Hadwiger's conjecture

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Abstract

This is a survey of Hadwiger's conjecture from 1943, that for all $t \geq 0$, every graph either can be t-coloured, or has a subgraph that can be contracted to the complete graph on $t+1$ vertices. This is a tremendous strengthening of the four-colour theorem, and is probably the most famous open problem in graph theory.

1 Introduction

The four-colour conjecture (or theorem as it became in 1976), that every planar graph is 4-colourable, was the central open problem in graph theory for a hundred years; and its proof is still not satisfying, requiring as it does the extensive use of a computer. (Let us call it the 4CT.) We would very much like to know the "real" reason the 4CT is true; what exactly is it about planarity that implies that four colours suffice? Its statement is so simple and appealing that the massive case analysis of the computer proof surely cannot be the book proof.

So there have been attempts to pare down its hypotheses to a minimum core, in the hope of hitting the essentials; to throw away planarity, and impose some weaker condition that still works, and perhaps works with greater transparency so we can comprehend it. This programme has not yet been successful, but it has given rise to some beautiful problems.

Of these, the most far-reaching is Hadwiger's conjecture. (One notable other attempt is Tutte's 1966 conjecture [78] that every 2-edge-connected graph containing no subdivision of the Petersen graph admits a "nowhere-zero 4-flow", but that is

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beyond the scope of this survey.) Before we state it, we need a few definitions. All graphs in this paper have no loops or parallel edges, and are finite unless we say otherwise. If G is a graph, any graph that can be obtained by moving to a subgraph of G and then contracting edges is called a *minor* of G. The complete graph on t vertices is denoted by K_t, and the complete bipartite graph with sides of cardinalities a, b is denoted by $K_{a,b}$.

By the Kuratowski-Wagner theorem [55, 82], planar graphs are precisely the graphs that do not contain K_5 or $K_{3,3}$ as a minor; so the 4CT says that every graph with no K_5 or $K_{3,3}$ minor is 4-colourable. If we are searching for the "real" reason for the four-colour theorem, then it is natural to exclude K_5 here, because it is not four-colourable; but why are we excluding $K_{3,3}$? What if we just exclude K_5, are all graphs with no K_5 minor four-colourable? And does the analogous statement hold if we change K_5 to K_{t+1} and four-colouring to t-colouring? That conjecture was posed by Hadwiger in 1943 [36] and is still open:

1.1 Hadwiger's conjecture: *For every integer $t \geq 0$, every graph with no K_{t+1} minor is t-colourable.*

Let HC(t) denote the statement "every graph with no K_{t+1} minor is t-colourable". Hadwiger proved HC(t) for $t \leq 3$ in 1943 when he introduced his conjecture. Wagner [82] had already shown that HC(4) is equivalent to the 4CT in 1937; and so HC(4) was finally proved when the 4CT was proved by Appel and Haken [4, 5] in 1976. Then in 1993, Robertson, Thomas and I proved HC(5) [70]; one step further than the 4CT! And the proof did not use a computer (although it did assume the 4CT itself). HC(6) remains open.

There have been numerous weakenings and variations proved, of various types, and strengthenings proposed, some of which still survive; and this is an attempt to survey them. Incidentally, there is an excellent 1996 survey on Hadwiger's conjecture by Toft [77], which is particularly informative on the early history of the problem.

2 The proved special cases

Let us first go through the results just mentioned more carefully. HC(0) and HC(1) are trivial. Graphs with no K_3 minor are forests, which are 2-colourable, so HC(2) holds. The first case that is not quite obvious is HC(3). How do we show that graphs with no K_4 minor are 3-colourable? Hadwiger [36] showed that every non-null graph with no K_4 minor has a vertex of degree at most two, which implies that all such graphs are 3-colourable; and there are later theorems of Dirac [23] and Duffin [25] on the same topic. This assembly of results can be expressed in several different ways, but here is one that is convenient for us. Take two graphs G_1, G_2, and for $i = 1, 2$ let C_i be a clique (that is, a subset of vertices, all pairwise adjacent) of G_i, where $|C_1| = |C_2|$. Choose some bijection between the cliques, and identify each vertex of C_1 with the corresponding vertex of C_2. We obtain a graph H say, with two subgraphs isomorphic to G_1, G_2 respectively, overlapping on a clique. Now let G be obtained from H by deleting some edges (or none) of the clique; we say that G is a *clique-sum* of G_1, G_2, and if the clique has size k, we also call it a *k-sum*.

It is easy to see that if G is a clique-sum of G_1, G_2, and both G_1, G_2 are t-colourable, then so is G. So if G can be built by repeated clique-sums starting from some basic class of graphs that are all t-colourable, then so is G. This gives us a slick proof of HC(t) for $t \leq 3$, because of the following:

2.1 Theorem: *For $0 \leq t \leq 3$, the graphs with no K_{t+1} minor are precisely the graphs that can be built by repeated clique-sums, starting from graphs with at most t vertices.*

HC(4) implies the 4CT, so we should not expect 2.1 to extend to $t = 4$. And it doesn't; large grids have no K_5 minor and yet cannot be built from 4-vertex graphs by clique-sums. (Indeed, let us say G has *tree-width k* if k is minimum such that G can be built by clique-sums from pieces with at most $k+1$ vertices; then the $n \times n$ grid has tree-width n.) Nevertheless, we can describe all the graphs with no K_5 minor in this language. Let V_8 be the graph obtained from a cycle of length 8 by adding four edges joining the four opposite pairs of vertices of the cycle. Wagner [82] essentially proved the following in 1937.

