An even better Density Increment Theorem and its application to Hadwiger's Conjecture

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Dedicated to the memory of Robin Thomas

Abstract

In 1943, Hadwiger conjectured that every graph with no K_t minor is $(t-1)$-colorable for every $t \geq 1$. In the 1980s, Kostochka and Thomason independently proved that every graph with no K_t minor has average degree $O(t\sqrt{\log t})$ and hence is $O(t\sqrt{\log t})$-colorable. Recently, Norin, Song and the author showed that every graph with no K_t minor is $O(t(\log t)^{\beta})$-colorable for every $\beta > 1/4$, making the first improvement on the order of magnitude of the $O(t\sqrt{\log t})$ bound. More recently, the author showed that every graph with no K_t minor is $O(t(\log t)^{\beta})$-colorable for every $\beta > 0$; more specifically, they are $t \cdot 2^{O((\log\log t)^{2/3})}$-colorable. In combination with that work, we show in this paper that every graph with no K_t minor is $O(t(\log\log t)^6)$-colorable.

1 Introduction

All graphs in this paper are finite and simple. Given graphs H and G, we say that G has an H *minor* if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges. We denote the complete graph on t vertices by K_t.

In 1943 Hadwiger made the following famous conjecture.

Conjecture 1.1 (Hadwiger's conjecture [Had43]). *For every integer $t \geq 1$, every graph with no K_t minor is $(t-1)$-colorable.*

Hadwiger's conjecture is widely considered among the most important problems in graph theory and has motivated numerous developments in graph coloring and graph minor theory. For an overview of major progress on Hadwiger's conjecture, we refer the reader to [NPS19], and to the recent survey by Seymour [Sey16] for further background.

The following is a natural weakening of Hadwiger's conjecture, which has been considered by several researchers.

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Conjecture 1.2 (Linear Hadwiger's conjecture [RS98, Kaw07, KM07])**.** *There exists a constant $C > 0$ such that for every integer $t \geq 1$, every graph with no K_t minor is Ct-colorable.*

For many decades, the best general bound on the number of colors needed to properly color every graph with no K_t minor had been $O(t\sqrt{\log t})$, a result obtained independently by Kostochka [Kos82, Kos84] and Thomason [Tho84] in the 1980s. The results of [Kos82, Kos84, Tho84] bound the "degeneracy" of graphs with no K_t minor. Recall that a graph G is *d-degenerate* if every non-empty subgraph of G contains a vertex of degree at most d. A standard inductive argument shows that every d-degenerate graph is $(d+1)$-colorable. Thus the following bound on the degeneracy of graphs with no K_t minor gives a corresponding bound on their chromatic number and even their list chromatic number.

Theorem 1.3 ([Kos82, Kos84, Tho84])**.** *Every graph with no K_t minor is $O(t\sqrt{\log t})$-degenerate.*

Very recently, Norin, Song and the author [NPS19] improved this with the following theorem.

Theorem 1.4 ([NPS19])**.** *For every $\beta > \frac{1}{4}$, every graph with no K_t minor is $O(t(\log t)^{\beta})$-colorable.*

In [NS19b], Norin and Song extended Theorem 1.4 to odd minors. In [NP20], Norin and the author extended Theorem 1.4 to list coloring. Even more recently, the author in [Pos20] further improved the bound in Theorem 1.4 as follows.

Theorem 1.5. *Every graph with no K_t minor is $t \cdot 2^{O((\log \log t)^{2/3})}$-colorable. Hence for every $\beta > 0$, every graph with no K_t minor is $O(t(\log t)^{\beta})$-colorable.*

The main result of this paper is the following.

Theorem 1.6. *Every graph with no K_t minor is $O(t(\log \log t)^6)$-colorable.*

1.1 A better density increment theorem

The key to the improvement is a nearly optimal density increment theorem as follows.

