoted to star coloring (introduced in 1973 by Grünbaum [99]), which is proper coloring with no 2-colored paths on four vertices. This term is motivated by the obvious fact that every component of every bicolored subgraph in a star coloring of a graph is a star. We also note that every star coloring is an acyclic coloring, and every 2-distance coloring is a star coloring.

1.4. A few general remarks

The purpose of this section is to present various important facts, techniques, and ideas that are well known to the experts in the area but could inspire and prepare a broader audience to navigate through the rest of this survey.

As distinct from ordinary coloring, in list coloring, introduced by Vizing [182] and Erdős, Rubin, and Taylor [96], every vertex v is assigned an individual list $L(v)$ of admissible colors, and one should choose a color $c(v)$ from $L(v)$ for each v so that the resulting coloring of the whole graph is proper; that is, any two neighboring vertices get different colors. To show the difference, Vizing [182] constructed, for any integer k with $k \geq 2$, a bipartite graph and a collection of lists of cardinality k on its vertices such that we cannot choose a proper coloring from this collection, as follows.

We take a complete bipartite graph K_{k,k^k}. The k vertices of the small part are assigned disjoint lists $L_1, \ldots, L_k$ of k colors each, and the vertices of the large part are assigned all possible k^k lists $\{\alpha_1, \ldots, \alpha_k\}$ such that $\alpha_i \in L_i$ for each i. We see that for any coloring of the small part of our K_{k,k^k} there is a vertex in the large part that cannot be colored.

In contrast with the 4CT, Voigt [187] presented a non-4-choosable plane graph. On the other hand, Thomassen [177] proved that every planar graph G is 5-choosable. Both these statements were conjectured in [96].

List coloring has become very popular among graph theorists; nowadays, almost all coloring problems are also considered in their list (choosability) versions. We will trace both versions within corresponding sections. The following simple but important and often-used fact was first published in [96]:

Proposition 1.1. *Every cycle of even length is* 2*-choosable.*

Proof. Let the cycle have vertices $v_1, \ldots, v_{2l}$ in order, and let $L(v_i) = \{\alpha_i, \beta_i\}$ for all i. If $L(v_1) = \cdots = L(v_{2l}) = \{\alpha, \beta\}$, then put $c(v_1) = \cdots = c(v_{2l-1}) = \alpha$ and $c(v_2) = \cdots = c(v_{2l}) = \beta$. Otherwise, we may assume by symmetry that $L(v_{2l}) \neq L(v_1)$. Put $c(v_1) \notin L(v_{2l})$, and then color $v_2, \ldots, v_{2l}$ in this order. (The latter is easy, because any vertex has fewer restrictions on the choice of color than colors available at the moment of its coloration.) $\square$

A lot of research is devoted to coloring *sparse planar graphs*. A traditional measure of sparseness of a planar graph G is its girth $g(G)$, which is the length of the shortest cycle in G. Another measure, suggested by Erdős (see [175]), is the absence of cycles of length from 4 to a certain constant. More generally, given a set S of integers, a graph is S*-free* if it has no cycle with length from S. For example, the Steinberg conjecture (see [6]) suggests that every $\{4, 5\}$-free planar graph is 3-colorable. In general, sparser planar graphs have smaller chromatic numbers. Other measures of sparseness have also been used, such as the minimum distance between triangles or the maximum average degree. The *maximum average degree* of a graph G, denoted by $\mathrm{mad}(G)$, is the maximum (over all subgraphs) of the average vertex degree. Some coloring results on planar graphs with large girth extend to nonplanar graphs with the comparable restriction on $\mathrm{mad}(G)$; it follows easily from Euler's Formula that $\mathrm{mad}(G) < \frac{2g(G)}{g(G)-2}$ when G is a planar graph with girth $g(G)$. In what follows, we will present only planar versions of such more general results. By the way, the last inequality can be rewritten as $(\mathrm{mad}(G)-2)(g(G)-2) < 4$, which shows a curious duality between $\mathrm{mad}(G)$ and $g(G)$.

An induced cycle $v_1 \cdots v_{2k}$ in a graph is t*-alternating* (Borodin [19]) if $d(v_1) = d(v_3) = \cdots = d(v_{2k-1}) = t$. Usually $t = \delta$, while the degrees of the other vertices are close or equal to Δ. This notion, along with its more sophisticated analogues (t-alternating subgraph, 3-alternator, cycle consisting of 3-paths, etc.), turns out to be useful for the study of coloring, since it sometimes provides crucial reducible configurations in coloring and partition problems (more often, on sparse graphs). Its first application was to show that the total choosability of planar graphs with $\Delta \geq 14$ equals $\Delta + 1$ (see [19]).

It is hard to say who was the first to use the following idea of *coloring diffusion*: let a part G_1 of a graph G be colored with colors $1, \ldots, s$, and let $G_2 = G - V(G_1)$. Let G_2 be colored with $s+1, \ldots, k$. Sometimes it is possible to erase the colors $t+1, \ldots, k$, where $s \leq t < k$, from G_2 so that the colors $1, \ldots, s$ can "percolate" on the whole uncolored part G_2' of G_2; that is, there exists a coloring of the subgraph of G induced by $V(G_1) \cup V(G_2')$ with $1, \ldots, s$. If so, then we obtain a t-coloring of G. In Sections 4 and 5, we will give several examples of using this idea.

In the proofs of certain results on circular, list acyclic, 2-distance, and oriented colorings in Borodin et al. [44], Borodin–Ivanova [45,46], and Borodin, Ivanova, Kostochka [59], some portions of charge are transferred along "feeding paths" over an unlimited distance in the hypothetic minimal counterexample to the statement proved. This new idea of *global discharging* opens additional possibilities in solving some coloring problems when the usual local discharging does not work.

Sometimes it is convenient to search for a USRC not in the whole minimal counterexample to the statement proved, but in a certain carefully chosen part of it; examples can be found in Borodin [18,20] and in [45]. In the simplest case, this part may just consist of a pendant block, which happens in some list coloring problems, where the minimum counterexample is not obliged to be 2-connected, in contrast to most non-list coloring problems.

Although the discharging method is the dominant tool in solving coloring problems on plane graphs, it is not exclusive. We mention only two important alternative techniques. One is due to Thomassen [177,178] and is based on cleverly using the "cyclic structure" of plane graphs (see celebrated Theorems 9.2 and 8.1). Nice examples of using a probabilistic argument for planar graph coloring and further references can be found in the remarkable paper by Havet et al. [107] and references in [107].

2. Examples of structural theorems useful for colorings

As shown by Lebesgue [134] in 1940, every normal plane map with $\delta \geq 4$ has an edge whose end-points have degree summing to at most 11, which bound is tight. Back in 1955, Kotzig [129] defined the *weight* $w(e)$ of an edge e in a graph G to be the degree sum $d(u) + d(v)$ of its end-points u and v. Let $w_e(G) = \min\{w(e) : e \in E(G)\}$.

Theorem 2.1 (*Kotzig [129]*). *Every 3-connected planar graph G satisfies $w_e(G) \leq 13$, and this bound is tight.*

We note that the class of 3-connected planar graphs (which are precisely the graphs of 3-dimensional polytopes due to Steinitz's famous theorem from 1922) is not closed under deleting edges. Therefore, Theorem 2.1 does not allow us to use induction on the number of edges in coloring proofs even if we restrict ourselves to 3-polytopes. However, it turns out that some extensions of Kotzig's Theorem, like Theorem 2.2 below, can be applied to total and entire colorings of arbitrary plane graphs, as well as to other coloring problems (see Section 5).

Theorem 2.2 (*Borodin [19]*). *Every normal plane map M satisfies $w_e(M) \leq 13$; moreover, if M does not contain a 3-alternating 4-cycle, then $w_e(M) \leq 11$, where both bounds are tight.*

The Total Coloring Conjecture by Behzad [14] and Vizing [183] asserts that every graph has a simultaneous coloring of its vertices and edges with $\Delta + 2$ colors, which can be written as $\chi_{ve} \leq \Delta + 2$. Up to now, it has been confirmed only for some narrow classes of graphs. The simple Theorem 2.2 (for the proof see Section 3.1) already makes it possible to prove that every planar graph with $\Delta \geq 9$ is totally $(\Delta + 2)$-colorable (and even totally $(\Delta + 2)$-choosable; that is, it satisfies $\chi_{ve}^{\text{list}} \leq \Delta + 2$; see Section 5.3).

To reach the best-possible upper bound $\Delta + 1$ for the total chromatic number of planar graphs, we have to bound $w_e(G)$ from above for planar graphs G having $\delta(G) \geq 2$, since pendant vertices are easily reducible. Since $w_e(K_{2,k}) = k+2$, we have to impose some restrictions on the vertices of degree 2 or forbid 4-cycles if we wish to describe natural classes of planar graphs with bounded w_e.

The following simple theorem already makes it possible to prove, for example, that every planar graph G with $\Delta(G) \geq 16$ satisfies $\chi_{ve}(G) = \Delta(G) + 1$.

Theorem 2.3 (*Borodin [21]*). *Every planar $K_{2,k}$-free graph G with $k \geq 2$ and $\delta(G) \geq 2$ satisfies $w_e(G) \leq 5k + 7$, and this bound is tight.*

Putting $k = 2$ in Theorem 2.3 (that is, forbidding 2-alternating 4-cycles), we get $w_e(G) \leq 17$. Forbidding all 2-alternating cycles gives us a bit more.

Theorem 2.4 (*Borodin [19]*). *Every planar graph G with $\delta(G) \geq 2$ and without 2-alternating cycles satisfies $w_e(G) \leq 15$, which is tight.*

From the structural Theorem 2.4 we easily deduce, in particular, the following coloring result.

Corollary 2.1. *Every planar graph G with $\Delta(G) \geq 14$ satisfies $\chi_{ve}^{\text{list}}(G) = \Delta(G) + 1$.*

Proof. It suffices to show that neither an edge e with $w(e) \leq 15$ nor a 2-alternating cycle C can belong to an edge-minimal graph G that is not totally $(\Delta + 1)$-choosable.

Indeed, if $w(e) \leq 15$, then we color $G - e$ from its list. Let the end-points of e be u and v. If $d(u) \leq 7$, then we uncolor u. Now e has at most $1 + 15 - 2$ restrictions on the choice of color and at least $14 + 1$ colors available, so e can be colored. It remains to color u, but it has at most 2×7 restrictions.

Now suppose a 2-alternating cycle C exists in G, with vertices $v_1, \ldots, v_{2l}$ in order. We first color $G - E(C)$ and uncolor the 2-vertices $v_1, v_3, \ldots, v_{2l-1}$. Now every edge of C has at least $\Delta + 1 - 1 - (\Delta - 2)$ admissible colors, so $E(C)$ can be colored by Proposition 1.1. It remains to color the 2-vertices of C, but each has only 2×2 restrictions on the choice of color. □

Similar proofs based on k-alternating cycles appear in several dozens of papers on coloring, beginning with [19].

In order to achieve the best-possible upper bound $\Delta + 2$ for entire choosability χ_{vef}^{list} (here, the vertices, edges, and faces are colored simultaneously; see Section 5.1), we must find an edge with low weight that is incident with a face of degree at most 5. An edge in a plane graph is *weak* (*semiweak*) if it is incident with two 3-faces (respectively, with at least one 3-face).

Theorem 2.5 (*Borodin [22]*)**.** *Every* 3*-connected plane graph contains one of the following:*

(C1) *a weak edge joining a* 3*-vertex to a* 10^-*-vertex;*
(C2) *a weak edge joining a* 4*-vertex to a* 7^-*-vertex;*
(C3) *a weak edge joining a* 5*-vertex incident with at least four* 3*-faces to a* 6^-*-vertex;*
(C4) *a semiweak edge joining a* 3*-vertex to a* 8^-*-vertex;*
(C5) *a semiweak edge joining a* 4*-vertex to a* 5^-*-vertex;*
(C6) *an edge incident with a* 4*-face and joining a* 3*-vertex to a* 5^-*-vertex;*
(C7) *a* 5*-face incident with at least four* 3*-vertices.*

Moreover, each of configurations (C1)–(C7) *may occur alone and with the highest value of the parameter allowed for it (that is of the degree of the vertex mentioned in* (C1)–(C6) *and with exactly four* 3*-vertices in* (C7)*).*

This gives a certain precise description of the structure of neighborhoods of an edge in 3-polytopes. It is formulated in basic terms, and therefore it can appropriately be called a structural theorem about plane graphs. On the other hand, it gives nothing for coloring. (The situation is the same as for Kotzig's Theorem 2.1 on 3-polytopes and its extensions to broader classes of planar graphs given in Theorems 2.2–2.4 above.) A more sophisticated extension in [22] of Theorem 2.5 to plane graphs with $\delta = 2$ implies $\chi_{vef}^{\text{list}} \le \Delta + 2$ for all plane graphs with $\Delta \ge 12$ (which is attained, for instance, by the star $K_{1,\Delta}$). However, this extension does not have attainable bounds of its numerical parameters and therefore cannot be regarded as a contribution to the structural theory of plane graphs.

The next simple structural result, along with the discharging used in its proof, was a starting point for quite a lot of more difficult results on the 3-coloring and 3-choosability of planar graphs (see Sections 7 and 8).

Theorem 2.6 (*Borodin [23]*)**.** *Every connected plane graph G with $\delta(G) \ge 3$ and without two adjacent* 3*-faces has either an i-face whenever $4 \le i \le 9$ or a* 10*-face incident with ten* 3*-vertices and adjacent to five* 3*-faces.*

By itself, Theorem 2.6 immediately implies that every planar graph without cycles of length from 4 to 9 is 3-colorable, as follows. A minimal non-3-colorable graph G satisfies $\delta(G) \ge 3$ and is 2-connected. If we color the graph obtained by deleting from G the edges of a 10-cycle formed by 3-vertices, then every vertex of the 10-cycle will have at least two admissible colors, and therefore this 10-cycle can be colored by Proposition 1.1.

Moreover, the following unpublished minor refinement of Theorem 2.6 implies that every planar graph without cycles of length from 4 to 9 is 3-choosable. To formulate it, we need a few definitions. Let G be a connected plane graph. A *facial walk* is a closed walk along the *boundary* $\partial(f)$ of a face $f \in F(G)$. (A cut-edge is passed twice when we go along the boundary of a face.) It is easy to see that $\partial(f)$ can be split into cycles and paths. The former are *facial cycles* in G. The latter consist of cut-edges of G. It is easy to see that for 2-connected plane graphs, the boundary of every face is its facial cycle.

Theorem 2.7. *Every connected plane graph G with $\delta(G) \ge 3$ and without two adjacent* 3*-faces has either a facial i-cycle, where $4 \le i \le 9$, or a facial* 10*-cycle incident with ten* 3*-vertices and adjacent to five* 3*-faces.*

For the proof of Theorem 2.7 see Section 3.2. We note that a minimal non-3-choosable graph G can have cut-vertices, in which case we can easily find the desired "sunflower" 10-cycle strictly inside any pendant block B of G, where "strictly inside" means that the cut-vertex of B is not used. It is therefore generally accepted that the 3-choosability of planar graphs without cycles of length from 4 to 9 was actually proved in [23] by means of Theorem 2.6. Still, to be rigorous, in Section 3.2 we give a formal proof of Theorem 2.7, which differs insignificantly from that of Theorem 2.6 in [23]. We note that since 1996 it remains open whether every planar graph without 4- to 8-cycles is 3-choosable.

3. Two simple applications of discharging

3.1. Proof of Theorem 2.2

The tightness of bounds 13 and 11 follows from the Archimedean solids (3, 10, 10) and (5, 6, 6), respectively. To construct a plane representation of the first of these, we take a 5-regular plane triangulation (that is the graph of the icosahedron), put a vertex inside every face, and join it to the three vertices of its face. The weight of any edge in the resulting graph is at least 13. The second construction arises from the dodecahedron: we take a cubic plane graph with twelve 5-faces, put a vertex inside every face, and join the new vertices to the five nearest old vertices. This time, the weight of any edge is at least 11.

Suppose that M is a counterexample to the main statement of Theorem 2.2 and M is maximal with respect to addition of edges. First we prove that M is a triangulation. Suppose there is a 4^+-face f with boundary $v_1v_2\cdots$ in M. If $d(v_1)+d(v_3) \ge 12$, then adding an edge v_1v_3 yields another counterexample to Theorem 2.2, since this operation neither creates 3-vertices (and hence 3-alternating 4-cycles), nor decreases the weight of "old" edges; a contradiction. By symmetry, we may assume that $d(v_3) \le 5$. Since $w_e(M) \ge 12$, it follows that $d(v_2) \ge 7$ and $d(v_4) \ge 7$. Now, adding edge v_2v_4 contradicts the maximality of M. This contradiction proves that M is a triangulation.