2.2 Theorem: *The graphs with no K_5 minor are precisely the graphs that can be built by repeated 0-, 1-, 2-, and 3-sums, starting from planar graphs and copies of V_8.*

Consequently the 4CT implies HC(4), as Wagner points out in his 1937 paper [82]. (Of course, this does not yet provide the profound insight into the four-colour theorem we hope for, because not only does the *proof* of HC(4) use the 4CT, but the graphs it concerns are themselves basically planar.)

What about HC(5)? One might imagine that since the curve of difficulty versus t has recently had such a steep slope, HC(5) would be impossible (or false); but that is not so. Suppose it is false, and look at a smallest counterexample G. Robertson, Thomas and I [70] showed, without using a computer and without assuming the four-colour theorem, that G must be an *apex* graph, that is, there is a vertex whose deletion makes it planar. If so, then since the 4CT implies that the planar part of G is 4-colourable, we still have a colour left for the vertex we deleted, so G is 5-colourable after all.

The proof that G is apex is (very roughly) as follows. One can show that G is 6-connected, and in particular all vertices have degree at least six; and vertices of degree six belong to K_4 subgraphs, and it follows that there are not many of them (in fact at most two), or else we could piece together all these K_4's to make a K_6 minor. On the other hand, a theorem of Mader says that the average degree of G is less than eight, and we cannot make the average degree bigger then eight even if we cleverly contract edges. That implies that there are edges that are in several triangles or squares. If, say, there is an edge uv in four triangles, then there is no K_4 minor of $G \setminus \{u, v\}$ on the four surviving vertices of the triangles (since G has no K_6 minor), and graphs with this property are well-understood; basically they have to be planar with the four special vertices on the infinite region. So $G \setminus \{u,v\}$ is planar, and now a little more thought shows that one of $G \setminus u, G \setminus v$ is planar, and hence G is apex.

Proving that graphs with no K_7-minor are 6-colourable is thus the first case of Hadwiger's conjecture that is still open. Albar and Gonçalves[2] proved:

2.3 Theorem: *Every graph with no K_7 minor is 8-colourable, and every graph with no K_8 minor is 10-colourable.*

3 Average degree

If we are stuck trying to prove Hadwiger's conjecture itself, what *can* we show about the chromatic number of graphs with no K_{t+1} minor? As Wagner [81] proved in 1964, all graphs with no K_{t+1} minor are 2^t-colourable. The proof is as follows: we may assume G is connected; fix some vertex z, and for each i let L_i be the set of vertices at distance i from z; since G has no K_{t+1} minor, the subgraph induced on L_i has no K_t minor (because the union of all the earlier levels would provide one more vertex in the minor); inductively each level L_i induces a subgraph that is 2^{t-1}-colourable; and now alternate colours in even and odd levels to get a 2^t-colouring of G.

Wagner's result has been considerably improved, but most of these improvements depend on "degeneracy", so let us first discuss that. We say G is *k-degenerate* if every non-null subgraph has a vertex of degree at most k. For instance, forests are 1-degenerate, series-parallel graphs (the graphs with no K_4 minor) are 2-degenerate, and planar graphs are 5-degenerate. By deleting a vertex of degree at most k and applying an inductive hypothesis, we have:

3.1 Theorem: *If G is k-degenerate then its chromatic number is at most $k+1$.*

So, if we can bound the degeneracy of the graphs with no K_{t+1} minor, we also bound their chromatic number. (This gives us another proof of HC(t) for $t \leq 3$, because for $t \leq 3$ every graph with no K_{t+1} minor is $(t-1)$-degenerate.)

The simplest way to bound the degeneracy is to bound the average degree. How many edges an n-vertex graph with no K_t minor can have is a much-studied question. Mader [58, 59] showed in 1967 that:

3.2 Theorem: *For every graph H there exists c such that $|E(G)| \leq c|V(G)|$ for every graph G with no H minor.*

But when $H = K_t$ for small values of t, we know the answer exactly:

- for $n \geq 1$, n-vertex graphs with no K_3 minor (forests) have at most $n-1$ edges;
- for $n \geq 2$, graphs with no K_4 minor have at most $2n-3$ edges;
- for $n \geq 3$, graphs with no K_5 minor have at most $3n-6$ edges.

Here is an example: for $n \geq t-2$, take the complete bipartite graph $K_{t-2,n-t+2}$, and add edges joining all pairs of vertices on the side of cardinality $t-2$. This has no K_t minor, and has n vertices and $(t-2)n-(t-1)(t-2)/2$ edges. Thus for $t \le 5$, this graph has the maximum number of edges possible, and if this were so for all t, it would prove Hadwiger's conjecture within a factor of 2. Mader [59] showed that the same holds for $t = 6,7$:

3.3 Theorem: *For $t \le 7$ and all $n \ge t-2$, every n-vertex graph G with no K_t minor satisfies*

$$|E(G)| \le (t-2)n-(t-1)(t-2)/2.$$

But for $t \ge 8$ the pattern fails. If $n_1,\dots,n_t > 0$, we denote by $K_{n_1,\dots,n_t}$ the complete t-partite graph with parts of cardinality $n_1,\dots,n_t$. Mader pointed out that $K_{2,2,2,2,2}$ has no K_8 minor, and does not satisfy the formula of 3.3.