Theorem 1.7. *There exists a constant $C = C_{1.7} > 0$ such that the following holds. Let G be a graph with $\mathsf{d}(G) \geq C$, and let $D > 0$ be a constant. Let $s = D/\mathsf{d}(G)$ and let $g_{1.7}(s) := C(1 + \log s)^6$. Then G contains at least one of the following:*

(i) *a minor J with $\mathsf{d}(J) \geq D$, or*

(ii) *a subgraph H with $\mathsf{v}(H) \leq g_{1.7}(s) \cdot \frac{D^2}{\mathsf{d}(G)}$ and $\mathsf{d}(H) \geq \frac{\mathsf{d}(G)}{g_{1.7}(s)}$.*

In [NS19a], Norin and Song had proved Theorem 1.7 with $g(s) = s^{\alpha}$ for any $\alpha > \frac{\log(2)}{\log(3/2)} - 1 \approx .7095$. Using that result, they showed that every graph with no K_t minor is $O(t(\log t)^{0.354})$-colorable. Shortly thereafter, in [Pos19], the author improved this to $g(s) = s^{o(1)}$. That result was then combined in [NPS19] with the earlier work to yield Theorem 1.4. The function $g(s)$ in [NPS19] was not explicitly found. It is not hard to derive an explicit function of $g(s) = 2^{O((\log s)^{2/3}+1)}$ from Lemma 2.5 in [NPS19]. The main result of this paper is the new bound listed above.

1.2 Proof of Theorem 1.6

We need the following theorem proved in [Pos20].

Theorem 1.8 (Theorem 2.2 in [Pos20])**.** *Every graph with no K_t minor has chromatic number at most*

$$O\left(t \cdot \left(g_{1.7}\left(3.2 \cdot \sqrt{\log t}\right) + (\log \log t)^2\right)\right).$$

Theorem 1.6 follows immediately from Theorems 1.7 and 1.8.

1.3 Outline of Paper

In Section 2, we introduce our more technical main theorem, Theorem 2.1, and derive Theorem 1.7 from it. In Section 3, we outline the proof of Theorem 2.1 while reviewing some preliminary definitions; namely, Theorem 2.1 follows from two other main theorems, Theorems 3.6 and 3.7. In Section 4, we prove Theorem 3.6 which shows that a very unbalanced bipartite graph of high minimum degree has either a small, dense subgraph or an $(\ell+1)$-bounded minor with density almost ℓ times the original. In Section 5, we prove Theorem 3.7 which shows that a graph of high density has either a small, dense subgraph, or a very unbalanced bipartite graph of high density, or a k-bounded minor with density almost k. As mentioned above, we combine in Section 3 these results to prove Theorem 2.1 by choosing k and ℓ appropriately. Finally in Section 6, we discuss impediments to improving the bound in Theorem 1.6.

1.4 Notation

We use largely standard graph-theoretical notation. We denote by $\mathsf{v}(G)$ and $\mathsf{e}(G)$ the number of vertices and edges of a graph G, respectively, and denote by $\mathsf{d}(G) = \mathsf{e}(G)/\mathsf{v}(G)$ the *density* of a non-empty graph G. We use $\chi(G)$ to denote the chromatic number of G, and $\kappa(G)$ to denote the (vertex) connectivity of G. The degree of a vertex v in a graph G is denoted by $\deg_G(v)$ or simply by $\deg(v)$ if there is no danger of confusion. We denote by $G[X]$ the subgraph of G induced by a set $X \subseteq V(G)$. For $F \subseteq E(G)$, we denote by G/F the minor of G obtained by contracting the edges of F. If A and B are disjoint subsets of $V(G)$, then we let $G(A,B)$ denote the bipartite subgraph with $V(G(A,B)) = A \cup B$ and $E(G(A,B)) = \{uv \in E(G) : u \in A, v \in B\}$.

2 Outline of Proof of Density Increment Theorem

Recall that our goal in this paper is to prove Theorem 1.7. In fact, we prove the following more technical theorem which is an improvement over similar theorems in [NPS19, Pos19].