By Euler's formula, $|V(M)| - |E(M)| + |F(M)| = 2$. Using the Handshake Lemma

$$\sum_{v \in V(M)} d(v) = 2|E(M)| = \sum_{f \in F(M)} d(f),$$

we have

$$\sum_{v \in V(M)} (d(v) - 6) + \sum_{f \in F(M)} (2d(f) - 6) = -12. \tag{1}$$

Define the *initial charge* of every vertex $v \in V(M)$ to be $\mu(v) = d(v) - 6$. By (1), we have

$$\sum_{v \in V(M)} \mu(v) \leq -12. \tag{2}$$

A *new charge* $\mu^*(v)$ of $v \in V(M)$ is defined by applying the following rule of discharging.

(R) Every vertex v such that $d(v) \leq 5$ receives charge $\frac{6-d(v)}{d(v)}$ from every adjacent vertex.

We now show that every vertex v satisfies $\mu^*(v) \geq 0$, which obviously contradicts (2).

Since no two recipients of charge are adjacent to each other, we see that $\mu^*(v) = 0$ holds if $d(v) \leq 5$ and every 7^+-vertex v makes at most $\lfloor \frac{d(v)}{2} \rfloor$ donations by our rule. Moreover, if M has no 3-alternating 4-cycles, then no 9^+-vertex v can make more than $\lfloor \frac{d(v)}{3} \rfloor$ donations of value 1 to 3-vertices.

If $d(v) \geq 12$, then $\mu^*(v) \geq \mu(v) - 1 \times \lfloor \frac{d(v)}{2} \rfloor \geq d(v) - 6 - 1 \times \frac{d(v)}{2} = \frac{d(v)-12}{2} \geq 0$, and if $d(v) = 11$, then $\mu^*(v) \geq 5 - 1 \times 5 = 0$.

If $d(v) = 10$, then v makes at most five donations altogether, including at most three donations of 1, so $\mu^*(v) \geq 4 - 3 \times 1 - 2 \times \frac{1}{2} \geq 0$. Suppose $d(v) = 9$. Now v makes at most four donations; if at most two of them are to 3-vertices, then $\mu^*(v) \geq 3 - 2 \times 1 - 2 \times \frac{1}{2} = 0$. If v makes at least three donations of 1, then v makes precisely three donations in all, and so $\mu^*(v) \geq 3 - 1 \times 1 = 0$.

If $d(v) = 8$, then donations of 1 from v are impossible since $w_e(M) \geq 12$, which implies that $\mu^*(v) \geq 2 - 4 \times \frac{1}{2} = 0$. If $d(v) = 7$, then donations from v are possible to 5-vertices only, so $\mu^*(v) \geq 1 - 3 \times \frac{1}{5} > 0$. Finally, if $d(v) = 6$, then $\mu^*(v) = \mu(v) = 0$.

3.2. *Proof of Theorem 2.7*

Suppose that G^* is a counterexample to Theorem 2.7. If G^* is 2-connected, then put $G = G^*$. Otherwise, let G be a pendant block of G^*, z be the cut-vertex of G^* that belongs to G, and f_∞ be the only face of G that is not a face of G^*. By definition, z belongs to the boundary of f_∞ and $d(z) \geq 2$ in G. Note that each facial walk in G is a cycle, and that f_∞ is the only face of G that can possibly have $4 \leq d(f_\infty) \leq 9$. Rewrite Euler's formula as follows:

$$\sum_{x \in V(G) \cup F(G)} (d(x) - 4) = \sum_{x \in V(G) \cup F(G)} \mu(x) = -8. \tag{3}$$

Let the initial charge $\mu(x)$ of every $x \in V(G) \cup F(G)$ now be $d(x) - 4$. This time, a new charge $\mu^*(x)$ of $x \in V(G) \cup F(G)$ is defined by applying the following rules:

(R1) Every nontriangular face f transfers to every incident vertex v, where $v \neq z$:
$\frac{2}{3}$ if $d(v) = 3$ and v is incident with a 3-face;
$\frac{1}{3}$ if $d(v) = 3$ but v is not incident with a 3-face;
$\frac{1}{3}$ if $d(v) = 4$ and v is incident either with two 3-faces or one 3-face not adjacent to f.

(R2) Every vertex transfers $\frac{1}{3}$ to every incident 3-face f.

Let $f \in F - f_\infty$. If $d(f) = 3$, then $\mu^*(f) = 3 - 4 - 3 \times \frac{1}{3} = 0$ by R2. If f is incident with z, then $\mu^*(f) \geq d(f) - 4 - (d(f) - 1) \times \frac{2}{3} = \frac{d(f)-10}{3} \geq 0$ by R1. So, assume that f is not incident with z, which means that the degrees of vertices incident with f are the same in G as in G^*. If $d(f) \geq 12$, then $\mu^*(f) \geq d(f) - 4 - d(f) \times \frac{2}{3} = \frac{d(f)-12}{3} \geq 0$. If $d(f) = 11$, then at least one vertex incident with f is not a triangular 3-vertex by parity, so $\mu^*(f) \geq 7 - 10 \times \frac{2}{3} - \frac{1}{3} = 0$. If $d(f) = 10$, then f can give $\frac{2}{3}$ to at most nine vertices by assumption. If at most eight vertices get $\frac{2}{3}$ from f, then $\mu^*(f) \geq 6 - 8 \times \frac{2}{3} - 2 \times \frac{1}{3} = 0$. Finally, if f is incident with precisely nine triangular 3-vertices, then the last vertex fails to receive any charge from f by R1, and so $\mu^*(f) \geq 6 - 9 \times \frac{2}{3} = 0$.

Now assume $v \in V - z$ and recall that v is incident with at most $\lfloor \frac{d(v)}{2} \rfloor$ triangles. If $d(v) \leq 4$, then we have five cases, and in each of them $\mu^*(v) = 0$ by R1 and R2. If $d(v) \geq 5$, then $\mu^*(v) \geq d(v) - 4 - \frac{\lfloor d(v)/2 \rfloor}{3} \geq \frac{5(d(v)-5)}{6} \geq 0$.

We also have $\mu^*(z) \geq d(z) - 4 - \frac{d(z)}{3} > -4$ and $\mu^*(f_\infty) \geq d(f_\infty) - 4 - d(f_\infty) \times \frac{2}{3} > -4$. This contradicts (3):

$$-8 < \sum_{x \in V(G) \cup F(G)} \mu^*(x) = \sum_{x \in V(G) \cup F(G)} \mu(x) = -8.$$

4. The three facets of the Six Color Theorem

It happens sometimes that a mathematical result allows several seemingly independent reformulations. This is the case with the Six Color Problem on the plane posed by Ringel [163] in 1965 and solved by Borodin [17] in 1984; it occurs in coloring of 1-planar graphs, in simultaneous coloring, and in cyclic coloring of plane graphs. (By a coincidence, about the same time, Bodendiek, Schumacher, and Wagner [16] published a paper in a broad mathematical journal with an appeal to a younger generation of mathematicians to get busy with this problem.)

The first solution of Ringel's problem (in [17]) used 35 reducible configurations (RCs). Another proof (Borodin [24]), using only 18 RCs, was published in 1995. Fortunately, the most complicated RCs from the first solution happen to be redundant. In other words, a smaller unavoidable subset of RCs was found. This substantially simplifies the reducibility part of the proof at the cost of a more careful discharging argument. As a result, the second proof is logically much simpler, but not much shorter.

A graph is 1-*planar* if it can be drawn in the plane so that every edge intersects at most one other edge in an interior point. Ringel [163] proved that every 1-planar graph is 7-colorable and conjectured that it is 6-colorable. Since the complete graph K_6 is 1-planar, fewer than six colors will not suffice. Ringel [163] also stated his problem in terms of *simultaneous* and 4-*cyclic* colorings of plane graphs. The former amounts to coloring the vertices and faces so that any two incident or adjacent elements get distinct colors. Such a coloring is called *coupled*, and the *coupled chromatic number* is denoted by χ_{vf}. The 4-cyclic coloring problem asks to color the vertices so that around every 4-face no color appears more than once.

Theorem 4.1 (*Borodin [17,24]*).

(i) *Every* 1-*planar graph is* 6-*colorable.*
(ii) *Every plane graph satisfied* $\chi_{vf} \leq 6$.
(iii) *Every plane graph is* 4-*cyclically* 6-*colorable.*

Each of the first and third forms of Theorem 4.1 is easily reduced to the other and implies the second form. (We note that the graph of adjacency and incidence of the vertices and faces of any plane graph is 1-planar.) It is not known if there is any easy way to deduce (i) from (ii). The following two complementary special cases of Theorem 4.1(ii) were solved some time ago.

Theorem 4.2 (*Ringel [163]*). *Every plane triangulation satisfies* $\chi_{vf} \leq 6$.

Theorem 4.3 (*Archdeacon [13]*). *Every plane triangle-free graph satisfies* $\chi_{vf} \leq 6$.

We prove a stronger list coloring version, $\chi_{vf}^{\text{list}}(T) \leq 6$, of Theorem 4.2. First, we can easily color the vertices of our triangulation T, since T is 5-degenerate. Now every triangle has three admissible colors and is in touch with three other triangles. Since $\chi_{vf}(K_4) = 4$, we can further assume that $T \neq K_4$. This implies that now the faces of T can be colored with their admissible colors by the list coloring version of Brooks' Theorem, proved independently by Vizing [182] and Erdős, Rubin, and Taylor [96].

Archdeacon's proof of Theorem 4.3 is one of the first instances of applying the idea of coloring diffusion (see Section 1.3) and can be presented as follows:

Proof of Theorem 4.3. Color the vertices with colors 1, 2, and 3 by the Grötzsch Theorem [98]. Erase color 3 from the vertices and "blow up" the uncolored vertices into small faces so as to preserve the adjacency of "old" faces. (Thus every uncolored k-vertex becomes a small k-face.) We color all faces with colors 3, 4, 5, and 6 by the dual version of the 4CT. Finally, contract the new faces back to vertices, preserving their colors. This gives the desired coloring. □

Every plane graph that is both even and bipartite clearly has a coupled $(2 + 2)$-coloring. Using the coloring diffusion, Archdeacon [13] proved the following.

Theorem 4.4 (*Archdeacon [13]*). *Every plane graph that is even or bipartite has a coupled* 5-*coloring.*

Borodin et al. [66] proved that every 1-planar graph is list acyclically 20-colorable and showed that 6 colors do not suffice.

Problem 4.1 (*Borodin, Kostochka, Raspaud, Sopena [66]*). Is every 1-planar graph list acyclically 7-colorable?

Problem 4.2. Is every 1-planar triangle-free graph 5-colorable?

It is easy to prove that every 1-planar graph is 7-degenerate, which immediately implies that every 1-planar graph is list 8-colorable. Wang and Lih [195] proved that every 1-planar graph is list 7-colorable and constructed a plane graph with $\chi_{vf} \neq \chi_{vf}^{\text{list}}$.

Problem 4.3 (*Wang–Lih [195]*). Is every 1-planar graph list 6-colorable?

5. Edge, total, edge-face, and entire colorings

As observed in Section 4, Theorem 4.1(ii) says $\chi_{vf} \leq 6$: every planar graph may be colored with six colors so that every two neighbor vertices and faces get distinct colors.

In the other types of simultaneous coloring, the edges take part; we thus have vertex-edge-face-coloring, called *entire coloring*, vertex-edge-coloring, called *total coloring*, and edge-face-coloring. Again, every two elements to be colored should be colored with distinct colors if they are adjacent or incident. The minimum number of colors needed for the corresponding coloring is a *simultaneous chromatic number* of the graph, denoted by χ_{vef}, χ_{ve}, or χ_{ef}, respectively. The *edge-chromatic number* χ_e is the minimum number of colors in a proper edge-coloring. We note that χ_{ve} and χ_e (often denoted by χ'' and χ', respectively, outside this survey) are meaningful for arbitrary graphs, but we confine ourselves to planar graphs. The *edge-choosability* χ_e^{list} and similarly denoted *simultaneous choosabilities* of plane graphs are also discussed in this section.

Since the edges at a Δ-vertex already need Δ different colors, we have $\chi_{vef} \geq \chi_{ve} \geq \Delta + 1$ and $\chi_{ef} \geq \Delta$. Moreover, the star $K_{1,\Delta}$ satisfies $\chi_{ve}(K_{1,\Delta}) = \chi_{ef}(K_{1,\Delta}) = \Delta + 1$, and $\chi_{vef}(K_{1,\Delta}) = \Delta + 2$ due to the presence of the infinite face.

5.1. Entire coloring

In 1973, Kronk and Mitchem [132] proved that every plane graph with $\Delta \leq 3$ satisfies $\chi_{vef} \leq 7$. We note that $\chi_{vef}(K_4) = 7$, since no three among the $4 + 6 + 4$ elements of a plane representation of K_4 can be colored the same, but 7 colors clearly suffice.

Conjecture 5.1 (*Kronk–Mitchem [132]*). *Every plane graph with $\Delta \geq 3$ satisfies $\chi_{vef} \leq \Delta + 4$.*

For a long time the case $\Delta = 3$ of this conjecture remained the only one confirmed. The first further result in this direction, with $\Delta \geq 12$, was obtained by Borodin [21] in 1987 by means of a simple structural theorem saying that the minimum weight of edges in planar graphs with $\delta \geq 3$ is at most 13 (see Theorem 2.2 in Section 2).

The application of this theorem is also very simple. First color the faces of our counterexample G by the 4CT. Now every vertex and edge has at least Δ and $\Delta + 2$ admissible colors, respectively. Let H be a minimal subgraph of G that cannot be colored from the resulting lists. If $\delta(H) \leq 2$, then we take $e \in E(H)$, where $e = uv$ and $d(u) \leq 2$, color $H - e$, uncolor u, color e (it has at most $1 + 1 + \Delta - 1$ restrictions on the choice of color), and then color u. Otherwise, there exists $e \in E(H)$ such that $w(e) \leq 13$. We first color $H - e$ and then color e (it has at most $13 - 2 + 2$ restrictions), a contradiction.

Later, the restriction on Δ was improved to 7 in [25] by using a stronger structural result. The proof in [25] is given in terms of χ_{vef}, but it works for list entire coloring without change (the same is true for Theorem 5.2 below).

Theorem 5.1 (*Borodin [25]*). *If a plane graph satisfies $\Delta \geq 7$, then $\chi_{vef}^{\text{list}} \leq \Delta + 4$.*

As mentioned earlier, for $\Delta = 3$ the bound $\chi_{vef} \leq \Delta + 4$ is sharp. However, for larger Δ the tight upper bound for χ_{vef} turned out to behave differently, as proved by Borodin [22] in 1993, using an extension of the structural Theorem 2.5.

Theorem 5.2 (*Borodin [22]*). *Every plane graph with $\Delta \geq 12$ satisfies $\chi_{vef} \leq \chi_{vef}^{\text{list}} \leq \Delta + 2$, and this bound is tight.*

We recall that $\chi_{vef}(K_{1,\Delta}) = \Delta + 2$ holds. Indeed, the central vertex, all Δ edges, and the infinite face are $\Delta + 2$ pairwise adjacent or incident elements. Adding to $K_{1,\Delta}$ any matching such that every edge remains incident with the infinite face, we again have $\chi_{vef} = \Delta + 2$. It is also easy to see that by identifying a pendant vertex of a Δ-star with any vertex of non-maximum degree in arbitrary plane graph, we obtain a graph such that $\chi_{vef} \geq \Delta + 2$. Thus the bound of Theorem 5.2 is attained by many graphs.

In light of Theorem 5.2, the main problem on the entire coloring of plane graphs now looks as follows.

Problem 5.1. Find sharp upper bounds on χ_{vef} and χ_{vef}^{list} for the plane graphs with $\Delta \leq 11$.

In particular, the following specific question seems to deserve first consideration.

Problem 5.2. Is it true that $\chi_{vef} \leq 13$ holds for every plane graph with $\Delta = 11$?

In 2000, Sanders and Zhao [165] claimed to have proved the case $\Delta = 6$ of the Kronk–Mitchem Conjecture, but recently Wang and Zhu [200] found a mistake in [165] and, fortunately, managed to correct it. (Thus it appears that solving the case $\Delta = 6$ should be attributed to both teams in [165,200].) Moreover, using an interesting version of coloring diffusion, Wang and Zhu [200] confirmed the remaining cases $\Delta = 4$ and $\Delta = 5$ of the Kronk–Mitchem Conjecture.