On the other hand, for $t = 8$ we understand all counterexamples to the formula. In the definition of a k-sum we are permitted to delete edges from the clique involved; if we do not delete any such edges let us call it a *pure* k-sum. Jørgensen [38] proved:

3.4 Theorem: *Let G be an n-vertex graph with no K_8 minor, with $n \ge 6$ and $|E(G)| > 6n-21$; then $|E(G)| = 6n-20$, and G can be built by pure 5-sums from copies of $K_{2,2,2,2,2}$.*

The same holds for K_9; Song and Thomas [71] proved:

3.5 Theorem: *Let G be an n-vertex graph with no K_9 minor, with $n \ge 7$ and $|E(G)| > 7n-28$; then $|E(G)| = 7n-27$, and either $G = K_{2,2,2,3,3}$, or G can be built by pure 6-sums from copies of $K_{1,2,2,2,2,2}$.*

But as t grows, the formula of 3.3 becomes completely wrong. For a graph H, let $\phi(H)$ be the infimum of all d such that every graph G with no H minor has average degree at most d, that is, satisfies $|E(G)| \le d|V(G)|/2$. (We are particularly concerned here with the case when H is a complete graph K_t, but $\phi(H)$ is of interest for non-complete graphs too.) Kostochka [48, 50] and de la Vega [16] proved that $\phi(K_t)$ is at least of order $t(\log t)^{1/2}$, and Kostochka [48, 50] and Thomason [72] proved the same was an upper bound; and in particular Kostochka[50] showed (logarithms are to base e):

3.6 Theorem: *For every integer $t > 0$,*

$$0.128 \le \frac{\phi(K_t)}{t(\log t)^{1/2}} \le 6.3.$$

Later Thomason [73] found the limit exactly: he proved (again with logarithms to base e):

3.7 Theorem: *Let $\lambda < 1$ be the solution of the equation $1-\lambda+2\lambda\log\lambda = 0$ and let*

$$\alpha = (1-\lambda)\log(1/\lambda)^{-1/2} \simeq 0.63817.$$

Then as $t \to \infty$, $\phi(K_t) = (\alpha+o(1))t(\log t)^{1/2}$.

This was extended to non-complete graphs by Myers and Thomason [62], who proved the following ($\mathbb{R}^+$ denotes the set of nonnegative real numbers, and α is as before):

3.8 Theorem: *Let H be a graph with t vertices, and let $\gamma(H)$ be the minimum of $\frac{1}{t}\sum_{u\in V(H)} w(u)$ over all functions $w : V(H) \to \mathbb{R}^+$ such that*

$$\sum_{uv\in E(H)} t^{-w(u)w(v)} \le t.$$

Then as $t \to \infty$, $\phi(H) = (\alpha\gamma(H) + o(1))t(\log t)^{1/2}$.

For classes of graphs H with $\gamma(H)$ bounded away from zero (such as regular graphs with degree ct^{ε} where $c, \varepsilon > 0$), this determines $\phi(H)$ asymptotically; but for some classes of graphs (such as those with a linear number of edges) it does not. This gap is addressed by two theorems of Reed and Wood [66]:

3.9 Theorem: *There is a constant d_0 such that $\phi(H) \le 3.895(\log d)^{1/2}t$ for every graph H with t vertices and average degree $d \ge d_0$.*

3.10 Theorem: *For every graph H, $\phi(H) \le |V(H)| + 6.291|E(H)|$.*

The Myers-Thomason theorem implies that $\phi(H)$ is not linear in t for graphs with t vertices and with a quadratic number of edges; but the second Reed-Wood theorem implies that if $|E(H)|$ is linear in t then so is $\phi(H)$.

For some graphs H we can determine exactly the maximum number of edges in graphs with no H minor, but those theorems are thinner on the ground. We already mentioned the cases when $H = K_t$; and the same can be done for many graphs H with at most six vertices, such as $K_{3,3}$; and there are two theorems doing it for larger graphs H. Chudnovsky, Reed and I [14] answered it for $K_{2,t}$ (extending a result of Myers [61]), and Kostochka and Prince [51] did it for $K_{3,t}$ (and the $K_{1,t}$ result is obvious):

3.11 Theorem: *Let G be an n-vertex graph with no H minor.*

- *If $H = K_{1,t}$ then $|E(G)| \le \frac{1}{2}(t-1)n$;*
- *if $H = K_{2,t}$ then $|E(G)| \le \frac{1}{2}(t+1)(n-1)$; and*
- *if $H = K_{3,t}$ and $t \ge 6300$ and $n \ge t+3$ then $|E(G)| \le \frac{1}{2}(t+3)(n-2)+1$.*

All three results are exact for infinitely many values of n. (By the way, when $H = K_{1,t}$, if we restrict to connected graphs G then the answer is quite different, namely $|E(G)| \le n + (t+1)(t-2)/2$ if $n \ge t+2$; see [21].)

What about $H = K_{s,t}$ in general, if $t \ge s$? For fixed s and large t, the value of $\phi(K_{s,t})$ is not determined by 3.8, so this is an interesting case. It turns out to be more natural to exclude $K^*_{s,t}$ instead; this is the graph obtained from $K_{s,t}$ by adding edges joining all pairs of vertices in the side of cardinality s. Extrapolating from 3.11, one might hope that if an n-vertex graph has no $K_{s,t}$ minor then

$$|E(G)| \le \frac{1}{2}(2s+t-3)n - \frac{1}{2}(s-1)(s+t-1),$$

because again this can be attained with equality for infinitely many n (take many disjoint copies of K_t and add $s-1$ extra vertices adjacent to everything). But this is not true, at least for $s > 18$. Kostochka and Prince [52, 53] proved (and see also [54] for a related result) that with the function $\phi(H)$ as before (here logarithms are binary):

3.12 Theorem: *Let s, t be positive integers with $t > (180s\log s)^{1+6s\log s}$. Then*

$$3s - 5s^{1/2} + t \leq \phi(K_{s,t}) \leq \phi(K^*_{s,t}) < 3s + t.$$

All these results tell us that the graphs with a certain minor H excluded have average degree at most some constant, and therefore have minimum degree at most the same constant; and that gives us a bound on their degeneracy. In particular, from 3.6, every graph with no K_t minor has degeneracy at most $O(t(\log t)^{1/2})$, and therefore chromatic number at most the same. For large t, this is the best bound known on the chromatic number of graphs excluding K_t.