Theorem 2.1. *Let $k \geq 100$ be an integer. Let G be a graph with $d = \mathsf{d}(G) \geq k^2$. Then G contains at least one of the following:*

(i) *a subgraph H with $\mathsf{v}(H) \leq 12 \cdot k^4 \cdot d$ and $\mathsf{d}(H) \geq \frac{d}{24k^6}$, or*

(ii) *an m-bounded minor G' with $\mathsf{d}(G') \geq m \cdot \left(1 - \frac{16}{m}\right) \cdot d$ for some integer $m \in [\frac{k}{6}, k]$.*

The proof of Theorem 2.1 occupies Sections 3, 4 and 5. Now we are ready to derive Theorem 1.7 from Theorem 2.1. We restate Theorem 1.7 for convenience.

Theorem 1.7. *There exists a constant $C = C_{1.7} > 0$ such that the following holds. Let G be a graph with $\mathsf{d}(G) \geq C$, and let $D > 0$ be a constant. Let $s = D/\mathsf{d}(G)$ and let $g_{1.7}(s) := C(1+\log s)^6$. Then G contains at least one of the following:*

(i) *a minor J with $\mathsf{d}(J) \geq D$, or*

(ii) *a subgraph H with $\mathsf{v}(H) \leq g_{1.7}(s) \cdot \frac{D^2}{\mathsf{d}(G)}$ and $\mathsf{d}(H) \geq \frac{\mathsf{d}(G)}{g_{1.7}(s)}$.*

Proof of Theorem 1.7. Let $C_{1.7} = 2^{9 \cdot 5} = 2^{45}$. We proceed by induction on s. If $s \leq 1$, then $J = G$ is a minor of G with $\mathsf{d}(J) = \mathsf{d}(G) \geq s \cdot \mathsf{d}(G) = D$ and (i) holds as desired. So we may assume that $s > 1$. Hence $D \geq d \geq C_{1.7}$.

If $g_{1.7}(s) \geq d$, then let $H = uv$ where $uv \in E(G)$ and (ii) holds as desired since $D \geq C_{1.7} \geq 2$. So we may assume that $d \geq g_{1.7}(s)$.

Let $k = \frac{1}{2} \cdot (C_{1.7})^{\frac{1}{6}} \cdot (1+\log s) = 2^8 \cdot (1+\log s)$. Since $\log s \geq 0$, we have that

$$k \geq 2^8 = 256 > 100 > e^3.$$

Since $d \geq g_{1.7}(s) \geq (2k)^6$, we find that

$$d \geq k^2.$$

We apply Theorem 2.1 to G with this k. Note that $d \geq k^2$ as needed.

First suppose that Theorem 2.1(i) holds. That is, there exists a subgraph H with $\mathsf{v}(H) \leq 12 \cdot k^3 \cdot d$ and $\mathsf{d}(H) \geq \frac{d^2}{24k^5}$. Note that

$$12 \cdot k^4 \leq 24 \cdot k^6 \leq (2k)^6 \leq C_{1.7}(1+\log s)^6 = g_{1.7}(s).$$

Hence

$$\mathsf{v}(H) \leq 12 \cdot k^4 \cdot d \leq g_{1.7}(s) \cdot d \leq g_{1.7}(s) \cdot \frac{D^2}{\mathsf{d}(G)}$$

since $s \geq 1$ and furthermore

$$\mathsf{d}(H) \geq \frac{d}{24k^6} \geq \frac{d}{g_{1.7}(s)}.$$

But then (ii) holds as desired.

So we may assume that Theorem 2.1(ii) holds. That is, there exists an m-bounded minor G' of G with

$$d(G') \geq m \cdot \left(1 - \frac{16}{m}\right) \cdot d$$

for some integer $m \in [\frac{k}{6}, k]$. Let $d' = \mathsf{d}(G')$ and $s' = D/d'$. Note that since $k \geq 6 \cdot 32$, we have that $m \geq \frac{k}{6} \geq 32$. Hence

$$d' \geq m \cdot \left(1 - \frac{16}{m}\right) \cdot d \geq \frac{m}{2} \cdot d > d,$$

and reciprocally

$$s' \leq \frac{s}{m \cdot \left(1 - \frac{16}{m}\right)} \leq \frac{2s}{m} < s.$$

Since $s' < s$, we have by induction that at least one of (i) or (ii) holds for G'.