Problem 5.3 (*Wang–Zhu [200]*). Is it true that every plane graph satisfies $\chi_{vef} = \chi_{vef}^{\text{list}}$?

5.2. Edge-face-coloring

The first result about this type of simultaneous coloring was obtained in 1969 by Jucovic [119].

Theorem 5.3 (*Jucovic [119]*). *Every plane cubic 3-connected graph satisfies $\chi_{ef} \leq 6$.*

In 1975, by analogy with the Behzad–Vizing Total Coloring Conjecture $\chi_{ve} \leq \Delta + 2$ and the Kronk–Mitchem Conjecture $\chi_{vef} \leq \Delta + 4$, Mel'nikov [141] conjectured that every plane graph satisfies $\chi_{ef} \leq \Delta + 3$.

The next theorem was stated in [26] in terms of χ_{ef}, but the proof works for χ_{ef}^{list} without changes.

Theorem 5.4 (*Borodin [26]*). *If a plane graph satisfies $\Delta \geq 10$, then $\chi_{ef} \leq \chi_{ef}^{\text{list}} \leq \Delta + 1$, and this bound is best possible.*

The bound $\chi_{ef} \leq \Delta + 1$ is attained by the same graphs, related to $K_{1,\Delta}$, as the bound $\chi_{vef} \leq \Delta + 2$ in Section 5.1. We note also this fact: $\chi_{ef}(C_3) = \Delta(C_3) + 3 = 5$.

Problem 5.4 (*Borodin [26]*). Find precise upper bounds for χ_{ef} and χ_{ef}^{list}.

In 1997, Mel'nikov's Conjecture was confirmed by Sanders and Zhao [166] and, independently, by Waller [189] by means of coloring diffusion.

Theorem 5.5 (*Sanders–Zhao [166], Waller [189]*). *Every plane graph satisfies $\chi_{ef} \leq \Delta + 3$.*

Proof. Color the edges with $1, \dots, \Delta + 1$ by Vizing's Edge-Coloring Theorem [184], uncolor the edges colored with Δ and $\Delta + 1$, and color the faces with colors in L, where $L = \{\Delta, \dots, \Delta + 3\}$, by the 4CT. Note that the uncolored edges are split into even cycles and paths. Since every uncolored edge has at least two admissible colors in L, we can color these edges using colors from L by Proposition 1.1. □

Later, Sanders and Zhao [167] proved $\chi_{ef} \leq 5$ if $\Delta = 3$, which is tight due to the example of a triangle with a pendant vertex attached. Recently, the tight bound $\chi_{ef} \leq \Delta + 1$ was proved by Sereni and Stehlik [173] for $\Delta = 9$, and then by Kang, Sereni, and Stehlik [121] for $\Delta = 8$. Hence the non-list part of Problem 5.4 remains open when $4 \leq \Delta \leq 7$. The best-known results in this direction are $\chi_{ef} \leq \Delta + 3$ whenever $4 \leq \Delta \leq 6$ [166,189] and $\chi_{ef} \leq 9$ for $\Delta = 7$ (Sanders–Zhao [168]).

Theorem 5.6 (*Luo–Zhang [137]*). *Every 2-connected simple plane graph such that $\Delta \geq 24$ satisfies $\chi_{ef} = \Delta$.*

Wang and Lih [196] proved that every plane graph satisfies $\chi_{ef}^{\text{list}} \leq \Delta + 3$ and constructed a plane graph such that $\chi_{ef} \neq \chi_{ef}^{\text{list}}$. The choosability part of Problem 5.4 remains open whenever $3 \leq \Delta \leq 9$; the best-known results are $\chi_{ef}^{\text{list}} \leq \Delta + 3$ whenever $3 \leq \Delta \leq 7$ [196] and $\chi_{ef} \leq 11$ if $8 \leq \Delta \leq 9$ (follows from Theorem 5.4 by adding two or one pendant vertex to a Δ-vertex, respectively).

5.3. *Total coloring*

This is the most studied type of simultaneous coloring and the only one defined for a graph not embedded on a surface.

Conjecture 5.2 (*Behzad [14], Vizing [183]*). *Every graph satisfies $\chi_{ve} \leq \Delta + 2$.*

This famous Total Coloring Conjecture by now has only been confirmed for $\Delta \leq 3$ (back in 1971, independently by N. Vijayaditya and M. Rosenfeld) and if $4 \leq \Delta \leq 5$ by Kostochka [123,124]. Later, Kostochka [125] found a new proof for $\Delta = 5$.

For planar graphs, the bound $\chi_{ve} \leq \Delta + 2$ was first proved in 1987 by Borodin [21] for $\Delta \geq 11$ by means of Theorem 2.2, and then for $\Delta \geq 9$ [19], which was strengthened to $\Delta \geq 8$ by Jensen and Toft [118] and to $\Delta \geq 7$ by Sanders and Zhao [169].

The last two results are obtained by means of coloring diffusion based on Vizing's Theorem [185] saying that every planar graph with $\Delta \geq 8$ satisfies $\chi_e = \Delta$ and its strengthening to $\Delta \geq 7$ in Sanders–Zhao [170] and Zhang [207]. The proof in [169] may be expressed as follows:

For $\Delta \geq 7$, we color the edges with $1, \dots, \Delta$ by [170,207], uncolor the edges colored with $\Delta - 1$ and Δ, and color the vertices with colors in L, where $L = \{\Delta - 1, \dots, \Delta + 2\}$, by the 4CT. Since every uncolored edge has at least two admissible colors in L, these edges can be colored from L either directly, when an uncolored component is a path, or by Proposition 1.1 applied to the edges rather than to the vertices of an even cycle.

Problem 5.5 (*Sanders–Zhao [169]*). Is every planar graph with $\Delta = 6$ totally 8-colorable?

Moreover, it turns out that every planar graph with large enough Δ satisfies $\chi_{ve} = \Delta + 1$. This was first proved in 1987 by Borodin [21] for $\Delta \geq 16$ (by means of Theorem 2.3), and then the restriction on Δ was lowered to 14 (Borodin [19]), 12 and 11 (Borodin, Kostochka, Woodall [67,68]), 10 (Wang [190]), and 9 (Kowalik, Sereni, Škrekovski [130]).

Problem 5.6 (*Kowalik, Sereni, Škrekovski [130]*). What is the smallest D_0 such that every planar graph with $\Delta \geq D_0$ satisfies $\chi_{ve} = \Delta + 1$?

Since $\chi_{ve}(K_4) = 5 = \Delta(K_4) + 2$, we have $4 \leq D_0 \leq 9$ due to [130].

As for list total coloring, the bound $\chi_{ve}^{\text{list}} \leq \Delta + 2$ was proved by Borodin first for $\Delta \geq 11$ in [21], and then for $\Delta \geq 9$ in [19]. Again, every planar graph with large enough Δ satisfies $\chi_{ve}^{\text{list}} = \Delta + 1$, which was proved by Borodin [21] for $\Delta \geq 16$, and then the restriction on Δ was lowered to 14 (Borodin [19]) and to 12 (Borodin, Kostochka, Woodall [67]).

Problem 5.7 (*Borodin, Kostochka, Woodall [67]*)**.** Prove that every planar graph with $\Delta = 11$ satisfies $\chi_{ve}^{\text{list}} = \Delta + 1$.

On the other hand, what is now called the List Total Coloring Conjecture, saying that every (not necessarily planar) graph satisfies $\chi_{ve}^{\text{list}} = \chi_{ve}$, was suggested independently in Borodin, Kostochka, Woodall [67] and Juvan, Mohar, Škrekovski [120]. D.R. Woodall (personal communication, 2009) does not exclude the possibility that the following holds.

Conjecture 5.3 (*Woodall, 2009*)**.** *Every planar graph with $\Delta \geq 4$ satisfies $\chi_{ve}^{\text{list}} = \Delta + 1$.*

These two conjectures are much stronger than what has been reached so far. All results on χ_{ve} and χ_{ve}^{list} for planar graphs with small Δ are in fact about graphs that are sparse in some sense (often, they are defined in terms of girth g or collections of forbidden cycles). We list only a few such results.

Juvan, Mohar, and Škrekovski [120] proved $\chi_{ve}^{\text{list}} = \chi_{ve} \leq 4$ if $\Delta = 2$ and $\chi_{ve}^{\text{list}} \leq 5$ if $\Delta = 3$. For $\Delta \geq 3$ and $g \geq 10$, we have $\chi_{ve} = \Delta + 1$ (Borodin, Kostochka, Woodall [69]), which was strengthened to $\chi_{ve}^{\text{list}} = \Delta + 1$ in Woodall [203]. (We recall that $\chi_{ve}^{\text{list}} = \Delta + 1$ is equivalent to $\chi_{ve}^{\text{list}} = \chi_{ve} = \Delta + 1$.) For $\Delta \geq 4$, the equality $\chi_{ve} = \Delta + 1$ was proved by Chen and Wu [84] for $g \geq 8$, by Wang and Wu [199] under the absence of 4- to 14-cycles, and then for $g \geq 6$ in [69]; the equality $\chi_{ve}^{\text{list}} = \Delta + 1$ was first proved for $g \geq 10$ (Borodin, Kostochka, Woodall [67]) and then for $g \geq 6$ in [203].

For $\Delta \geq 5$, we have $\chi_{ve}^{\text{list}} = \Delta + 1$ if $g \geq 5$ (Woodall [203]) or if there are no 4- to 8-cycles (Hou, Liu, Cai [112]). Suppose $\Delta \geq 6$. Now sufficient conditions for $\chi_{ve} \leq \Delta + 2$ deserve consideration due to Problem 5.5; these are: no adjacent 3-cycles (Sun et al. [176]), no 5-cycles (Hou, Liu, Wu [113]), and no 6-cycles [113]. Also, we have $\chi_{ve}^{\text{list}} = \Delta + 1$ if $g \geq 5$ [67] or if there are no 4- or 5-cycles [112].

If $\Delta \geq 7$, then, in particular, $\chi_{ve}^{\text{list}} = \Delta + 1$ provided that $g \geq 4$ (Borodin, Kostochka, Woodall [67]) or under the absence of 4-cycles (Hou, Liu, Cai [112]). For $\Delta \geq 8$, let us mention the following results: $\chi_{ve}^{\text{list}} = \Delta + 1$ if there are no intersecting 3-cycles (Wu–Wang [204]), no 5-cycle, or no 6-cycle (Ma, Wu, Yu [140]). Finally, $\Delta \geq 9$ implies $\chi_{ve}^{\text{list}} = \Delta + 1$ if there are no adjacent 3-cycles [204].

Dozens of other partial contributions to confirming the super-strong Conjecture 5.3 by Woodall can be found in the references to [112,113,203,204,176,140].

5.4. Edge-coloring

By the celebrated Vizing Edge-Coloring Theorem [184], every graph (not necessarily planar) satisfies $\chi_e \leq \Delta + 1$. Using strong properties of graphs critical with respect to edge-coloring, Vizing [185] proved that every planar graph with $\Delta \geq 8$ satisfies $\chi_e = \Delta$.

Problem 5.8 (*Vizing [185]*)**.** Is it true that every planar graph such that $6 \leq \Delta \leq 7$ satisfies $\chi_e = \Delta$?

A planar graph with $\Delta \leq 5$ and $\chi_e = \Delta + 1$ is obtained from any Δ-regular planar graph with an even number of vertices (say, icosahedron, octahedron, or K_4) by subdividing one of its edges. The case $\Delta = 7$ of Problem 5.8 was settled in the positive independently by Sanders and Zhao [169] and Zhang [207]. For the case $\Delta = 6$ only partial results are known. For example, $\chi_e = \Delta$ holds if $\Delta \geq 5$ and $g \geq 4$, or $\Delta \geq 4$ and $g \geq 5$, or $\Delta \geq 3$ and $g \geq 8$ (Li–Luo [135]).

As for list edge-coloring, the situation resembles that described in Section 5.3 for total coloring. On the one hand, according to the folklore List Edge-Coloring Conjecture, every graph (not only planar) satisfies $\chi_e^{\text{list}} = \chi_e$. Back in 1976, Vizing [182] suggested the next conjecture; being confirmed, it would trivially imply a relaxation, $\chi_{ve} \leq \Delta + 3$, of the Total Coloring Conjecture $\chi_{ve} \leq \Delta + 2$ for arbitrary graphs. Conjecture 5.4 was also one of the reasons for him to introduce the notion of list coloring (personal communication of V.G. Vizing to O.V. Borodin and A.V. Kostochka, 1975).

Conjecture 5.4 (*Vizing [182]*)**.** *Every graph satisfies $\chi_e^{\text{list}} \leq \Delta + 1$.*

On the other hand, not much has been proved as yet. The case $\Delta = 3$ was settled in Vizing [182] and, independently, Erdős, Rubin, Taylor [96] by proving the choosability version of the Brooks Theorem; the case $\Delta = 4$ is due to Juvan, Mohar, and Škrekovski [120].

For planar graphs, the bound $\chi_e^{\text{list}} \leq \Delta + 1$ was proved by Borodin [27] for $\Delta \geq 9$, by means of Theorem 2.2. Moreover, $\chi_e^{\text{list}} = \Delta$ (that is $\chi_e^{\text{list}} = \chi_e = \Delta$) holds for $\Delta \geq 14$ (Borodin [27], using Theorem 2.4), which was strengthened to $\Delta \geq 12$ in Borodin, Kostochka, Woodall [67] by further developing the idea of 2-alternating cycles.

Problem 5.9 (*Borodin [27]*)**.** Prove the bound $\chi_e^{\text{list}} \leq \Delta + 1$ for all planar graphs with $\Delta = 8$.

Problem 5.10 (*Borodin, Kostochka, Woodall [67]*)**.** Prove the equality $\chi_e^{\text{list}} = \Delta$ for all planar graphs with $\Delta = 11$.

Let us mention a few recent partial results on $\chi_e^{\text{list}} \leq \Delta + 1$ for $5 \leq \Delta \leq 8$. This bound holds if there are no 4-cycles (Shen [174]); if $\Delta \neq 5$ and there are no chordal 5-cycles, or if $\Delta(G) = 5$ and there are neither chordal 4-cycles nor chordal 6-cycles (Chen, Zhu, Wang [85]); if $\Delta \neq 5$ and there are no chordal 4-cycles (Cranston [87]). Many other similar and related results can be traced through the references to [174,85,87].

For $\chi_e^{\text{list}} = \Delta$ to hold when $\Delta \le 11$, the following sufficient conditions are known: (i) $\Delta \ge 7$ and $g \ge 4$, (ii) $\Delta \ge 5$ and $g \ge 5$, (iii) $\Delta \ge 4$ and $g \ge 6$, and (iv) $\Delta \ge 3$ and $g \ge 10$ (Borodin, Kostochka, Woodall [67]). Now if k is an integer and there are no i-cycles whenever $4 \le i \le k$, then $\chi_e^{\text{list}} = \Delta$ holds in each of the cases: (v) $\Delta \ge 7$ and $k = 4$, (vi) $\Delta \ge 6$ and $k = 5$, (vii) $\Delta \ge 5$ and $k = 8$, and (viii) $\Delta \ge 4$ and $k = 14$ (Hou, Liu, Cai [112]).

It looks like list edge-coloring of planar graphs with $\Delta \le 11$ will remain an active area of research for a long time.

6. Cyclic coloring

This generalization of proper coloring was introduced in 1969 by Ore and Plummer [158] and indirectly considered earlier by Ringel [163] for $k = 4$.

A coloring is called *k-cyclic* if any two vertices lying in a common face of size at most k have different colors. The *k-cyclic chromatic number* of a plane graph G, denoted by $\chi_{c,k}(G)$, is the minimum number of colors in a k-cyclic coloring of G.

It is easy to see that 3-cyclic coloring corresponds to proper coloring, while 4-cyclic coloring corresponds to proper coloring of 1-planar graphs. Also, every graph having a k-face satisfies $\chi_{c,k} \ge k$.

The only results known for the *k-cyclic choosability* $\chi_{c,k}^{\text{list}}$ are $\chi_{c,3}^{\text{list}} \le 5$ (Thomassen [177]) and $\chi_{c,4}^{\text{list}} \le 7$ (Wang–Lih [195]).