Incidentally, bounding minimum degree by average degree is natural, but it might not give the right answer. For instance, graphs with no K_4 minor can have average degree > 3; and yet they always have minimum degree at most 2. When we exclude K_5, average degree gives the true bound for minimum degree; but what happens with K_6? Graphs with no K_6 minor can have average degree more than 7, but can they have minimum degree 7? I think this is open.

4 Stability number

One possible cause of the intractability of Hadwiger's conjecture is that we need to use the fact that the chromatic number is large, and graphs can have large chromatic number for obscure reasons. What if we make our lives easier, and look at graphs that have large chromatic number for obvious reasons? The *stability number* $\alpha(G)$ of a graph G is the size of the largest stable set (a set of vertices is *stable* if no two of its members are adjacent). (This is different from Thomason's α, which we do not need any more.) Every n-vertex graph G has chromatic number at least $\lceil n/\alpha(G)\rceil$, and should contain a clique minor of this size if Hadwiger's conjecture is true. Can we prove this at least?

The signs are not good; the only known proof that every n-vertex planar graph has stability number at least $n/4$ is via the 4CT. Nevertheless, there are some results. There is an elegant argument by Duchet and Meyniel [24] proving:

4.1 Theorem: *Every n-vertex graph G has a K_t minor where $t \geq n/(2\alpha(G)-1)$.*

Their argument can also be used to show a result that seems to have been overlooked:

4.2 Theorem: *For every graph G with no K_{t+1} minor, there exists a t-colourable induced subgraph containing at least half the vertices of G.*

4.1 is within a factor of 2 of what should be true, and there have been subsequent improvements, notably by Fox [28] (who proved a factor slightly less than 2) and then Balogh and Kostochka [6], who reduced Fox's factor a little further and currently have the record. They showed the following:

4.3 Theorem: *Every n-vertex graph G has a K_t minor where* $t \geq 0.51338n/\alpha(G)$.

A different strengthening, better than 4.3 when α is small, was proved by Kawarabayashi and Song [46]:

4.4 Theorem: *Every n-vertex graph G with* $\alpha(G) \geq 3$ *has a K_t minor where* $t \geq n/(2\alpha(G)-2)$.

Returning to 4.1: it implies that if G has no K_{t+1} minor then some stable set has cardinality at least $n/(2t)$. Suppose we give each vertex of G a nonnegative real weight. Hadwiger's conjecture would imply that there is a stable set such that the total weight of its members is at least $1/t$ times the sum of all weights. One might hope to prove a weighted version of 4.1 (without the -1 in the denominator) and this turns out to be true, though more difficult to prove. Say the *fractional chromatic number* of a graph G is the minimum real number k such that for some integer $s > 0$, there is a list of ks stable sets of G such that every vertex is in s of them. Via linear programming duality, the weighted Duchet-Meyniel statement is equivalent to the following, proved by Reed and myself [64]:

4.5 Theorem: *Every graph with no K_{t+1} minor has fractional chromatic number at most* $2t$.

The proof also gives a corresponding extension of 4.2:

4.6 Theorem: *In every graph G with no K_{t+1} minor, there is a non-null list of t-colourable subsets of $V(G)$, such that every vertex is in exactly half of the sets in the list.*

Graphs G with $\alpha(G) = 2$ are particularly interesting, because these graphs are more tractable for colouring; for instance, there is a polynomial-time algorithm to find the chromatic number of such a graph (just find the largest matching in the complement graph). Here is another nice feature of them: say a *seagull* in a graph G is an induced 3-vertex path. If $\alpha(G) = 2$ and S is a seagull in G then every other vertex of G has a neighbour in S, and so finding many disjoint seagulls is a way to find a large clique minor. In [15], Chudnovsky and I proved there is a min-max formula for the maximum number of disjoint seagulls in a graph G with $\alpha(G) = 2$.

For an n-vertex graph G with $\alpha(G) = 2$, the Duchet-Meyniel theorem implies that there is a K_t minor with $t \geq n/3$. This was strengthened by Böhme, Kostochka and Thomason [9], who proved (for graphs with arbitrary stability number):

4.7 Theorem: *Every n-vertex graph with chromatic number k has a K_t minor where* $t \geq (4k-n)/3$.

But Hadwiger's conjecture implies that if $\alpha(G) = 2$ then there should be a K_t minor with $t \geq n/2$. This seems to me to be an excellent place to look for a counterexample. My own belief is, if it is true for graphs with stability number two then it is probably true in general, so it would be very nice to decide this case. Despite some intensive effort the following remains open:

4.8 Open question: *Does there exists $c > \frac{1}{3}$ such that every graph G with $\alpha(G) = 2$ has a K_t minor where $t \geq c|V(G)|$?*

A graph is *claw-free* if no vertex has three pairwise nonadjacent neighbours. Thus graphs with stability number two are claw-free. Fradkin [30] proved:

4.9 Theorem: *Every n-vertex connected claw-free graph G with $\alpha(G) \geq 3$ has a K_t minor where $t \geq n/\alpha(G)$.*

Chudnovsky and Fradkin [13] proved:

4.10 Theorem: *Every claw-free graph G with no K_{t+1} minor is $\lfloor 3t/2 \rfloor$-colourable.*

Line graphs are claw-free, so these last two results are related to a theorem of Reed and myself; we proved [65] that Hadwiger's conjecture is true for line graphs (of multigraphs).