First suppose that (i) holds for G'. That is, there exists a minor J of G' with $\mathsf{d}(J) \geq D$. But then J is also a minor of G and (i) holds for G as desired.

So we may assume that (ii) holds for G'. That is, there exists a subgraph H' of G' with

$$\mathsf{v}(H') \leq g_{1.7}(s') \cdot \frac{D^2}{d'}$$

and

$$\mathsf{d}(H') \geq \frac{d'}{g_{1.7}(s')}.$$

But then H' corresponds to a subgraph H of G with $\mathsf{v}(H) \leq m \cdot \mathsf{v}(H')$ and $\mathsf{e}(H) \geq \mathsf{e}(H')$. Now

$$\mathsf{v}(H) \leq \ell \cdot \mathsf{v}(H') \leq m \cdot g_{1.7}(s') \cdot \frac{D^2}{d'} \leq \left(\frac{g_{1.7}(s')}{1 - \frac{16}{m}}\right) \cdot \frac{D^2}{d}.$$

Similarly

$$\mathsf{d}(H) = \frac{\mathsf{e}(H)}{\mathsf{v}(H)} \geq \frac{\mathsf{e}(H')}{\ell \cdot \mathsf{v}(H')} = \frac{\mathsf{d}(H')}{m} \geq \frac{d'}{m \cdot g_{1.7}(s')} \geq \left(\frac{1 - \frac{16}{m}}{g_{1.7}(s')}\right) \cdot d.$$

Note that

$$m \geq \frac{k}{6} \geq 16(1 + \log s).$$

Hence

$$\frac{1}{1 - \frac{16}{m}} \leq 1 + \frac{32}{m} \leq 1 + \frac{2}{1 + \log s},$$

where the first inequality follows since $\frac{16}{m} \leq \frac{1}{2}$ as $m \geq \frac{k}{6} \geq 32$. On the other hand,

$$\log(\ell) = \log(k) \geq \log(e^3) \geq 3,$$

since $k \geq e^3$. Hence

$$\log s' \leq \log\left(\frac{2s}{\ell}\right) \leq \log(s) + 1 - \log(\ell) \leq \log(s) - 2.$$

Thus

$$\frac{g_{1.7}(s')}{g_{1.7}(s)} \leq \frac{(1 + \log s')^6}{(1 + \log s)^6} \leq \frac{1 + \log s'}{1 + \log s} \leq \frac{1 + \log(s) - 2}{1 + \log s} = 1 - \frac{2}{1 + \log s}.$$

We now have that

$$\frac{g_{1.7}(s')}{1 - \frac{16}{m}} \leq \left(1 - \frac{2}{1 + \log s}\right)\left(1 + \frac{2}{1 + \log s}\right) g_{1.7}(s) \leq g_{1.7}(s).$$

Hence

$$\mathsf{v}(H) \leq g_{1.7}(s) \cdot \frac{D^2}{d},$$

and

$$\mathsf{d}(H) \geq \frac{d}{g_{1.7}(s)},$$

and (ii) holds as desired. ☐

3 Outline of the Proof of our Density Increment Theorem

In this section we introduce additional definitions used in the proof of Theorem 2.1, and outline its proof.

Definition 3.1. Let G be a graph, and let $K, d \geq 1$, $\varepsilon \in (0,1)$ be real. We say that

- a vertex of G is (K,d)-*small* in G if $\deg_G(v) \leq Kd$ and (K,d)-*big* otherwise;
- two vertices of G are (ε, d)-*mates* if they have at least εd common neighbors;
- G is $(K, \varepsilon_1, \varepsilon_2, d)$-*unmated* if every (K,d)-small vertex in G have strictly fewer than $\varepsilon_1 d$ (ε_2, d)-mates.

Here is a useful proposition and corollary.