6.1. Arbitrary plane graphs

The best-known lower bound on $\chi_{c,k}$ for graphs having a k-face, where $k \ge 4$, is $\chi_{c,k} \ge \lfloor \frac{3k}{2} \rfloor$; it is attained by the graph obtained from the 3-prism by replacing the three vertical edges by paths of length $\lfloor \frac{k}{2} \rfloor - 1$, $\lfloor \frac{k}{2} \rfloor - 1$, and $\lceil \frac{k}{2} \rceil - 1$, respectively (Borodin [17]).

Conjecture 6.1 (*Borodin [17]*). *Every plane graph is k-cyclically $\lfloor \frac{3k}{2} \rfloor$-colorable.*

Ore and Plummer [158] proved $\chi_{c,k} \le 2k$ for $k \ge 3$. There are tight bounds $\chi_{c,3} \le 4$ (by the 4CT) and $\chi_{c,4} \le 6$ (by Theorem 4.1). For larger k, Ore and Plummer's bound was first improved by Borodin [28] in 1992 to $\chi_{c,5} \le 9$, $\chi_{c,6} \le 10$, $\chi_{c,7} \le 12$, and $\chi_{c,k} \le 2k-3$ for $k \ge 8$, and then further improved in Borodin, Sanders, Zhao [73] to $\chi_{c,5} \le 8$ and $\chi_{c,k} \le \lfloor \frac{9k}{5} \rfloor$ for $k \ge 6$. Recently Havet, Sereni, and Škrekovski [106] proved $\chi_{c,7} \le 11$.

Theorem 6.1 (*Sanders–Zhao [171]*). *Every plane graph satisfies $\chi_{c,k} \le \lceil \frac{5k}{3} \rceil$.*

Borodin et al. [31] introduced a new parameter k^*, the maximum number of vertices which two faces have in common, and proved $\chi_{c,k} \le \max\{k + 3k^* + 2, k + 14, 3k^* + 6, 18\}$ for all $k \ge 3$ and $k^* \ge 2$, which implies $\chi_{c,k} \le k + 3k^* + 2$ if $k \ge 4$ and $k^* \ge 4$.

Conjecture 6.2 (*Borodin, Broersma, Glebov, van den Heuvel [31]*). *If k and k^* are large enough, then $\chi_{c,k} \le k+k^*$, which implies $\chi_{c,k} \le \lfloor \frac{3}{2}k \rfloor$ if k is large enough.*

6.2. 3-connected planar graphs

For the polyhedral graphs, which by Steinitz' Theorem of 1922 correspond to 3-connected planar graphs, Plummer and Toft [160] proved $\chi_{c,k} \le k + 9$ whenever $k \ge 3$.

Conjecture 6.3 (*Plummer–Toft [160]*). *Every polyhedral graph satisfies $\chi_{c,k} \le k + 2$.*

The case $k = 4$ follows from Theorem 4.1(iii). For $k \ge 24$, the Plummer–Toft Conjecture was confirmed in Horňák–Jendrol' [110], and recently Horňák and Zlámalová [111] settled the cases $18 \le k \le 23$.

For large k, Borodin and Woodall [74] established the tight bound $\chi_k \le k + 1$ (in fact, for $k \ge 122$), which is attained by the k-pyramid. Enomoto, Horňák, and Jendrol' [95], using an idea from [74], proved $\chi_k \le k + 1$ for $k \ge 60$. It would be interesting to explore the choosability version of these results:

Problem 6.1. Is it true that every polyhedral graph satisfies $\chi_{c,k}^{\text{list}} \le k + 1$ for all large enough k?

In fact, the only known k for which the bound $\chi_{c,k} \le k + 2$ is tight is $k = 4$, and this prompts the next question.

Problem 6.2. Is it true that every polyhedral graph satisfies $\chi_{c,k} \le k + 1$ for all $k \ge 5$?

Trivially, $\chi_{c,k} \ge k$ when G has a k-face. It would be interesting to know when equality holds. Back in 1996, O.V. Borodin and D.R. Woodall suggested as a possibility (unpublished) that the only obstacle for a 3-polytope to satisfy $\chi_{c,k} = k$, where k is large enough, is the presence of what they called a *class monitor*: a vertex not incident with a k-face f but lying in common k^--faces with every vertex in the boundary of f.

7. Three Color Problem

It is well known that the problem of deciding whether a planar graph is 3-colorable is NP-complete. A lot of research has been devoted to sufficient conditions for a planar graph to be 3-colorable. Due to the famous Grötzsch Three Color Theorem [98], all further sufficient conditions allow 3-cycles.

Theorem 7.1 (*Grötzsch [98]*). *Every plane graph without* 3*-cycles is* 3*-colorable; moreover, every proper* 3*-coloring of a* 4*- or* 5*-cycle can be extended to a* 3*-coloring of the whole graph.*

Thomassen [179] gave a short elegant proof of Theorem 7.1. Grünbaum [100] extended Grötzsch's Theorem by allowing at most three 3-cycles, which is best possible due to K_4. To prove this extension, he used an auxiliary statement on the possibility of coloring extension. However, a counterexample to Grünabum's auxiliary statement was found by T. Gallai. Aksenov [3] proved that the important first part of Grünbaum's theorem is true. Borodin [20] gave a new short proof, free of coloring extension arguments, of what is now called the Grünbaum–Aksenov Theorem. Erdős (see [175]) posed a problem of describing all edge-minimal 4-chromatic planar graphs with four triangles.

The *distance* $d(T_1, T_2)$ between triangles T_1 and T_2 in a graph is the length of a shortest path joining T_1 and T_2. In particular, $d(T_1, T_2) = 0$ if T_1 and T_2 share a vertex. A cycle C is *triangular* if it is adjacent to a triangle other than C. We note that a triangular k-cycle without non-triangular chords either is contained in a triangular $(k+1)$-cycle if it does not have a triangular chord, or contains a triangular $(k-1)$-cycle otherwise. Let d^{∇} denote the minimum distance between distinct triangles.

As early as 1970, Havel [103] asked if every planar graph with large enough d^{∇} is 3-colorable. There are 4-chromatic planar graphs with $d^{\nabla} = 1$ and $d^{\nabla} = 2$ (Havel [104,103]) and $d^{\nabla} = 3$ (Aksenov and Mel'nikov [6], modifying Havel's constructions, and Steinberg, using a different idea (see [6])). The first breakthrough in the positive direction of Havel's Problem was made only in 2003 by Borodin and Raspaud [72], who proved that every planar graph with $d^{\nabla} \geq 4$ and no 5-cycles is 3-colorable. Recently, Havel's Problem was solved by Dvořák, Král, and Thomas [93] in the positive.

Theorem 7.2 (*Dvořák, Král, Thomas [93]*). *There exists a constant d such that every planar graph with $d^{\nabla} \geq d$ is* 3*-colorable.*

Despite this outstanding achievement, the following strongest possible version of Havel's Problem deserves attention:

Conjecture 7.1 (*Borodin–Raspaud [72]*). *Every planar graph with $d^{\nabla} \geq 4$ is* 3*-colorable.*

The first step towards confirming Conjecture 7.1 was made in Borodin–Raspaud [72]. A joint extension of the Grötzsch Theorem and the result in [72] is expressed by the following.

Theorem 7.3 (*Borodin, Glebov, Jensen [39]*). *Every planar graph with $d^{\nabla} \geq 4$ and without triangular* 5*-cycles is* 3*-colorable.*

The authors of [39] believe that the next conjecture is about halfway from Theorem 7.3 to Conjecture 7.1.

Conjecture 7.2 (*Borodin, Glebov, Jensen [39]*). *Every planar graph with $d^{\nabla} \geq 4$ and without triangular* 4*-cycles is* 3*-colorable.*

Another important direction in 3-coloring was initiated by Steinberg (see [175]).

Conjecture 7.3 (*Steinberg, 1976*). *Every planar graph without* 4*-cycles and without* 5*-cycles is* 3*-colorable.*

There had been no progress in Conjecture 7.3 for a long time, until Erdős (see [175]) suggested a relaxation of this problem: does there exist a constant C such that the absence in a planar graph of cycles of length from 4 to C guarantees its 3-colorability? Abbott and Zhou [1] proved that such a C exists, with $C \leq 11$. This result was later on improved to $C \leq 9$ in Borodin [23] and Sanders–Zhao [172], and to $C \leq 7$ in Borodin et al. [43].

Theorem 7.4 (*Borodin, Glebov, Raspaud, Salavatipour [43]*). *Every planar graph without cycles of length from* 4 *to* 7 *is* 3*-colorable.*

Borodin et al. [41] improved the result in [43] by proving that every planar graph without 5- and 7-cycles and without adjacent triangles is 3-colorable; they also showed counterexamples to the proof of the same result given in Xu [205].

Conjecture 7.4 (*Borodin–Raspaud [72]*). *Every planar graph with $d^{\nabla} \geq 1$ and without* 5*-cycles is* 3*-colorable.*

This *Bordeaux 3-Color Conjecture* (Bx3CC) obviously has common features with Havel's and Steinberg's Problems. It follows from well-known constructions that both Steinberg's Conjecture and Bx3CC are tight, and their assumptions cannot be relaxed. Indeed, a 4-chromatic graph without 5-cycles can be obtained from a 7-cycle $v_1 \cdots v_7$ by adding vertices w_{2k}, where $1 \leq k \leq 3$, and joining every w_{2k} to $v_{2k-1}, v_{2k}, v_{2k+1}$. A construction of a 4-chromatic C_4-free graph with $d^{\nabla} = 1$ is: replace every edge $a_i a_j$ of K_4 by Havel's graph [104], which is obtained from a 6-cycle $a_i u_{ij} u'_{ij} a_j w_{ij} w'_{ij}$ by adding a vertex x_{ij} joined to u_{ij} and u'_{ij} and vertex y joined to x_{ij}, w_{ij}, w'_{ij}.

The above-mentioned result in Borodin–Raspaud [72] was strengthened to $d^{\nabla} \geq 3$ in Borodin–Glebov [36] and Xu [206]. Recently, the penultimate step towards Bx3CC was made:

Theorem 7.5 (*Borodin–Glebov [37]*). *Every planar graph with $d^\nabla \geq 2$ and without 5-cycles is 3-colorable.*

We note that the above-mentioned results towards Steinberg's Conjecture and Bx3CC [1,23,36,37,43,72,172,206] do not imply Grötzsch's Theorem [98]. A common generalization of the Grötzsch Theorem and that in [23,172] was obtained in Borodin et al. [40]:

Theorem 7.6 (*Borodin, Glebov, Jensen, Raspaud [40]*). *Every planar graph without triangular cycles of length from 4 to 9 is 3-colorable.*

The following *Novosibirsk 3-Color Conjecture* (Nsk3CC) is posed in [40]:

Conjecture 7.5 (*Borodin, Glebov, Jensen, Raspaud [40]*). *Every planar graph without triangular cycles of length 4 or 5 (or, equivalently, 3 or 5) is 3-colorable.*

We note that Nsk3CC is stronger than both Steinberg's Conjecture and Bx3CC. Theorems 7.3 and 7.6 confirm relaxations of Nsk3CC. Another relaxation of Nsk3CC is proved in Borodin, Glebov, Raspaud [42]:

Theorem 7.7 (*Borodin, Glebov, Raspaud [42]*). *Every planar graph without triangular cycles of length from 4 to 7 (or, which is equivalent, without triangular cycles of length from $\{3, 5, 7\}$ or $\{4, 5, 7\}$) is 3-colorable.*

In particular, Theorem 7.7 implies the 3-colorability of every planar graph such that the set of lengths of its cycles is disjoint from one of the sets $\{4, 5, 7\}$, $\{4, 6, 7\}$ or $\{4, 6, 8\}$, and thus absorbs the results in [1,23,40,41,43,71,80,82,136,192,193,198].

Aksenov [4] and, independently, Jensen and Thomassen [117] proved the following strengthening of the Grötzsch Theorem: for any vertices x and y in a plane triangle-free graph G, there is a 3-coloring of G such that x and y are colored differently. Aksenov, Borodin, and Glebov [5] proved a similar stronger result: for any nonadjacent vertices x and y there is a 3-coloring such that x and y are colored the same.

We conclude this section with a classical result from the nineteenth century (see [118, p. 6]).

Theorem 7.8 (*Heawood, 1898*). *A plane triangulation is 3-colorable if and only if all its vertices have even degrees.*

8. List 3-coloring

In contrast with Grötzsch's Theorem, Voigt [186] presented a triangle-free planar graph with $\chi^{\text{list}} = 4$. On the other hand, Thomassen [178] proved the following choosability version of Grötzsch's Theorem.

Theorem 8.1 (*Thomassen [178]*). *Every planar graph without 3- and 4-cycles is 3-choosable.*

Voigt [188] constructed a non-3-choosable planar graph without cycles of length 4 and 5, so Steinberg's Conjecture does not extend to 3-choosability. Montassier, Raspaud, and Wang [147] showed that Bx3CC does not hold for 3-choosability either. The proof of the next result follows easily from the simple structural Theorem 2.7, but, surprisingly, the bound 9 in Theorem 8.2 remains best-known for already more than 15 years.

Theorem 8.2 (*Borodin [23]*). *Every planar graph without cycles of length from 4 to 9 is 3-choosable.*

Problem 8.1 (*Borodin [23]*). Prove that every planar graph without cycles of length from 4 to 8 is 3-choosable.

Augmenting the argument in [23], we easily deduce the following unpublished refinement of Theorem 8.2.

Corollary 8.1. *Every planar graph without 4-cycles, 5-cycles, and without k-cycles adjacent to at least $k - 6$ triangles, where $6 \leq k \leq 9$, is 3-choosable.*

Proof. In [23], a k-face f, where $6 \leq k \leq 9$, has initial charge $k-4$ and gives $\frac{2}{3}$ to any incident 3-vertex that is incident with a 3-face and at most $\frac{1}{3}$ to each other incident vertex. Therefore, the new charge of f is at least $k-4-2(k-6) \times \frac{2}{3} - \frac{k-2(k-6)}{3}$, which is 0. □

On the other hand, combining results in certain $\binom{4}{2}$ papers (for the references, see Wang, Lu, Chen [197]) yields another refinement of Theorem 8.2.

Theorem 8.3 (*Ming Chen, H. Lu, L. Shen, Y. Wang, B. Wu, L. Zhang*). *Every planar graph without cycles of lengths 4, 9, and any two lengths from $\{5, 6, 7, 8\}$ is 3-choosable.*

The following sufficient conditions for 3-choosability in terms of the minimum distance between small cycles are known.

Theorem 8.4 (*Montassier, Raspaud, Wang, Wang [150]*). *Every planar graph in which the 5^--cycles are at distance at least 4 from each other is 3-choosable.*

Theorem 8.5 (*Montassier, Raspaud, Wang [147]*). *A planar graph is 3-choosable if:*

(i) $d^{\nabla} \geq 4$, *and it has neither 4- nor 5-cycles, or*
(ii) $d^{\nabla} \geq 3$, *and it has no 4- to 6-cycles.*

We note that Theorem 8.5(ii) follows also from Corollary 8.1.

Problem 8.2 (*Montassier, Raspaud, Wang [147]*).

(A) Find best-possible restrictions on d^{∇} in Theorem 8.5.
(B) Is it true that every planar C_5-free graph with large enough d^{∇} is 3-choosable?

Some other results and problems on planar 3-choosability can be found in the references to the papers mentioned in this section. We conclude with a famous result of Alon and Tarsi [11].

Theorem 8.6 (*Alon–Tarsi [11]*). *Every bipartite planar graph is 3-choosable.*

9. Acyclic coloring

In 1973, Grünbaum [99] defined a proper coloring of a graph to be *acyclic* if every cycle uses at least three colors and proved that every planar graph is acyclically 9-colorable. This bound was improved to 8, 7, and 6 in Mitchem [142], Albertson–Berman [7], and Kostochka [126], respectively.

Theorem 9.1 (*Borodin [18]*). *Every planar graph is acyclically 5-colorable.*

The bound of 5 is best possible; moreover, there are bipartite 2-degenerate planar graphs that are not acyclically 4-colorable (Kostochka–Mel'nikov [127]). We first note that in any acyclic 4-coloring of $K_{2,4}$, the two 4-vertices must have distinct colors. Now take a 5-cycle $x_1 \cdots x_5$ and replace every edge $x_i x_{i+1}$, where $1 \leq i \leq 5$ (addition modulo 5), by a copy of $K_{2,4}$ with the 4-vertices x_i and x_{i+1}. Finally, we add vertices y and z and join each of them with all x_i's. The resulting graph needs three colors for $\{x_1, \ldots, x_5\}$ and two more colors for y and z.