5 Weakenings

The statement of Hadwiger's conjecture is:

For all $t \geq 0$ and every graph G, either G has a K_{t+1} minor or $V(G)$ can be partitioned into t stable sets.

How can we weaken this and still have something non-trivial? Section 2 covered changing "For all $t \geq 0$" to "For a few $t \geq 0$"; section 3 did changing "t stable sets" to "$f(t)$ stable sets"; and section 4 covered changing "partitioned into stable sets" to "fractional chromatic number"; but there are several other ways to weaken the statement. Here are some.

Change "every graph G" to "almost every graph G". (The meaning of "almost every" here is that the proportion of n-vertex graphs that satisfy the statement tends to 1 as $n \to \infty$.) This weakening is true. It follows from a combination of a theorem of Bollobás, Catlin and Erdős [10] and a theorem of Grimmett and McDiarmid [34]:

5.1 Theorem: *For all $d > 2$, almost every n-vertex graph has a K_t minor where $t \geq n/((\log n)^{1/2} + 4)$, and has chromatic number at most $2n/\log n$.*

Change "K_{t+1}" to something else. If we hope to prove that every graph with no H minor has chromatic number at most t, then H had better have at most $t+1$ vertices, or else taking $G = K_{t+1}$ is a counterexample. So, which subgraphs H of K_{t+1} work? Kostochka [47, 49] proved the following.

5.2 Theorem: *For all s there exists t_0 such that for all $t \geq t_0$, every graph with no $K^*_{s,t}$ minor is $(s+t-1)$-colourable.*

Change "stable sets" to something else. Here is a recent theorem of Edwards, Kang, Kim, Oum and myself [26]:

5.3 Theorem: *For all t there exists k such that if G has no K_{t+1} minor then $V(G)$ can be partitioned into t sets $X_1,\ldots,X_t$, such that for $1 \le i \le t$, $G[X_i]$ has maximum degree at most k.*

(For $X \subseteq V(G)$, $G[X]$ denotes the subgraph induced on X.) This result is quite easy, but it has two attractive features; first, it is best possible in that if we ask for a partition into $t-1$ sets there is no such k; and second, it and 6.8 below are the only results known that derive a partition into t sets with *any* non-trivial property from the absence of a K_{t+1} minor.

There are more weakenings to describe yet, but they deserve a new section.

6 Bounded component-size

What if we try to improve 5.3? Let us say $X \subseteq V(G)$ has *component-size k* if the largest component of $G[X]$ has k vertices. Thus having bounded component-size is more restrictive than have bounded maximum degree (though less than what we really want, being stable). Instead of just saying that each $G[X_i]$ has bounded maximum degree, what if we ask that each of them has bounded component-size? It has not been proved that for graphs G with no K_{t+1} minor, we can partition into t sets with this property, but there has been a series of papers proving that $V(G)$ can be partitioned into a linear number of parts each with bounded component-size. Initially Kawarabayashi and Mohar [42] proved:

6.1 Theorem: *For all $t \ge 0$ there exists k such that if G has no K_t minor, then $V(G)$ can be partitioned into at most $f(t)$ parts each with component-size at most k, where $f(t) = \lceil 31t/2 \rceil$.*

Wood [83] proved the same with $f(t) = \lceil 7t/2 - 3/2 \rceil$ (using 10.6, an unpublished theorem of Norin and Thomas which we discuss later), and there have been further improvements which we describe below.

There is a set of lemmas here that can be combined in various ways. DeVos, Ding, Oporowski, Reed, Sanders, Vertigan and I [17] proved:

6.2 Theorem: *For all t there exists w such that for every graph G with no K_t minor, there is a partition of $V(G)$ into two parts, such that the subgraph induced on each part has tree-width at most w.*

Alon, Ding, Oporowski and Vertigan [3] showed:

6.3 Theorem: *For all w,d and for every graph G with tree-width at most $w \ge 3$ and maximum degree at most $d \ge 1$, there is a partition of $V(G)$ into two parts each with component-size at most $24wd$.*

Wood [84] improved this, replacing $24kd$ with $5(k+1)(7d-2)/4$. Liu (unpublished) has recently proved a list-colouring version:

6.4 Theorem: *For all w,d there exists k such that for every graph G with tree-width at most w and maximum degree at most d, and every assignment of a set L_v*

with $|L_v| \geq 2$ to each vertex v, there is a choice of $c(v) \in L_v$ for each v such that for each x, the set of all vertices v with $c(v) = x$ has component-size at most k.

Incidentally, an interesting asymmetric version was proved by Ding and Dziobiak [20]:

6.5 Theorem: *For all $t \geq 0$ there exists $w \geq 0$ such that for every graph G with no K_t minor, $V(G)$ can be partitioned into two sets X, Y, where $G[X]$ has tree-width at most w, and $G[Y]$ is $(t+1)$-degenerate.*

By combining 6.2 and 6.3, Alon et al. deduced:

6.6 Theorem: *For all t, d there exists k such that for every graph G with no K_t minor and maximum degree at most d, there is a partition of $V(G)$ into four parts each with component-size at most k.*

Recently, Liu and Oum [57] improved this, replacing "four" by "three". If we then combine their result with 5.3 we deduce an improvement of 6.1 with $f(t) = 3(t-1)$. Even more recently, Norin used a different approach to do better. He proved the following lemma [63]:

6.7 Theorem: *For all $t, w \geq 0$ there exists N with the following property. Let G be a graph with $|V(G)| \geq N$, with tree-width at most w and with no K_t minor. Then for every $S \subseteq V(G)$ with $|S| \leq 2w$, there exists $I \subseteq V(G) \setminus S$, nonempty, such that at most $2w$ vertices in $V(G) \setminus I$ have a neighbour in I, and every vertex in I has at most $t-2$ neighbours in $V(G) \setminus I$.*

With the aid of this lemma, an easy inductive argument yields:

6.8 Theorem: *For all $t, w \geq 0$ there exists k such that for every graph G with tree-width at most w and no K_t minor, there is a partition of $V(G)$ into $t-1$ parts such that each part has component-size at most k.*

Then this, combined with 6.2, yields an improvement of 6.1 with $f(t) = 2(t-1)$.

7 Odd minors

We have finished with weakenings of Hadwiger's conjecture now; time to turn to strengthenings.

Graphs that are not 2-colourable not only have a K_3 minor (or equivalently, a cycle); they have an odd cycle. It is tempting to try to make some corresponding strengthening of Hadwiger's conjecture. Here is what seems to be the most natural way to do it. If G is a graph and $X \subseteq V(G)$, $\delta(X)$ denotes the set of edges of G with one end in X and the other in $V(G) \setminus X$. We say that $F \subseteq E(G)$ is a *cut* of G if $F = \delta(X)$ for some $X \subseteq V(G)$. Now let G, H be graphs. We say that H is an *odd minor* of G if H can be obtained from a subgraph G' of G by contracting a set of edges that is a cut of G'. (Note that $\emptyset$ is a cut.) Thus a graph is not 2-colourable if and only if it contains K_3 as an odd minor. In 1979, Catlin [12] proved:

7.1 Theorem: *If G has no K_4 odd minor then G is 3-colourable.*

Incidentally, a much stronger statement than this has now been proved. Say a *fully odd K_4* in G is a subgraph of G which is obtained from K_4 by replacing each edge of K_4 by a path of odd length (the *length* of a path is the number of edges in it) in such a way that the interiors of these six paths are disjoint. Toft [76] conjectured in 1975 and Zang [85] proved in 1998 (and, independently, Thomassen [75] proved in 2001) that:

7.2 Theorem: *If G contains no fully odd K_4 then G is 3-colourable.*

Returning to odd minors: there is a result giving a construction for all graphs with no K_4 odd minor, due to Lovász, Schrijver, Truemper and myself. It is rather awkward to state, and not published, although it was proved many years ago (in the early 1980's, and mostly on a riverboat in Bonn, if I remember correctly). We omit its statement here; see [32].

In view of its truth for $t \leq 3$, Gerards and I conjectured the following strengthening of Hadwiger's conjecture (see [37]):

7.3 Conjecture: *For every $t \geq 0$, if G has no K_{t+1} odd minor, then G is t-colourable.*

Several of the results mentioned earlier approaching Hadwiger's conjecture have extensions to odd minors. For instance, Guenin [35] announced at a meeting in Oberwolfach in 2005 that:

7.4 Theorem: *Every graph with no K_5 odd minor is 4-colourable.*

Geelen, Gerards, Reed, Vetta and I [31] proved (see also [40] for a simpler proof):

7.5 Theorem: *If G has no K_t odd minor then $\chi(G) \leq O(t(\log t)^{1/2})$.*

Kawarabayashi and Song [46] proved an odd minor version of 4.1, namely:

7.6 Theorem: *Every n-vertex graph G has a K_t odd minor where $t \geq n/(2\alpha(G)-1)$.*

Kawarabayashi and Reed [44] proved:

7.7 Theorem: *Every graph with no K_t odd minor is fractionally $2t$-colourable.*

Kawarabayashi and Song [46] proved an odd minor relative of 10.2:

7.8 Theorem: *For every $t \geq 0$, there exists N such that for every $(496t+13)$-connected graph G with at least N vertices, either G has a K_t odd minor, or there exists $X \subseteq V(G)$ with $|X| \leq 8t$ such that $G \setminus X$ is bipartite.*

Kawarabayashi [39] proved:

7.9 Theorem: *If G has no K_t odd minor, then there is a partition of $V(G)$ into $496t$ parts such that each part induces a subgraph of bounded maximum degree.*

8 Other strengthenings

There have been some other strengthenings of Hadwiger's conjecture proposed, but they have mostly not fared so well as 7.3. For instance, say G is a *subdivision* of H if G can be obtained from H by replacing each edge by a path, where the paths have disjoint interiors. Hajós conjectured in the 1940's (but did not publish it) that

8.1 False conjecture: *For all $t \geq 0$, if no subgraph of G is a subdivision of K_{t+1} then G is t-colourable.*

This is true for $t \leq 3$, but it is still open for $t = 4,5$, and Catlin [12] gave a counterexample for all $t \geq 6$. Indeed, Erdős and Fajtlowicz [27] proved that almost all graphs are counterexamples, because of the following:

8.2 Theorem: *There are constants C_1, C_2 such that for almost every n-vertex graph G, no subgraph of G is a subdivision of K_t for $t \geq C_1 n^{1/2}$, and the chromatic number of G is at least $C_2 n/\log(n)$.*

Erdős and Fajtlowicz conjectured and Fox, Lee and Sudakov [29] proved that the ratio of the chromatic number over the clique subdivision number over all n-vertex graphs is maximized up to a constant factor by the random graph on n vertices; in other words, the uniform random graph is essentially the strongest counterexample to the Hajós conjecture.