Acyclic coloring turns out to be useful in obtaining results about other types of colorings and partitions. For example, Hakimi, Mitchem, and Schmeichel [102] solved a problem posed in Algor–Alon [9] by proving that the star arboricity of planar graphs is at most 5 (which means that the edges can be split into five forests such that every connected component is a star), as follows. We acyclically 5-color the vertices by Theorem 9.1 and orient every 2-colored tree away from an arbitrary root. The i-th star forest now consists of all arcs leaving the vertices colored i, where $1 \leq i \leq 5$.

In Sections 10 and 14, we consider best-known upper bounds for the oriented and star chromatic numbers of planar graphs deduced, respectively, from Theorem 9.1 in Raspaud–Sopena [161] and from Theorem 9.1 combined with the 4CT in Albertson et al. [8].

Two more applications of the *acyclic chromatic number* χ_a in general and of the bound $\chi_a \leq 5$ for planar graphs in particular are: every m-coloring of the edges of any graph G can be homomorphically mapped to an m-coloring of the edges of a graph with at most $\chi_a m^{\chi_a - 1}$ vertices (Alon–Marshall [10]), and every mixed graph G (that is a graph having both edges and arcs) with any m-coloring of its edges combined with any n-coloring of its arcs can be homomorphically mapped to an m-coloring of a mixed graph with at most $\chi_a(2n+m)^{\chi_a - 1}$ vertices (Nešetřil–Raspaud [152]).

We recall that every edgeless graph is 0-degenerate, and every forest is 1-degenerate. An acyclic coloring is *strong* if every 3-colored subgraph is 2-degenerate, and every 4-colored subgraph is 3-degenerate. Recently, Rautenbach [162] proved that 18 colors in the next old conjecture suffice.

Conjecture 9.1 (*Borodin [18]*). *Every planar graph has a strong acyclic 5-coloring.*

It is easy to see that every planar graph G is 6-choosable. Indeed, every subgraph of G has a vertex of degree at most 5 due to (1) in Section 3.1, which means that G is 5-degenerate and we can color it by induction. In contrast with the 4CT, Voigt [187] presented a non-4-choosable planar graph. As already mentioned, the following famous Five Choosability Theorem by Thomassen [177] has the extraordinary feature that its proof does not use Euler's formula.

Theorem 9.2 (*Thomassen [177]*). *Every planar graph is 5-choosable.*

Borodin et al. [35] proved that every planar graph is acyclically 7-choosable and conjectured a common extension of Theorems 9.1 and 9.2.

Conjecture 9.2 (*Borodin, Fon-Der-Flaass, Kostochka, Raspaud, Sopena [35]*). *Every planar graph is acyclically 5-choosable.*

This challenging conjecture seems to be difficult. As yet, it has been verified only for several restricted classes of planar graphs: those of girth at least 5 (Montassier, Ochem, Raspaud [145]), without 4- and 5-cycles (Montassier, Raspaud, Wang [148]), without 4- and 6-cycles [148], with neither 4-cycles nor chordal 6-cycles (Zhang–Xu [208]), with neither 4-cycles nor two 3-cycles at distance less than 3 (Chen–Wang [83]), and without 4-cycles and intersecting 3-cycles (Chen–Raspaud [77]). Wang and Chen [194] proved that all planar graphs without 4-cycles are acyclically 6-choosable.

Recently, Borodin and Ivanova [47] proved that a planar graph is acyclically 5-choosable if it does not contain an i-cycle adjacent to a j-cycle, where $3 \leq j \leq 5$ if $i = 3$ and $4 \leq j \leq 6$ if $i = 4$, which absorbs the above-mentioned results in [145,148,208]. Also, Borodin and Ivanova [48] proved a common strengthening of the above-mentioned results in [83,77, 145,148,194,208]:

Theorem 9.3 (*Borodin–Ivanova [48]*). *Every planar graph without* 4*-cycles is acyclically* 5*-choosable.*

Problem 9.1 (*Borodin and Ivanova*). Is it true that every planar graph without 4-cycles adjacent to 3-cycles or 4-cycles is acyclically 5-choosable?

Some sufficient conditions are also obtained for a planar graph to be acyclically j-colorable or j-choosable, where $j \in \{3, 4\}$. Borodin, Kostochka, and Woodall [70] showed that $\chi_a \leq 4$ if $g \geq 5$ and $\chi_a \leq 3$ if $g \geq 7$. Recently, $\chi_a^{\text{list}} \leq 3$ was proved if $g \geq 7$ (Borodin et al. [34]) or if G has no cycles of length from 4 to 12 (Borodin [29] and, independently, Hocquard–Montassier [109]), which was strengthened to the absence of 4- to 11-cycles in Borodin–Ivanova [45].

Conjecture 9.3 (*Borodin, Chen, Ivanova, Raspaud [34]*). *Every planar graph with girth at least* 5 *is acyclically* 3*-choosable.*

Problem 9.2 (*Borodin–Ivanova [45]*). Find the smallest k such that every planar graph without cycles of length from 4 to k is acyclically 3-choosable.

The bound $\chi_a^{\text{list}} \leq 4$ was proved in the following cases: $g \geq 5$ (Montassier [144]), there are no 4-, 5-, and 6-cycles (Montassier, Raspaud, Wang [149]), there are no 4-, 6-, and 7-cycles (Chen, Raspaud, Wang [81]), and there are no 4-, 6-, and 8-cycles [81]. Recently, Borodin, Ivanova, and Raspaud [63] gave a common extension of the results in [81,144,149] by proving $\chi_a^{\text{list}} \leq 4$ under the absence of 4-cycles and triangular 6-cycles.

Furthermore, Montassier, Raspaud, and Wang [149] proved $\chi_a^{\text{list}} \leq 4$ for every planar graph without 4-, 5-, and 7-cycles, or without 4-, 5- and intersecting 3-cycles, while Chen and Raspaud [78] proved this assuming that the graph under consideration has neither 4- and 5-cycles nor 8-cycles with a triangular chord. Borodin [30] proved $\chi_a \leq 4$ for all planar graphs having neither 4- nor 5-cycles. The above-mentioned results in [30,78,145,149] are subsumed by the following result, obtained independently in [49,79].

Theorem 9.4 (*Borodin–Ivanova [49], Chen–Raspaud [79]*). *Every planar graph without* 4*- and* 5*-cycles is acyclically* 4*-choosable.*

Conjecture 9.4 (*Montassier, Raspaud, Wang [149]*). *Every planar graph without* 4*-cycles is acyclically* 4*-choosable.*

We recall that there are bipartite planar graphs that are not acyclically 4-colorable (Kostochka–Mel'nikov [127]). These graphs in [127] contain many 4-cycles. Therefore, to obtain acyclically 4-choosable planar graphs, we must impose restrictions on 4-cycles. It can be observed that in all previously mentioned results on $\chi_a^{\text{list}} \leq 4$, 4-cycles are completely forbidden. In the next theorem, 4-cycles are allowed provided that they are not adjacent to relatively short cycles.

Theorem 9.5 (*Borodin–Ivanova [50]*). *A planar graph is acyclically* 4*-choosable if it does not contain an i-cycle adjacent to a j-cycle, where $3 \leq j \leq 6$ if $i = 3$ and $4 \leq j \leq 7$ if $i = 4$.*

It is easy to see that Theorem 9.5 is a common strengthening of the results in [63,81,144,149]. This suggests a common strengthening of Conjecture 9.4 and Problem 9.1:

Problem 9.3 (*Borodin–Ivanova [50]*). Is it true that every planar graph without 4-cycles adjacent to 3- or 4-cycles is acyclically 4-choosable?

Proposition 9.1. *For every integer $n \geq 3$ there is a nonplanar graph with $\chi_a = 3$ and $\chi_a^{\text{list}} > n$.*

Proof. We combine ideas in Vizing [182] and Kostochka–Mel'nikov [127]. Take a complete bipartite graph K_{n,n^n} with partite sets $X = \{x_1, \dots, x_n\}$ and $Z = \{z_1, \dots, z_{n^n}\}$. The vertices of X are assigned disjoint lists $L_1, \dots, L_n$ of n colors each, and the vertices of Z are assigned all possible n^n lists $\{\alpha_1, \dots, \alpha_n\}$ such that $\alpha_i \in L_i$ for every i. Now replace every edge $x_i z_k$ by vertices $y_{i,j,k}$, where $1 \leq j \leq n+1$, each of which is joined to x_i and z_k. We obtain $(n+1)n^{n+1}$ independent vertices $y_{i,j,k}$ and assign to all of them the same arbitrary list L of cardinality n.

Our tripartite graph is acyclically 3-colorable, since we can color every x_i with 1, every $y_{i,j,k}$ with 2, and every z_k with 3. Now suppose there is an acyclic coloring c chosen from the collection of lists described above. By construction, there is a k^* such that the list on z_{k^*} is $\{c(x_1), \dots, c(x_n)\}$, and hence there is an i^* such that $c(x_{i^*}) = c(z_{k^*})$. Furthermore, there are j_1 and j_2 such that $c(y_{i^*,j_1,k^*}) = c(y_{i^*,j_2,k^*})$. Hence we have found a 2-colored 4-cycle in c, a contradiction. □

Problem 9.4. Is it true that every planar graph satisfies $\chi_a = \chi_a^{\text{list}}$?

10. Oriented coloring

Oriented graphs are directed graphs that are orientations of simple graphs. An *oriented k-coloring* of an oriented graph G is a function $\varphi: V(G) \rightarrow \{1, \ldots, k\}$ such that there are no arcs (x_1y_1) and (y_2x_2) in G such that $\varphi(x_1) = \varphi(x_2)$ and $\varphi(y_1) = \varphi(y_2)$. The *oriented chromatic number* is denoted by χ_{or}. Raspaud and Sopena [161] bounded χ_{or} in terms of the acyclic chromatic number.

Theorem 10.1 (*Raspaud–Sopena [161]*)**.** *Every simple graph satisfies* $\chi_{or} \leq \chi_a 2^{\chi_a - 1}$.

By means of monadic second order logic, Courcelle [86] proved that every planar graph satisfies $\chi_{\text{or}} \leq 3^{63}$, which was improved to $\chi_{\text{or}} \leq 80$ in Raspaud–Sopena [161] by combining Theorems 9.1 and 10.1. If $g \geq 4$, then $\chi_{\text{or}} \leq 59$ (Ochem [155]) and $\chi_{\text{or}} \leq 47$ (Borodin–Ivanova [51]). Borodin et al. [65] proved $\chi_{\text{or}} \leq 19$ if $g \geq 5$, $\chi_{\text{or}} \leq 11$ if $g \geq 6$, $\chi_{\text{or}} \leq 7$ if $g \geq 8$, and $\chi_{\text{or}} \leq 5$ if $g \geq 14$, where 14 was replaced by 13 in Borodin et al. [64]. Best-known upper bounds can be summarized as follows.

Theorem 10.2. *The following bounds for the oriented chromatic number of planar graphs hold:*

- *if* $g \geq 3$, *then* $\chi_{or} \leq 80$ *(Raspaud–Sopena* [161]*),*
- *if* $g \geq 4$, *then* $\chi_{or} \leq 40$ *(Ochem–Pinlou* [156]*),*
- *if* $g \geq 5$, *then* $\chi_{or} \leq 16$ *(Pinlou* [159]*),*
- *if* $g \geq 6$, *then* $\chi_{or} \leq 11$ *(Borodin, Kostochka, Nešetřil, Raspaud, Sopena* [65]*),*
- *if* $g \geq 7$, *then* $\chi_{or} \leq 7$ *(Borodin–Ivanova* [52]*),*
- *if* $g \geq 11$, *then* $\chi_{or} \leq 6$ *(Ochem–Pinlou* [157]*),*
- *if* $g \geq 12$, *then* $\chi_{or} \leq 5$ *(Borodin, Ivanova, Kostochka* [59]*).*

On the other hand, there are planar graphs with $\chi_{\text{or}} = 5$ and arbitrarily large g (Nešetřil, Raspaud, Sopena [153]), with $\chi_{\text{or}} = 11$ and $g = 4$ (Ochem [155]), and with $\chi_{\text{or}} = 17$ and $g = 3$ (Marshall [139]).

11. Circular coloring

A *circular* (k, d)*-coloring* of a graph G, introduced by A. Vince in 1988, is a function $\phi: V(G) \rightarrow \{0, \ldots, k-1\}$ such that $d \leq |\phi(u) - \phi(v)| \leq k - d$ for every edge $uv \in E(G)$. Such a coloring is "circular" in the sense that we may view the k colors as points on a circle, where the colors on adjacent vertices must be at least d positions apart on the circle. We note that circular $(k, 1)$-coloring is the same as proper k-coloring. In the literature on circular coloring, "circular (k, d)-coloring" is commonly called "(k, d)-coloring"; here we add "circular" to distinguish it from other two-parameter coloring models.

It is not hard to see that G has a circular $(2t + 1, t)$-coloring if and only if it has a homomorphism into the cycle C_{2t+1} (in this coloring, the neighbors of a vertex colored, say, with 0 are allowed to be colored only either with t or with $t + 1$). A relaxation for planar graphs of Jaeger's Conjecture [116] on nowhere-zero flows states the following:

Conjecture 11.1 (*Jaeger [116]*)**.** *For every positive integer* t*, every planar graph with girth at least* $4t$ *has a circular* $(2t + 1, t)$*-coloring.*

When $t = 1$, Conjecture 11.1 is confirmed by Grötzsch's Three Color Theorem. The conjecture is sharp if true, as shown by M. DeVos (2000) as follows:

Take a wheel consisting of a central vertex z adjacent to every vertex of a cycle C with $4t - 1$ vertices. Replace each edge from z to C by a path of length $2t - 1$. The resulting graph G has girth $4t - 1$. If G has a circular $(2t + 1, t)$-coloring φ, then by symmetry we may assume $\varphi(z) = 0$. Along each path from z to C, the set of allowable colors increases by one with each step, but in fact the paths forbid color 0 from C. Since color 0 cannot be used on C, φ must provide a homomorphism from the odd cycle C into a bipartite graph, which is impossible.

Nešetřil and Zhu [154] proved a relaxation of Conjecture 11.1 with $g \geq 10t - 4$, which was improved to $g \geq 8t - 3$ by Zhu [209].

Theorem 11.1 (*Borodin, Kim, Kostochka, West [64]*)**.** *For every positive integer* t*, every planar graph with girth at least* $\frac{20t-2}{3}$ *has a circular* $(2t + 1, t)$*-coloring.*

For $t = 2$ we can achieve more, using the recent idea of global discharging explained at the end of Section 1.4.

Theorem 11.2 (*Borodin, Hartke, Ivanova, Kostochka, West [44]*)**.** *Every planar graph with* $g \geq 12$ *has a circular* $(5, 2)$*-coloring.*

12. 2-distance coloring

A coloring $\varphi: V(G) \to \{1, \ldots, k\}$ of G is 2-*distance* if any two vertices at distance at most 2 from each other get different colors. The minimum number of colors in 2-distance colorings of G is its 2-*distance chromatic number*, denoted by $\chi_2(G)$.

Note that $\chi_2(G) = \chi(G^2)$, where G^2 is the square of G. This is the most explored special case ($p = q = 1$) of $L(p, q)$-labeling. Results on χ_2 often can be extended to larger p and q, and also to the choosability version of these problems.

Back in 1977, Wegner [201] posed the following conjecture, which still remains open for all Δ.[1]

Conjecture 12.1 (*Wegner [201]*). *Every planar graph satisfies*
$\chi_2 \le 7$ *if* $\Delta = 3$,
$\chi_2 \le \Delta + 5$ *if* $4 \le \Delta \le 7$, *and*
$\chi_2 \le \lfloor \frac{3\Delta}{2} \rfloor + 1$ *otherwise.*

The following upper bounds were established: $\lceil \frac{9\Delta}{5} \rceil + 1$ for $\Delta \ge 749$ (Agnarsson–Halldórsson [2]), $\lceil \frac{9\Delta}{5} \rceil + 1$ for $\Delta \ge 47$ (Borodin et al. [32,33]), $\lceil \frac{5\Delta}{3} \rceil + 78$ for all Δ (Molloy–Salavatipour [143]), $\lceil \frac{5\Delta}{3} \rceil + 25$ for $\Delta \ge 241$ [143], and $\frac{3\Delta}{2} + o(\Delta)$ (Havet et al. [107]). The strong result in [107] was proved by a probabilistic argument.