Another strengthening of Hadwiger's conjecture was proposed by Borowiecki [11]. A graph G is t-*choosable* if for every assignment of a t-element set L_v to each vertex v of G, it is possible to select a member $c(v) \in L_v$ for each v such that $c(u) \neq c(v)$ if u, v are adjacent. Borowiecki asked whether

8.3 False conjecture: *Every graph with no K_{t+1} minor is t-choosable.*

This is true for $t \leq 3$, but false for $t = 4$; Voigt [80] gave a planar graph that is not 4-choosable. Thomassen [74] proved that all planar graphs are 5-choosable, but an additive constant adjustment is not enough to repair the conjecture in general; Barát, Joret and Wood [7] showed that for all $t \geq 1$ there is a graph with no K_{3t+2} minor that is not $4t$-choosable. Kawarabayashi and Mohar [42] conjectured:

8.4 Conjecture: *For all t, every graph with no K_t minor is $3t/2$-choosable.*

A third strengthening was proposed by Ding, Oporowski, Sanders and Vertigan [22]:

8.5 Conjecture: *For all integers $t \geq s \geq 2$, if G has no K_t minor, then there is a partition of $V(G)$ to $t-s+1$ parts, such that the subgraph induced on each part has no K_s minor.*

For $s = 2$ this is Hadwiger's conjecture, but it has not been disproved for any values of s, t. For $s \geq t-1$ it is easy, and it was proved for $s = t-2$ by Gonçalves [33].

Reed and I proposed a fourth variation in [64]:

8.6 Conjecture: *For every graph G with no no K_{t+1} minor, there is a partition of its vertex set, such that each part induces a connected bipartite graph, and contracting all the edges within the parts yields a graph with no induced cycle of length more than three.*

This would imply that all graphs with no K_{t+1} minor are $2t$-colourable. It remains open.

A fifth possible extension is to infinite graphs. (Henceforth, graphs may be infinite in this section.) By compactness, if t is an integer and HC(t) holds for finite graphs then it also holds for infinite graphs; but we could try to extend Hadwiger's conjecture to allow infinitely many colours. One might hope that

8.7 False conjecture: *For every cardinal t, every graph with no K_t minor has chromatic number less than t;*

but this is trivially false (let G be the disjoint union of infinitely many finite cliques, one of each size; then G cannot be coloured with finitely many colours, but has no infinite clique minor). A better formulation is:

8.8 Conjecture: *For every cardinal t, let s be the least cardinal larger than t; every graph with no K_s minor is t-colourable.*

I believe 8.8 remains open, but van der Zypen [79] proved the following:

8.9 Theorem: *For every infinite cardinal t, every graph with no subgraph which is a subdivision of K_t is t-colourable.*

Van der Zypen's proof uses the fact that when t is an infinite cardinal, one can give a construction of the graphs that contain no subdivision of K_t, a result due to Robertson, Thomas and myself [69]. It is also possible [68] to do the same for graphs that contain no K_t minor when t is an infinite cardinal.

9 Immersions

There is an interesting conjecture, parallel to Hadwiger's conjecture, that was proposed by Lescure and Meyniel [56], (and independently, by Abu-Khzam and Langston [1], later). Let G,H be graphs. An *immersion* of H in G is a choice $\eta(v) \in V(G)$ for each $v \in V(H)$, all distinct, and a choice $\eta(e)$ for each $e \in E(G)$, where for $e = uv$, $\eta(e)$ is a path of G between $\eta(u)$ and $\eta(v)$, and all the paths $\eta(e)$ are pairwise edge-disjoint (they may share vertices; and an end-point of one path may be an internal vertex of another). Let us say G *immerses* H if there is an immersion of H in G. Lescure and Meyniel proposed:

9.1 Conjecture: *For every integer $t \geq 0$, every graph that does not immerse K_{t+1} is t-colourable.*

This neither implies nor is implied by Hadwiger's conjecture, since immersing K_{t+1} neither implies nor is implied by having a K_{t+1} minor; but it is in some respects similar. (In one respect it is very different: planar graphs can immerse huge complete graphs.) 9.1 was proved for $t \leq 6$ by Lescure and Meyniel (though they did not publish the proof for $t = 6$), and more recently DeVos, Kawarabayashi, Mohar and Okamura [19] published a proof for $t = 6$. Both sets of authors used the same approach, proving the stronger statement that for $t \leq 6$, every simple graph with minimum degree at least t immerses K_t. For $t \geq 9$, it is not true that every graph with minimum degree at least t immerses K_t; but DeVos, Dvořák, Fox, McDonald, Mohar and Scheide [18] proved the following (in fact they proved it for "strong"immersion, in which the vertices $\eta(v)$ are not permitted to be internal vertices of the paths $\eta(e)$):

9.2 Theorem: *For all $t \geq 0$, every graph of minimum degree at least $200t$ immerses K_t.*

It follows that

9.3 Theorem: *Every graph that does not immerse K_t is $200t$-colourable.*

10 Big graphs

The constructions of Kostochka and de la Vega mentioned earlier show that there are graphs with no K_t minor with average degree of the order of $t(\log t)^{1/2}$, and indeed their minimum degree and connectivity are also of this order. But there is a feeling that honest, sensible graphs with no K_t minor are not really like this; they will have vertices of degree about t. How can we make this intuition closer to a true statement?