For planar graphs with given girth, it is sometimes possible to prove sharp bounds. For every graph, we have $\chi_2^{\text{list}} \ge \chi_2 \ge \Delta + 1$, simply because any vertex of maximum degree along with its neighbors need this number of colors. It turns out that $\chi_2^{\text{list}} = \chi_2 = \Delta + 1$ if $g \ge 7$ and Δ is large enough (Borodin, Ivanova, Neustroeva [60], Dvořák et al. [91], Ivanova[114]). More specifically, $\chi_2^{\text{list}} = \Delta + 1$ holds in each of the following cases:

(i) $\Delta \ge 16$ and $g = 7$;
(ii) $\Delta \ge 10$ and $8 \le g \le 9$;
(iii) $\Delta \ge 6$ and $10 \le g \le 11$;
(iv) $\Delta = 5$ and $g \ge 12$ (all cases $\Delta \ge 5$ are due to Ivanova [114]);
(v) $\Delta = 4$ and $g \ge 15$ (Borodin, Ivanova, Neustroeva [61]).

On the other hand, for $g = 6$ there are planar graphs such that $\chi_2 = \Delta + 2$ (Borodin et al. [38] and, independently, Dvořàk et al. [92]). Dvořàk et al. [91] proved $\chi_2 \le \Delta + 2$ if $\Delta \ge 8821$, for which the restriction on Δ was lowered to 18 in Borodin–Ivanova [53]. Also, $\chi_2^{\text{list}} \le \Delta + 2$ if $\Delta \ge 24$ (Borodin–Ivanova [54,55]).

However, as proved in Borodin, Ivanova, Neustroeva [62], $\chi_2 = \Delta + 1$ holds for $g = 6$ whenever $\Delta \ge 31$ under the additional assumption that every edge is incident with a vertex of degree 2. Such graphs can be obtained from planar graphs with $\delta \ge 3$ by putting *at least* one vertex of degree 2 on every edge. The *1-subdivision* of a graph is obtained by putting *precisely* one 2-vertex on every edge. Note that a total coloring of G is a 2-distance coloring of the 1-subdivision of G. Thus results on total coloring can sometimes be extended to 2-distance coloring of arbitrarily subdivided graphs (with worse restrictions on Δ; see Section 5.3). For example, Borodin, Kostochka, and Woodall [69] proved $\chi_{ve} = \Delta + 1$ if $g \ge 6$ and $\Delta \ge 4$ (compare with $\Delta \ge 31$ in [62]).

Kostochka and Woodall [128] extended to 2-distance coloring the List Total Coloring Conjecture that every graph satisfies $\chi_{ve}^{\text{list}} = \chi_{ve}$:

Conjecture 12.2 (*Kostochka–Woodall [128]*). *For every graph,* $\chi_2^{\text{list}} = \chi_2$.

Since there are planar graphs with $g \le 4$ and $\chi_2 = \lfloor \frac{3\Delta}{2} \rfloor + 1$ for arbitrarily large Δ (join vertices u, v, and w by paths ux_iv, uy_jw, and vz_kw, where $1 \le i, j \le \lfloor \frac{\Delta}{2} \rfloor$, $1 \le k \le \lceil \frac{\Delta}{2} \rceil - 1$, and join v with w by an edge), whereas the bound $\chi_2^{\text{list}} \le \Delta + 2$ for $g = 6$ and large enough Δ is sharp, the next question seems intriguing.

Problem 12.1 (*Dvořák, Kràl, Nejedlỳ, Škrekovski [91], Borodin–Ivanova [54]*). Is it true that every planar graph with $g = 5$ satisfies $\chi_2^{\text{list}} \le \Delta + 2$, or at least $\chi_2 \le \Delta + 2$, for all large enough Δ?

For all C_4-free planar graphs, Wang and Cai [191] proved $\chi_2 \le \Delta + 48$, so it is natural to ask this question:

Problem 12.2 (*Wang–Cai [191]*). Find the smallest integer k such that every planar graph without 4-cycles satisfies $\chi_2 \le \Delta + k$.

A lot of research has been devoted to various colorings of graphs with maximum degree 3 (called *subcubic graphs*). Already in 1977, Wegner [201] proved that eight colors suffice for 2-distance coloring of such planar graphs. Cranston and Kim [88] proved $\chi_2^{\text{list}} \le 8$ and $\chi_2^{\text{list}} \le 7$ if $g \ge 7$. It was proved in Havet [105] and [88] that $\chi_2^{\text{list}} \le 6$ holds if $g \ge 9$. The bound $\chi_2 \le 5$ was obtained for $g \ge 14$ (Montassier–Raspaud [146] and Dvořák, Škrekovski, Tancer [94]), after which the restriction was weakened to $g \ge 13$ (Ivanova–Solov'eva [115] and Havet [105]), and to $g \ge 12$ (Borodin–Ivanova [46]). Finally, $\chi_2 = 4$ was proved for $g \ge 24$ (Borodin, Ivanova, Neustroeva [61] and Dvořák, Škrekovski, Tancer [94]), and for $g \ge 22$ (Borodin–Ivanova [56]).

[1] As reported in Kramer, Kramer [131], some authors announced in their papers that recently C. Thomassen had solved this conjecture for $\Delta = 3$, but C. Thomassen confirmed to the authors of [131] in private communication that the problem must still be considered as an open problem.

Problem 12.3. For every integer k, find the smallest integers $g(k)$ and $g^{\text{list}}(k)$ such that every planar subcubic graph is 2-distance k-colorable if its girth g is at least $g(k)$ and is 2-distance k-choosable if $g \geq g^{\text{list}}(k)$.

13. Injective coloring

A coloring of a graph is *injective* if any two vertices having a common neighbor get different colors. The parameters for the number of colors needed to be available at each vertex in the list and non-list versions are the *injective chromatic number* $\chi_{\text{in}}^{\text{list}}$ and *injective choosability* χ_{in}, respectively. By definition, $\chi_{\text{in}}^{\text{list}} \geq \chi_{\text{in}} \geq \Delta$ holds for every graph. Injective coloring originated in complexity theory, and it is used in coding theory (see [101]) and in the designing of computer networks [15].

We note that an injective coloring is not necessarily proper, and this is its only difference from 2-distance coloring. In fact, it is the only improper coloring considered in this survey.

Injective coloring of planar graphs has been studied much less than 2-distance coloring. For triangle-free graphs, this coloring is just the case $p = 0, q = 1$ of $L(p, q)$-labeling. Also, proper coloring can be regarded as $L(1, 0)$-labeling.

It is easy to construct a planar graph with $g = 4$, arbitrary even Δ, and $\chi_{\text{in}} = \frac{3}{2}\Delta$ (it suffices to replace every edge of a triangle by a copy of $K_{\Delta/2,2}$). A cycle C_{4n-1} satisfies $\chi_{\text{in}}(C_{4n-1}) = \Delta(C_{4n-1}) + 1 = 3$, while a graph with $g = 6$ and $\chi_{\text{in}} = \Delta + 1$ for arbitrarily large Δ is obtained by joining vertices u and x by paths uv_iw_ix, $1 \leq i \leq \Delta$.

For planar graphs, the following sufficient conditions for $\chi_{\text{in}} = \Delta$ were obtained: $\Delta \geq 71$ and $g \geq 7$ (Bu et al. [75]); $\Delta \geq 4$ and $g \geq 13$ (Cranston, Kim, Yu [89]); $\Delta \geq 3$ and $g \geq 19$ (Lužar, Škrekovski, Tancer [138]). The following results are also known: $\chi_{\text{in}} \leq \Delta + 1$ if $g \geq 6$ and $\Delta \geq 18$ (Borodin–Ivanova [57]); $\chi_{\text{in}} \leq \Delta + 4$ if $g \geq 5$ [138]; $\chi_{\text{in}} \leq \Delta + 2$ if $g \geq 8$ [75]. For $\Delta \geq 4$, we have $\chi_{\text{in}} \leq \Delta + 1$ if $g \geq 9$ [75,89,138], and $\chi_{\text{in}} \leq \Delta + 2$ if $g \geq 7$ (Cranston, Kim, Yu [90]).

Theorem 13.1 (*Borodin–Ivanova [58]*). *A planar graph satisfies $\chi_{in} \leq \chi_{in}^{\text{list}} \leq \Delta + 1$ if $\Delta \geq 24$ and $g \geq 6$, and $\chi_{in}^{\text{list}} = \chi_{in} = \Delta$ in each of the cases:*

(i) $\Delta \geq 16$ *and* $g \geq 7$;
(ii) $\Delta \geq 10$ *and* $8 \leq g \leq 9$;
(iii) $\Delta \geq 6$ *and* $10 \leq g \leq 11$;
(iv) $\Delta \geq 5$ *and* $g \geq 12$.

Problem 13.1 (*Borodin–Ivanova [58]*). Find precise upper bounds for the injective chromatic number and injective choosability of planar graphs with given girth and maximum degree.

For $g \leq 5$, Problem 13.1 remains completely open; for $g \geq 6$, it is open only for relatively small Δ (depending on g) due to Theorem 13.1.

14. Star coloring

In 1973, Grünbaum [99] defined a *star coloring* of a graph to be a proper coloring in which every 2-colored subgraph is a star forest. In other words, star coloring forbids 2-colored path on four vertices. By definition, every star coloring is acyclic, and every 2-distance coloring is a star coloring, so $\chi_a \leq \chi_s \leq \chi_2$, where χ_s is the star chromatic number. Grünbaum [99] proved that every planar graph satisfies $\chi_s \leq 2304$ and mentioned without the proof that every graph satisfies $\chi_s \leq \chi_a 2^{\chi_a - 1}$. Thus Grünbaum replaces every "main" color that comes from an acyclic coloring by 2^{k-1} "shadows" of this color, suitable for constructing a star coloring of the graph. The unpublished proof of Grünbaum may be explained as follows.

Let c be an acyclic k-coloring of an arbitrary orientation of our graph. Every vertex v with $c(v) = i$ is assigned a vector $Q(v) = (q_1(v), \dots, q_k(v))$ with $q_i(v) = i$. Every other component of $Q(v)$ will be either $\oplus$ or $\ominus$ and is defined by the following procedure. Let T be a tree colored with i and j, and let $v \in V(T)$. If $c(v) = i$, then our procedure defines $q_j(v)$; if $c(v) = j$, then it defines $q_i(v)$. We fix a root r in T. Without loss of generality, we assume that r is colored i; choose $q_j(r) \in \{\oplus, \ominus\}$ arbitrarily. Going along T away from r, we choose $\oplus$ or $\ominus$ at the next vertex as follows. If uw is an arc in T such that the pair $(q_i(u), q_j(u))$ is already determined, then the pair $(q_i(w), q_j(w))$ is determined, depending on the orientation of uw, by the following circular rule: $(i, \oplus) \to (\ominus, j) \to (i, \ominus) \to (\oplus, j) \to (i, \oplus)$. This yields a star 4-coloring of T by pairs of the form (q_i, q_j). Applying this operation to all 2-colored trees, we obtain a proper $k2^{k-1}$-coloring of our graph by the vectors Q in which the vertices colored with any Q's induce an oriented star forest.

Due to Theorem 9.1 on acyclic 5-colorability of planar graphs (Borodin [18]), Grünbaum's idea of splitting colors into vectors implies that every planar graph satisfies $\chi_s \leq 80$. In 2003, this bound was lowered to 30 by Nešetřil and Ossona de Mendez [151]. The best-known upper bound was soon after deduced by Albertson et al. [8] from the 4CT combined with Theorem 9.1.

Theorem 14.1 (*Albertson, Chappell, Kierstead, Kündgen, Ramamurthi [8]*). *Every planar graph satisfies $\chi_s \leq 20$.*

On the other hand, Albertson et al. [8] constructed a plane graph with $\chi_s = 10$ and plane graphs of arbitrarily large girth with $\chi_s = 4$.

Nešetřil and Ossona de Mendez [151] showed that every bipartite planar graph satisfies $\chi_s \leq 18$, which was recently improved to $\chi_s^{\text{list}} \leq 14$ in Kierstead, Kündgen, Timmons [122], where an example of a bipartite planar graph with $\chi_s = 8$ is also given.

Albertson et al. [8] proved $\chi_s \leq 9$ if $g \geq 7$ and $\chi_s \leq 16$ if $g \geq 5$. Timmons [180,181] proved $\chi_s \leq 4$ if $g \geq 14$, $\chi_s \leq 5$ if $g \geq 9$, $\chi_s \leq 7$ if $g \geq 7$, and $\chi_s \leq 8$ if $g \geq 6$. He also constructed planar graphs with $\chi_s = 6$ if $g = 5$ and $\chi_s = 5$ if $g = 7$. It is known that $\chi_s \leq 4$ if $g \geq 13$ (Bu et al. [76]).

Kündgen and Timmons [133] further obtained the following results for star choosability: $\chi_s^{\text{list}} \leq 6$ if $g \geq 8$, $\chi_s^{\text{list}} \leq 7$ if $g \geq 7$, and $\chi_s^{\text{list}} \leq 8$ if $g \geq 6$.

Problem 14.1 (*[8,151,122,180,181,76,133]*)**.** Find precise upper bounds for the star chromatic number and for the star choosability of planar graphs with given girth and of bipartite planar graphs.

Needless to say, the general Problem 14.1 is very difficult, so any new specific result towards it at this stage is valuable.

Recall that Borodin et al. [35] proved that every planar graph is acyclically 7-choosable.

Problem 14.2. Is it possible to give a general upper bound for χ_s^{list} as a function of χ_a^{list} only?

Problem 14.3. Does there exist a constant C such that every planar graph is star C-choosable?

Concluding remark

Once again, the author apologizes for not being able to include some relevant topics, papers, results, and conjectures due to the space limitations of this paper.

Acknowledgments

The author is greatly indebted to Douglas Woodall, Douglas West, and the anonymous referees for their careful reading and numerous suggestions on improving the presentation.

The author was supported by grants 12-01-00448 and 12-01-00631 of the Russian Foundation for Basic Research and by the Ministry of education and science of the Russian Federation (contract number 14.740.11.0868).

References

[1] H.L. Abbott, B. Zhou, On small faces in 4-critical graphs, Ars Combin. 32 (1991) 203–207.
[2] G. Agnarsson, M.M. Halldórsson, Coloring powers of planar graphs, SIAM J. Discrete Math. 16 (4) (2003) 651–662.
[3] V.A. Aksenov, The extension of a 3-coloring on planar graphs, Diskret. Analiz 26 (1974) 3–19 (in Russian).
[4] V.A. Aksenov, Chromatic connected vertices in planar graphs, Diskret. Analiz 31 (1977) 5–16 (in Russian).
[5] V.A. Aksenov, O.V. Borodin, A.N. Glebov, On the continuation of a 3-coloring from two vertices in a plane graph without 3-cycles, Diskretn. Anal. Issled. Oper. 9 (1) (2002) 3–26 (in Russian).
[6] V.A. Aksenov, L.S. Mel'nikov, Essay on the theme: the three-color problem, in: Combinatorics (Proc. Fifth Hungarian Colloq., Keszthely, 1976), Vol. I, in: Colloq. Math. Soc. Janos Bolyai, vol. 18, North-Holland, Amsterdam, New York, 1978, pp. 23–34.
[7] M.O. Albertson, D. Berman, Every planar graph has an acyclic 7-coloring, Israel J. Math. 28 (1977) 169–177.
[8] M.O. Albertson, G. Chappell, H.A. Kierstead, A. Kündgen, R. Ramamurthi, Coloring with no 2-colored P4's, Electron. J. Combin. 11 (2004) 13. Research Paper 26.
[9] I. Algor, N. Alon, The star arboricity of graphs, Discrete Math. 75 (1–3) (1989) 11–22.
[10] N. Alon, T.H. Marshall, Homomorphisms of edge-colored graphs and Coxeter groups, J. Algebraic Combin. 2 (1991) 277–289.
[11] N. Alon, M. Tarsi, Colorings and orientations of graphs, Combinatorica 12 (2) (1992) 125–134.
[12] K. Appel, W. Haken, Every planar map is four colorable, I, discharging, Illinois J. Math. 21 (1977) 429–490.
[13] D. Archdeacon, Coupled coloring of planar maps, Congr. Numer. 39 (1986) 89–94.
[14] M. Behzad, Graphs and their chromatic numbers, Doctoral Thesis, Michigan State University, 1965.
[15] A.A. Bertossi, M.A. Bonuccelli, Code assignment for hidden terminal interference avoidance in multihop packet radio networks, IEEE/ACM Trans. Networking 3 (1995) 441–449.
[16] R. Bodendiek, H. Schumacher, K. Wagner, Uber das Ringelsche Sechsfarbenproblem, Praxis Math. 25 (1983) 353–356.
[17] O.V. Borodin, Solution of the Ringel problem on vertex-face coloring of planar graphs and coloring of 1-planar graphs, Metody Diskret. Analiz 41 (1984) 12–26 (in Russian).
[18] O.V. Borodin, On acyclic colorings of planar graphs, Discrete Math. 25 (3) (1979) 211–236.
[19] O.V. Borodin, On the total coloring of planar graphs, J. Reine Angew. Math. 394 (1989) 180–185.
[20] O.V. Borodin, A new proof of Grünbaum's 3-colour theorem, Discrete Math. 169 (1–3) (1997) 177–183.
[21] O.V. Borodin, Coupled colorings of graphs on a plane, Metody Diskret. Analiz 45 (1987) 21–27 (in Russian).
[22] O.V. Borodin, The structure of neighborhoods of an edge in planar graphs and the simultaneous coloring of vertices, edges, and faces, Mat. Zametki 53 (5) (1993) 35–47 (in Russian).
[23] O.V. Borodin, Structural properties of plane graphs without adjacent triangles and an application to 3-colorings, J. Graph Theory 21 (2) (1996) 183–186.
[24] O.V. Borodin, A new proof of the 6-color theorem, J. Graph Theory 19 (1995) 507–521.
[25] O.V. Borodin, Structural theorem on plane graphs with application to the entire coloring, J. Graph Theory 23 (3) (1996) 233–239.
[26] O.V. Borodin, Simultaneous coloring of edges and faces of plane graphs, Discrete Math. 128 (1–3) (1994) 21–33.
[27] O.V. Borodin, A generalization of Kotzig's theorem on prescribed edge coloring of planar graphs, Mat. Zametki 48 (6) (1990) 22–28 (in Russian).
[28] O.V. Borodin, Cyclic coloring of plane graphs, Discrete Math. 100 (1–3) (1992) 281–289.

[29] O.V. Borodin, Acyclic 3-choosability of planar graphs without cycles of length from 4 to 12, Diskretn. Anal. Issled. Oper. 16 (5) (2009) 26–33 (in Russian).
[30] O.V. Borodin, Acyclic 4-colorability of planar graphs with neither 4-cycles nor 5-cycles, Diskretn. Anal. Issled. Oper. 17 (2) (2010) 20–38 (in Russian).
[31] O.V. Borodin, H.J. Broersma, A. Glebov, J. van den Heuvel, A new upper bound on the cyclic chromatic number, J. Graph Theory 54 (1) (2007) 58–72.
[32] O.V. Borodin, H.J. Broersma, A. Glebov, J. van den Heuvel, The structure of plane triangulations in terms of stars and bunches, Diskretn. Anal. Issled. Oper. 8 (2) (2001) 15–39 (in Russian).
[33] O.V. Borodin, H.J. Broersma, A. Glebov, J. van den Heuvel, Minimal degrees and chromatic numbers of squares of planar graphs, Diskretn. Anal. Issled. Oper. 8 (4) (2001) 9–33 (in Russian).
[34] O.V. Borodin, M. Chen, A.O. Ivanova, A. Raspaud, Acyclic 3-choosability of sparse graphs with girth at least 7, Discrete Math. 310 (17–18) (2010) 2426–2434.
[35] O.V. Borodin, D.G. Fon-Der-Flaass, A.V. Kostochka, A. Raspaud, E. Sopena, Acyclic list 7-coloring of planar graphs, J. Graph Theory 40 (2002) 83–90.
[36] O.V. Borodin, A.N. Glebov, A sufficient condition for the 3-colorability of plane graphs, Diskretn. Anal. Issled. Oper. Ser. 1 11 (1) (2004) 13–29. (in Russian).
[37] O.V. Borodin, A.N. Glebov, Planar graphs without 5-cycles and with minimal distance between triangles at least 2 are 3-colourable, J. Graph Theory 66 (1) (2011) 1–31.
[38] O.V. Borodin, A.N. Glebov, A.O. Ivanova, T.K. Neustroeva, V.A. Tashkinov, Sufficient conditions for the 2-distance $(\Delta + 1)$-colorability of plane graphs, Sib. Elektron. Mat. Izv. 1 (2004) 129–141. (in Russian).
[39] O.V. Borodin, A.N. Glebov, T.R. Jensen, A step towards the strong version of Havel's 3 Color Conjecture, J. Combin. Theory, Ser. B. 102 (2012) 1295–1320.
[40] O.V. Borodin, A.N. Glebov, T.R. Jensen, A. Raspaud, Planar graphs without triangles adjacent to cycles of length from 3 to 9 are 3-colorable, Sib. Elektron. Mat. Izv. 3 (2006) 428–440.
[41] O.V. Borodin, A.N. Glebov, M. Montassier, A. Raspaud, Planar graphs without 5- and 7-cycles and without adjacent triangles are 3-colorable, J. Combin. Theory Ser. B 99 (2009) 668–673.
[42] O.V. Borodin, A.N. Glebov, A. Raspaud, Planar graphs without triangles adjacent to cycles of length from 4 to 7 are 3-colorable, Thomassen's special issue of Discrete Math. 310 (2) (2010) 2584–2594.
[43] O.V. Borodin, A.N. Glebov, A. Raspaud, M.R. Salavatipour, Planar graphs without cycles of length from 4 to 7 are 3-colorable, J. Combin. Theory Ser. B 93 (2005) 303–311.
[44] O.V. Borodin, S.G. Hartke, A.O. Ivanova, A.V. Kostochka, D.B. West, (5, 2)-coloring of sparse graphs, Sib. Elektron. Mat. Izv. 5 (2008) 417–426.
[45] O.V. Borodin, A.O. Ivanova, Acyclic 3-choosability of planar graphs with no cycles of length from 4 to 11, Sib. Elektron. Mat. Izv. 7 (2010) 275–283.
[46] O.V. Borodin, A.O. Ivanova, List 2-facial 5-colorability of plane graphs with girth at least 12, Discrete Math. 312 (2012) 306–314.
[47] O.V. Borodin, A.O. Ivanova, Acyclic 5-choosability of planar graphs without adjacent short cycles, J. Graph Theory 68 (2) (2011) 169–176.
[48] O.V. Borodin, A.O. Ivanova, Acyclic 5-choosability of planar graphs without 4-cycles, Sibirsk. Mat. Zh. 52 (3) (2011) 522–541. (in Russian).
[49] O.V. Borodin, A.O. Ivanova, Acyclic 4-choosability of planar graphs with no 4- and 5-cycles, J. Graph Theory. http://dx.doi.org/10.1002/jgt.21647.
[50] O.V. Borodin, A.O. Ivanova, Acyclic 4-choosability of planar graphs without adjacent short cycles, Discrete Math. 312 (22) (2012) 3335–3341.
[51] O.V. Borodin, A.O. Ivanova, Oriented coloring of planar graphs with girth at least four, Sib. Elektron. Mat. Izv. 2 (2005) 239–249. (in Russian).
[52] O.V. Borodin, A.O. Ivanova, Oriented 7-coloring of planar graphs with girth at least seven, Sib. Elektron. Mat. Izv. 2 (2005) 222–229. (in Russian).
[53] O.V. Borodin, A.O. Ivanova, 2-distance $(\Delta + 2)$-coloring of planar graphs with girth six and $\Delta \geq 18$, Discrete Math. 309 (23–24) (2009) 6496–6502.
[54] O.V. Borodin, A.O. Ivanova, List 2-distance $(\Delta + 2)$-coloring of planar graphs with girth six, European J. Combin. 30 (5) (2009) 1257–1262.
[55] O.V. Borodin, A.O. Ivanova, List 2-distance $(\Delta + 2)$-coloring of planar graphs with girth six and $\Delta \geq 24$, Sibirsk. Mat. Zh. 6 (2009) 1216–1224. (in Russian).
[56] O.V. Borodin, A.O. Ivanova, 2-distance 4-colorability of planar subcubic graphs with girth at least 22, Discuss. Math. Graph Theory 32 (1) (2012) 141–151.
[57] O.V. Borodin, A.O. Ivanova, Injective $(\Delta + 1)$-coloring of planar graphs with girth six, Sibirsk. Mat. Zh. 52 (1) (2011) 30–38 (in Russian).
[58] O.V. Borodin, A.O. Ivanova, List injective colorings of planar graphs, Discrete Math. 311 (2–3) (2011) 154–165.
[59] O.V. Borodin, A.O. Ivanova, A.V. Kostochka, Oriented 5-coloring of vertices in sparse graphs, Diskretn. Anal. Issled. Oper. Ser. 1 13 (1) (2006) 16–32 (in Russian).
[60] O.V. Borodin, A.O. Ivanova, T.K. Neustroeva, A prescribed 2-distance $(\Delta + 1)$-coloring of planar graphs with a given girth, Diskretn. Anal. Issled. Oper. Ser. 1 14 (3) (2007) 13–30 (in Russian).
[61] O.V. Borodin, A.O. Ivanova, T.K. Neustroeva, 2-distance colouring of sparse planar graphs, Sib. Elektron. Mat. Izv. 1 (2004) 76–90 (in Russian).
[62] O.V. Borodin, A.O. Ivanova, T.K. Neustroeva, Sufficient conditions for the minimum 2-distance colorability of plane graphs with girth six, Sib. Elektron. Mat. Izv. 3 (2006) 441–450 (in Russian).
[63] O.V. Borodin, A.O. Ivanova, A. Raspaud, Acyclic 4-choosability of planar graphs with neither 4-cycles nor triangular 6-cycles, Discrete Math. 310 (21) (2010) 2946–2950.
[64] O.V. Borodin, S.-J. Kim, A.V. Kostochka, D.B. West, Homomorphisms from sparse graphs with large girth, dedicated to Adrian Bondy and U.S.R. Murty, J. Combin. Theory Ser. B 90 (1) (2004) 147–159.
[65] O.V. Borodin, A.V. Kostochka, J. Nešetřil, A. Raspaud, E. Sopena, On the maximal average degree and the oriented chromatic number of a graph, Discrete Math. 206 (1–3) (1999) 77–89.
[66] O.V. Borodin, A.V. Kostochka, A. Raspaud, E. Sopena, Acyclic coloring of 1-planar graphs, Diskretn. Anal. Issled. Oper. Ser. 1 6 (4) (1999) 20–35 (in Russian).
[67] O.V. Borodin, A.V. Kostochka, D.R. Woodall, List edge and list total colourings of multigraphs, J. Combin. Theory Ser. B 71 (2) (1997) 184–204.
[68] O.V. Borodin, A.V. Kostochka, D.R. Woodall, Total colourings of planar graphs with large maximal degree, J. Graph Theory 26 (1997) 53–59.
[69] O.V. Borodin, A.V. Kostochka, D.R. Woodall, Total colourings of planar graphs with large girth, European J. Combin. 19 (1998) 19–24.
[70] O.V. Borodin, A.V. Kostochka, D.R. Woodall, Acyclic colorings of planar graphs with large girth, J. Lond. Math. Soc. 60 (1999) 344–352.
[71] O.V. Borodin, M. Montassier, A. Raspaud, Planar graphs without adjacent cycles of length at most seven are 3-colorable, Discrete Math. 310 (1) (2010) 167–173.
[72] O.V. Borodin, A. Raspaud, A sufficient condition for planar graphs to be 3-colorable, J. Combin. Theory Ser. B 88 (2003) 17–27.
[73] O.V. Borodin, D. Sanders, Y. Zhao, On cyclic colorings and their generalizations, Discrete Math. 203 (1–3) (1999) 23–40.
[74] O.V. Borodin, D.R. Woodall, Cyclic colourings of 3-polytopes with large maximum face size, SIAM J. Discrete Math. 15 (2) (2002) 143–154.
[75] Y. Bu, D. Chen, A. Raspaud, W. Wang, Injective coloring of planar graphs, Discrete Appl. Math. 157 (4) (2009) 663–672.
[76] Y. Bu, D. Cranston, M. Montassier, A. Raspaud, W. Wang, Star coloring of sparse graphs, J. Graph Theory 62 (3) (2009) 201–219.
[77] M. Chen, A. Raspaud, A sufficient condition for planar graphs to be acyclically 5-choosable, J. Graph Theory 70 (2) (2012) 135–151.
[78] M. Chen, A. Raspaud, On acyclic 4-choosability of planar graphs without short cycles, Discrete Math. 310 (15–16) (2010) 2113–2118.
[79] M. Chen, A. Raspaud, Planar graphs without 4- and 5-cycles are acyclically 4-choosable, Discrete Appl. Math. (in press).
[80] M. Chen, A. Raspaud, W. Wang, Three-coloring planar graphs without short cycles, Inform. Process. Lett. 101 (3) (2007) 134–138.
[81] M. Chen, A. Raspaud, W. Wang, Acyclic 4-choosability of planar graphs without prescribed cycles (submitted for publication).
[82] M. Chen, W. Wang, On 3-colorable planar graphs without short cycles, Appl. Math. Lett. 21 (9) (2008) 961–965.
[83] M. Chen, W. Wang, Acyclic 5-choosability of planar graphs without 4-cycles, Discrete Math. 308 (24) (2008) 6216–6225.
[84] D.L. Chen, J.L. Wu, The total colouring of some graphs, in: Combinatorics, Graph Theory, Algorithms, and Applications (Beijing, 1993), World Sci. Publ., River Edge, NY, 1994, pp. 17–20.
[85] Y. Chen, W. Zhu, W. Wang, Edge choosability of planar graphs without 5-cycles with a chord, Discrete Math. 309 (8) (2009) 2233–2238.