The intuition comes mostly from the Graph Minors structure theorem of Robertson and myself [67], which says very roughly that to make graphs with no K_{t+1} minor, one takes graphs on surfaces of bounded genus and adds a bounded number of extra vertices; and if these extra vertices are not just attached to small parts of the surface, there had better not be many of them (or else we will get a K_{t+1} minor); in fact at most $t-4$ of them, and fewer if the surface is not the plane. But typical vertices in the surface have average degree (in the surface) less than six, so total degree less than $t+2$. There are have been several attempts to bring this very vague argument closer to reality, and in this section we discuss some of them.

The feeling is that the examples of Kostochka and de la Vega have only bounded size (which they do) in some essential way (which remains to be made precise). Of course we can make bigger examples by taking disjoint unions of the little ones, but then the connectivity is lost. What if we impose some connectivity restriction? Can there still be large examples?

Thomas and I conjectured:

10.1 Conjecture: *For all $t \geq 0$ there exists N such that every $(t-2)$-connected graph G with no K_t minor and with $n \geq N$ vertices satisfies*

$$|E(G)| \leq (t-2)n - (t-1)(t-2)/2.$$

This remains open. Böhme, Kawarabayashi, Maharry and Mohar [8] proved:

10.2 Theorem: *For all positive integers t, there exists N such that every $3t+2$-connected graph with no K_t minor and with at least N vertices has a vertex of degree less than $31(t+1)/2-3$.*

This is very encouraging: everything is linear, the frightening $(\log t)^{1/2}$ term has disappeared.

How can we arrange some decent connectivity? To prove HC(t) it is enough to prove the impossibility of minimal or minimum counterexamples to HC(t) (a counterexample is "minimal" if no proper minor of itself is a counterexample; and "minimum" if no counterexample is smaller.) What about the connectivity of minimal counterexamples? Kawarabayashi [41] proved:

10.3 Theorem: *For $t \geq 0$, every minimal counterexample to HC(t) is $\lceil 2(t+1)/27 \rceil$-connected, and every minimum counterexample to HC(t) is $\lceil (t+1)/3 \rceil$-connected.*

Mader [60] proved that for any value of t, if G is a minimal counterexample to HC(t) then G is 6-connected, and 7-connected if $t \geq 6$. When $t = 5$ this is particularly interesting, because it means that to prove HC(5) we only have to consider 6-connected graphs without K_6 minors. And Jørgensen [38] conjectured the following:

10.4 Conjecture: *Every 6-connected graph with no K_6 minor is apex.*

(We recall that a graph is *apex* if it can be made planar by deleting one vertex, and in particular all apex graphs are 5-colourable.) Thus if only we could prove Jørgensen's conjecture, we would obtain a much more appealing proof of HC(5). Unfortunately it remains open; but it might point a way to solve Hadwiger's conjecture in general, if we could only figure out an analogue of this conjecture for larger values of t (and then figure out how to prove it).

Kawarabayashi, Norin, Thomas and Wollan [43] proved that 10.4 itself is true in large graphs:

10.5 Theorem: *There exists N such that every 6-connected graph with at least N vertices and with no K_6 minor is apex.*

More recently Norin and Thomas have announced the following analogue of 10.5 for general values of t (this is a difficult result with a huge proof, and is still being written at this time):

10.6 Theorem: *For all $t \geq 0$ there exists N such that every $t+1$-connected graph with at least N vertices and with no K_{t+1} minor can be made planar by deleting $t-4$ vertices.*

So it would be nice to know that minimal counterexamples to HC(t) are $t+1$-connected; but we do not know this.

But recently it may have been shown that in fact there are no large minimal counterexamples to HC(t), using a feature of them slightly different from connectivity.

A *cutset* of G means (in this paper) a partition (A,B,C) of $V(G)$ with $A,B \neq \emptyset$, such that there are no edges between A and B. A *one-way clique cutset* of G means a cutset (A,B,C), and for each $v \in C$ a connected subgraph $X_v \subseteq B \cup C$ containing v, such that X_u, X_v are disjoint and some edge has an end in X_u and an end in X_v, for all distinct $u, v \in C$. In other words, we can turn C into a clique by contracting edges within $B \cup C$. Suppose that G is a minimal counterexample to HC(t), for some value of t. Then it is easy to show that:

- no vertex has degree at most t;
- no vertex of degree $t+1$ has three nonadjacent neighbours;
- there is no one-way clique cutset; and
- G cannot be made planar by deleting $t-4$ vertices.

Robertson and I announced (about 1993) that we proved:

10.7 Theorem: *For all $t \geq 0$ and for every graph G with no K_{t+1} minor, if G satisfies the four bullets above then G has bounded tree-width.*

This had all kinds of pleasing consequences, but the proof was very long, and was never written down, and now it is lost. Fortunately, almost the same thing, and with the same desirable consequences, has recently been announced by Kawarabayashi and Reed [45], and their proof seems more manageable, and may get written down. (At the moment the proof sketched in [45] has developed a few cracks, but Kawarabayashi maintains it can be fixed.) They added a fifth bullet to the four above:

- there do not exist a cutset (A,B,C) of G and disjoint connected subgraphs $X_1,\dots,X_k$ of of $G[A \cup C]$, each including a stable subset of C, such that if B' denotes the graph obtained from $B \cup X_1 \cup \cdots \cup X_t$ by contracting the edges of $X_1,\dots,X_t$, then every t-colouring of B' extends to one of G.

This evidently also holds in any minimal counterexample to HC(t). They claim:

10.8 Theorem: *For all $t \geq 0$ and for every graph G with no K_{t+1} minor, if G satisfies the five bullets above then G has bounded tree-width.*

This would have several consequences. The most important is probably an explicit function $f(t)$ such that for all t, every minimal counterexample to HC(t) has at most $f(t)$ vertices.

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