[86] B. Courcelle, The monadic second order logic of graphs VI: on several representations of graphs by relational structures, Discrete Appl. Math. 54 (1994) 117–149.
[87] D.W. Cranston, Edge-choosability and total-choosability of planar graphs with no adjacent 3-cycles, Discuss. Math. Graph Theory 29 (1) (2009) 163–178.
[88] D.W. Cranston, S.-J. Kim, List-coloring the square of a subcubic graph, J. Graph Theory 57 (2008) 65–87.
[89] D.W. Cranston, S.-J. Kim, G. Yu, Injective colorings of sparse graphs, Discrete Math. 310 (21) (2010) 2965–2973.
[90] D.W. Cranston, S.-J. Kim, G. Yu, Injective colorings of graphs with low average degree, Algorithmica 60 (3) (2011) 553–568.
[91] Z. Dvořák, D. Král', P. Nejedlý, R. Škrekovski, Coloring squares of planar graphs with girth six, European J. Combin. 29 (4) (2008) 838–849.
[92] Z. Dvořák, D. Král', P. Nejedlý, R. Škrekovski, Distance constrained labelings of planar graphs with no short cycles, Discrete Appl. Math. 157 (12) (2009) 2634–2645.
[93] Z. Dvořák, D. Král, R. Thomas, Coloring planar graphs with triangles far apart, http://people.math.gatech.edu/thomas/PAP/havel.pdf.
[94] Z. Dvořák, R. Škrekovski, M. Tancer, List-coloring squares of sparse subcubic graphs, SIAM J. Discrete Math. 22 (1) (2008) 139–159.
[95] H. Enomoto, M. Hořnák, S. Jendrol', Cyclic chromatic number of 3-connected plane graphs, SIAM J. Discrete Math. 14 (1) (2001) 121–137.
[96] P. Erdős, A. Rubin, H. Taylor, Choosability in graphs, Congr. Numer. 26 (1979) 125–157.
[97] Ph. Franklin, The four colour problem, Amer. J. Math. 44 (1922) 225–236.
[98] H. Grötzsch, Ein Dreifarbensatz für dreikreisfreie Netze auf der Kugel, Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe 8 (1959) 109–120.
[99] B. Grünbaum, Acyclic colorings of planar graphs, Israel J. Math. 14 (3) (1973) 390–408.
[100] B. Grünbaum, Grötzsch's theorem on 3-colorings, Michigan Math. J. 10 (1963) 303–310.
[101] G. Hahn, J. Kratochvíl, J. Širáň, D. Sotteau, On the injective chromatic number of graphs, Discrete Math. 256 (1–2) (2002) 179–192.
[102] S.L. Hakimi, J. Mitchem, E. Schmeichel, Star arboricity of graphs, Discrete Math. 149 (1–3) (1996) 93–98.
[103] I. Havel, O zbarvitelnosti rovinnych grafů theremi barvami, Math. Geometrie a theorie grafů (Praha) (1970) 89–91.
[104] I. Havel, On a conjecture of B. Grünbaum, J. Combin. Theory 7 (1969) 184–186.
[105] F. Havet, Choosability of the square of planar subcubic graphs with large girth, Discrete Math. 309 (11) (2009) 3553–3563.
[106] F. Havet, J. Sereni, R. Škrekovski, 3-facial coloring of plane graphs, SIAM J. Discrete Math. 22 (1) (2008) 231–247.
[107] F. Havet, J. van den Heuvel, C. McDiarmid, B. Reed, List colouring squares of planar graphs, Eurocomb07, Electron. Notes Discrete Math. 29 (2007) 515–519.
[108] H. Heesch, Untersuchungen zum Vierfarbenproblem, Bibliographisches Institut, Mannheim-Vienna-Zurich, 1969.
[109] H. Hocquard, M. Montassier, Every planar graph without cycles of lengths 4–12 is acyclically 3-choosable, Inform. Process. Lett. 109 (21–22) (2009) 1193–1196.
[110] M. Horňák, S. Jendrol', On a conjecture by Plummer and Toft, J. Graph Theory 30 (1999) 177–189.
[111] M. Horňák, J. Zlámalová, Another step towards proving a conjecture by Plummer and Toft, Discrete Math. 310 (3) (2010) 442–452.
[112] J. Hou, G. Liu, J. Cai, List edge and list total colorings of planar graphs without 4-cycles, Theoret. Comput. Sci. 369 (2006) 250–255.
[113] J. Hou, G. Liu, J. Wu, Some results on list total colorings of planar graphs, Lecture Notes in Comput. Sci. 4489 (2007) 320–328.
[114] A.O. Ivanova, List 2-distance $(\Delta + 1)$-coloring of planar graphs with girth at least 7, Diskretn. Anal. Issled. Oper. 17 (5) (2010) 22–36 (in Russian).
[115] A.O. Ivanova, A.S. Solov'eva, 2-distance $(\Delta + 2)$-coloring of sparse planar graphs with $\Delta = 3$, Math. Notes Yakutsk Univ. 16 (2) (2009) 32–41 (in Russian).
[116] F. Jaeger, On circular flows in graphs, in: Finite and Infinite Sets (Eger, 1981), in: Colloq. Math. Soc. J. Bolyai, vol. 37, North-Holland, 1984, pp. 391–402.
[117] T.R. Jensen, C. Thomassen, The color space of a graph, J. Graph Theory 34 (3) (2000) 234–245.
[118] T.R. Jensen, B. Toft, Graph Coloring Problems, Wiley Interscience, 1995.
[119] E. Jucovič, On a problem in map colouring, Mat. Casopis 19 (3) (1969) 225–227.
[120] M. Juvan, B. Mohar, R. Škrekovski, List total colourings of graphs, Combin. Probab. Comput. 7 (1998) 181–188.
[121] R.J. Kang, J.S. Sereni, M. Stehlik, Every plane graph of maximum degree 8 has an edge-face 9-coloring, SIAM J. Discrete Math. 25 (2) (2011) 514–533.
[122] H.A. Kierstead, A. Kündgen, C. Timmons, Star coloring bipartite planar graphs, J. Graph Theory 60 (1) (2009) 1–10.
[123] A.V. Kostochka, The total coloring of a multigraph with minimum degree 4, Discrete Math. 17 (2) (1977) 161–163.
[124] A.V. Kostochka, An analogue of Shannon's bound for total colorings, Metody Diskret. Analiz 30 (1977) 13–22 (in Russian).
[125] A.V. Kostochka, The total chromatic number of any multigraph with maximum degree five is at most seven, Discrete Math. 162 (1–3) (1996) 199–214.
[126] A.V. Kostochka, Acyclic 6-coloring of planar graphs, Diskret. Analiz 28 (1976) 40–56 (in Russian).
[127] A.V. Kostochka, L.S. Mel'nikov, Note to the paper of Grünbaum on acyclic colorings, Discrete Math. 14 (4) (1976) 403–406.
[128] A.V. Kostochka, D.R. Woodall, Choosability conjectures and multicircuits, Discrete Math. 240 (1–3) (2001) 123–143.
[129] A. Kotzig, Contribution to the theory of Eulerian polyhedra, Mat.-Fyz. Casopis 5 (1955) 101–113.
[130] L. Kowalik, J.S. Sereni, R. Škrekovski, Total colouring of plane graphs with maximum degree nine, SIAM J. Discrete Math. 4 (2008) 1462–1479.
[131] F. Kramer, H. Kramer, A survey on the distance-colouring of graphs, Discrete Math. 308 (2008) 422–426.
[132] H.V. Kronk, J. Mitchem, A seven-color theorem on the sphere, Discrete Math. 5 (1973) 253–260.
[133] A. Kündgen, C. Timmons, Star coloring planar graphs from small lists, J. Graph Theory 63 (4) (2010) 324–337.
[134] H. Lebesgue, Quelques conséquences simples de la formule d'Euler, J. Math. Pures Appl. 19 (1940) 27–43.
[135] X. Li, R. Luo, Edge coloring of embedded graphs with large girth, Graphs Combin. 19 (3) (2003) 393–401.
[136] X. Luo, M. Chen, W.F. Wang, On 3-colorable planar graphs without cycles of four lengths, Inform. Process. Lett. 103 (4) (2007) 150–156.
[137] R. Luo, C. Zhang, Edge-face chromatic number and edge chromatic number of simple plane graphs, J. Graph Theory 49 (3) (2005) 234–256.
[138] B. Lužar, R. Škrekovski, M. Tancer, Injective colorings of planar graphs with few colors, Discrete Math. 309 (18) (2009) 5636–5649.
[139] T.H. Marshall, Homomorphism bounds for oriented planar graphs, J. Graph Theory 55 (3) (2007) 175–190.
[140] Q. Ma, J.-L. Wu, X. Yu, Planar graphs without 5-cycles or without 6-cycles, Discrete Math. 309 (10) (2009) 2998–3005.
[141] L.S. Mel'nikov, Problem 9, in: M. Fiedler (Ed.), Recent Advances in Graph Theory, in: Proc. International Symposium, Prague, 1974, Academia Praha, 1975, p. 543.
[142] J. Mitchem, Every planar graph has an acyclic 8-coloring, Duke Math. J. 14 (1974) 177–181.
[143] M. Molloy, M.R. Salavatipour, A bound on the chromatic number of the square of a planar graph, J. Combin. Theory Ser. B 94 (2005) 189–213.
[144] M. Montassier, Acyclic 4-choosability of planar graphs with girth at least 5, Graph Theory Trends Math. (2006) 299–310.
[145] M. Montassier, P. Ochem, A. Raspaud, On the acyclic choosability of graphs, J. Graph Theory 51 (2006) 281–300.
[146] M. Montassier, A. Raspaud, A note on 2-facial coloring of plane graphs, Inform. Process. Lett. 98 (6) (2006) 235–241.
[147] M. Montassier, A. Raspaud, W. Wang, Bordeaux 3-color conjecture and 3-choosability, Discrete Math. 306 (6) (2006) 573–579.
[148] M. Montassier, A. Raspaud, W. Wang, Acyclic 5-choosability of planar graphs without small cycles, J. Graph Theory 54 (2007) 245–260.
[149] M. Montassier, A. Raspaud, W. Wang, Acyclic 4-choosability of planar graphs without cycles of specific lengths, in: Topics in Discrete Mathematics, in: Algorithms Combin., vol. 26, Springer, Berlin, 2006, pp. 473–491.
[150] M. Montassier, A. Raspaud, W. Wang, Y. Wang, A relaxation of Havel's 3-color problem, Inform. Process. Lett. 107 (3–4) (2008) 107–109.
[151] J. Nešetřil, P. Ossona de Mendez, Colorings and homomorphisms of minor closed classes, in: Discrete and Computational Geometry, in: Algorithms Combin., vol. 25, Springer, Berlin, 2003, pp. 651–664.
[152] J. Nešetřil, A. Raspaud, Colored homomorphisms of colored mixed graphs, J. Combin. Theory Ser. B 80 (1) (2000) 147–155.
[153] J. Nešetřil, A. Raspaud, E. Sopena, Colorings and girth of oriented planar graphs, Discrete Math. 165–166 (1997) 519–530.
[154] J. Nešetřil, X. Zhu, On bounded tree-width duality of graphs, J. Graph Theory 23 (1996) 151–162.
[155] P. Ochem, Oriented colorings of triangle-free planar graphs, Inform. Process. Lett. 92 (2) (2004) 71–76.

[156] P. Ochem, A. Pinlou, Oriented coloring of triangle-free planar graphs and 2-outerplanar graphs, in: LAGOS'11—VI Latin-American Algorithms, Graphs and Optimization Symposium, 123–128, in: Electron. Notes Discrete Math., vol. 37, Elsevier Sci. B.V., Amsterdam, 2011.
[157] P. Ochem, A. Pinlou, Oriented colorings of partial 2-trees, Inform. Process. Lett. 108 (2008) 82–86.
[158] O. Ore, M.D. Plummer, Cyclic coloration of plane graphs, in: Recent Progress in Combinatorics, Academic Press, New York, 1969, pp. 287–293.
[159] A. Pinlou, An oriented coloring of planar graphs with girth at least five, Discrete Math. 309 (8) (2009) 2108–2118.
[160] M.D. Plummer, B. Toft, Cyclic coloration of 3-polytopes, J. Graph Theory 11 (5) (1987) 507–515.
[161] A. Raspaud, E. Sopena, Good and semi-strong colorings of oriented planar graphs, Inform. Process. Lett. 51 (1994) 171–174.
[162] D. Rautenbach, A conjecture of Borodin and a coloring of Grünbaum, J. Graph Theory 58 (2) (2008) 139–147.
[163] G. Ringel, Ein Sechsfarbenproblem auf der Kugel, Abh. Math. Semin. Univ. Hambg. 29 (1965) 107–117.
[164] N. Robertson, D. Sanders, P. Seymour, R. Thomas, The four-color theorem, J. Combin. Theory B 70 (1997) 2–44.
[165] D.P. Sanders, Y. Zhao, On the entire colouring conjecture, Canad. Math. Bull. 43 (1) (2000) 108–114.
[166] D.P. Sanders, Y. Zhao, On simultaneous edge-face colorings of plane graphs, Combinatorica 17 (1997) 441–445.
[167] D.P. Sanders, Y. Zhao, On improving the edge-face colouring theorem, Graphs Combin. 17 (2001) 329–341.
[168] D.P. Sanders, Y. Zhao, A five-color theorem, Discrete Math. 220 (1–3) (2000) 279–281.
[169] D.P. Sanders, Y. Zhao, On total 9-colouring planar graphs of maximum degree seven, J. Graph Theory 31 (1999) 67–73.
[170] D.P. Sanders, Y. Zhao, Planar graphs of maximum degree seven are class I, J. Combin. Theory Ser. B 83 (2) (2001) 201–212.
[171] D.P. Sanders, Y. Zhao, A new bound on the cyclic chromatic number, J. Combin. Theory Ser. B 83 (1) (2001) 102–111.
[172] D.P. Sanders, Y. Zhao, A note on the three color problem, Graphs Combin. 11 (1995) 91–94.
[173] J.-S. Sereni, M. Stehlik, Edge-face colouring of plane graphs with maximum degree nine, J. Graph Theory 66 (4) (2011) 332–346.
[174] Y. Shen, G. Zheng, W. He, Y. Zhao, Structural properties and edge choosability of planar graphs without 4-cycles, Discrete Math. 308 (23) (2008) 5789–5794.
[175] R. Steinberg, The state of the three color problem, quo vadis, graph theory? J. Gimbel, J.W. Kennedy & L.V. Quintas (eds.), Ann. Discrete Math. 55 (1993) 211–248.
[176] X.-Y. Sun, J.-L. Wu, Y.-W. Wu, J.-F. Hou, Total colorings of planar graphs without adjacent triangles, Discrete Math. 309 (1) (2009) 202–206.
[177] C. Thomassen, Every planar graph is 5-choosable, J. Combin. Theory Ser. B 62 (1994) 180–181.
[178] C. Thomassen, 3-List-coloring planar graphs of girth 5, J. Combin. Theory Ser. B 64 (1995) 101–107.
[179] C. Thomassen, Grötzsch's 3-color theorem and its counterparts for the torus and the projective plane, J. Combin. Theory Ser. B 62 (2) (1994) 268–279.
[180] C. Timmons, Star coloring planar graphs, Masters Thesis, California State University San Marcos, May 2007.
[181] C. Timmons, Star coloring high girth planar graphs, Electron. J. Combin. 15 (1) (2008) 17. Research Paper 124.
[182] V.G. Vizing, Vertex colourings with given colours, Metody Diskret. Analiz 29 (1976) 3–10 (in Russian).
[183] V.G. Vizing, Some unsolved problems in graph theory, Uspekhi Mat. Nauk 23 (1968) 117–134 (in Russian).
[184] V.G. Vizing, On an estimate of the chromatic index of a p-graph, Diskret. Analiz 3 (1964) 25–30 (in Russian).
[185] V.G. Vizing, Critical graphs with given chromatic index, Diskret. Analiz 5 (1965) 9–17 (in Russian).
[186] M. Voigt, A not 3-choosable planar graph without 3-cycles, Discrete Math. 146 (1–3) (1995) 325–328.
[187] M. Voigt, List colourings of planar graphs, Discrete Math. 120 (1–3) (1993) 215–219.
[188] M. Voigt, A non-3-choosable planar graph without cycles of length 4 and 5, Discrete Math. 307 (7–8) (2007) 1013–1015.
[189] A.O. Waller, Simultaneously colouring the edges and faces of plane graphs, J. Combin. Theory Ser. B 69 (1997) 219–221.
[190] W. Wang, Total chromatic number of planar graphs with maximum degree ten, J. Graph Theory 54 (2007) 91–102.
[191] W. Wang, L. Cai, Labelling planar graphs without 4-cycles with a condition on distance two, Discrete Appl. Math. 156 (12) (2008) 2241–2436.
[192] W. Wang, M. Chen, Planar graphs without 4, 6, 8-cycles are 3-colorable, Sci. China A 50 (11) (2007) 1552–1562.
[193] W. Wang, M. Chen, On 3-colorable planar graphs without prescribed cycles, Discrete Math. 307 (22) (2007) 2820–2825.
[194] W. Wang, M. Chen, Planar graphs without 4-cycles are acyclically 6-choosable, J. Graph Theory 61 (2009) 307–323.
[195] W. Wang, K.-W. Lih, Coupled choosability of plane graphs, J. Graph Theory 58 (2008) 27–44.
[196] W. Wang, K.-W. Lih, The edge-face choosability of plane graphs, European J. Combin. 25 (2004) 935–948.
[197] Y. Wang, H. Lu, M. Chen, Planar graphs without cycles of length 4, 5, 8 or 9 are 3-choosable, Discrete Math. 310 (1) (2010) 147–158.
[198] Y.Q. Wang, X.H. Mao, H.J. Lu, W.F. Wang, On 3-colorability of planar graphs without adjacent short cycles, Sci. China Math. 53 (4) (2010) 1129–1132.
[199] P. Wang, J. Wu, A note on total colorings of planar graphs without 4-cycles, Discuss. Math. Graph Theory 24 (2004) 125–135.
[200] W. Wang, X. Zhu, Entire colouring of plane graphs, J. Combin. Theory Ser. B 101 (6) (2011) 490–501.
[201] G. Wegner, Graphs with given diameter and a coloring problem, Technical Report, University of Dortmund, Germany, 1977.
[202] P. Wernicke, Uber den Kartographischen Vierfarbensatz, Math. Ann. 58 (1904) 413–426.
[203] D.R. Woodall, Some totally 4-choosable multigraphs, Discuss. Math. Graph Theory 27 (2007) 425–455.
[204] J. Wu, P. Wang, List-edge and list-total colorings of graphs embedded on hyperbolic surfaces, Discrete Math. 308 (24) (2008) 6210–6215.
[205] B. Xu, On 3-colorable plane graphs without 5- and 7-cycles, J. Combin. Theory Ser. B 96 (2006) 958–963.
[206] B. Xu, A 3-color theorem on plane graph without 5-circuits, Acta Math. Sinica 23 (6) (2007) 1059–1062.
[207] L. Zhang, Every planar graph with maximum degree 7 is of class 1, Graphs Combin. 16 (2000) 467–495.
[208] H. Zhang, B. Xu, Acyclic 5-choosable planar graphs with neither 4-cycles nor chordal 6-cycles, Discrete Math. 309 (20) (2009) 6087–6091.
[209] X. Zhu, Circular chromatic number of planar graphs of large odd girth, Electronic J. Combin. 8 (1) (2001) 11. Research Paper 25.