Proposition 3.2. *For all $K, d \geq 1$, $\varepsilon_1, \varepsilon_2 \in (0,1)$ and every graph G at least one of the following holds:*

(i) *there exists a subgraph H of G with $\mathsf{v}(H) \leq 3Kd$ and $\mathsf{e}(H) \geq \varepsilon_1 \cdot \varepsilon_2 \cdot \frac{d^2}{2}$, or*

(ii) *G is $(K, \varepsilon_1, \varepsilon_2, d)$-unmated.*

Proof. Assume that G is not $(K, \varepsilon_1, \varepsilon_2, d)$-unmated. Then there exists $v \in V(G)$ with at least $\varepsilon_1 d$ (ε_2, d)-mates. Let $v_1, \ldots, v_{\lceil \varepsilon_1 d \rceil}$ be distinct (ε_2, d)-mates of v. Let $H = G[N(v) \cup \{v, v_1, \ldots, v_{\lceil \varepsilon_1 d \rceil}\}]$. Then $\mathsf{v}(H) \leq 1 + Kd + \lceil \varepsilon_1 d \rceil \leq 3Kd$ and $\mathsf{e}(H) \geq \varepsilon_1 \cdot \varepsilon_2 \cdot \frac{d^2}{2}$. Thus (i) holds, as desired. ☐

Corollary 3.3. *Let $K_0, k, d \geq 1$, $\varepsilon_1, \varepsilon_2 \in (0,1)$, and let G' be a k-bounded minor of a graph G. Then at least one of the following holds:*

(i) *there exists a subgraph H of G with $\mathsf{v}(H) \leq 3kK_0 d$ and $\mathsf{e}(H) \geq \varepsilon_1 \cdot \varepsilon_2 \cdot \frac{d^2}{2}$, or*

(ii) *G' is $(K_0, \varepsilon_1, \varepsilon_2, d)$-unmated.*

Proof. Assume that G' is not $(K_0, \varepsilon_1, \varepsilon_2, d)$-unmated. By Proposition 3.2 applied to G', there exists a subgraph H' of G' with $\mathsf{v}(H') \leq 3K_0 d$ and $\mathsf{e}(H') \geq \varepsilon_1 \cdot \varepsilon_2 \cdot \frac{d^2}{2}$. Since H' is a k-bounded minor of G, it corresponds to a subgraph H of G with $\mathsf{v}(H) \leq k \cdot \mathsf{v}(H') \leq 3kK_0 d$ and $\mathsf{e}(H) \geq \mathsf{e}(H') \geq \varepsilon_1 \cdot \varepsilon_2 \cdot \frac{d^2}{2}$. ☐

Definition 3.4. Let F be a non-empty forest in a graph G. Let $K, k, d, s \geq 1$ be real and let $\varepsilon_2, c \in (0,1)$. We say F is

- (K,d)-*small* if every vertex in $V(F)$ is (K,d)-small in G,
- (ε_2, d)-*mate-free* if no two distinct vertices in any component of F are (ε_2, d)-mates in G, and
- (c,d)-*clean* if $\mathsf{e}(G) - \mathsf{e}(G/F) \leq c \cdot d \cdot \mathsf{v}(F)$,

Definition 3.5. Let $\ell \geq 1$ be an integer. An ℓ-*star* is a star with ℓ leaves. An ℓ^--*star* is a star with at least one but at most ℓ leaves. Let G be a graph and let (A,B) be a partition of $V(G)$. Let $\ell \geq 1$ be an integer. We say a forest F is an ℓ^--*star-matching from B to A* if for every component T of F, then T is an ℓ^--star, the center of T is in B and the leaves of T are in A. Similarly we define ℓ-*star-matching*, ℓ-*claw-matching* and ℓ^--*claw-matching from B to A* as above if every component of F is an ℓ-star (resp. ℓ-claw and ℓ^--claw) instead of an ℓ^--star.

The proof of Theorem 2.1 is based on the following two theorems.

Theorem 3.6. *Let $K, \ell \geq 2$ be integers with $K \geq \ell(\ell+1)$, and let $\varepsilon_{1,0} \in \left(0, \frac{1}{\ell}\right], \varepsilon_{2,0} \in \left(0, \frac{1}{\ell^2}\right]$ and $d_0 \geq 1/\varepsilon_{2,0}$ be real. Let $G = (A,B)$ be a bipartite graph such that $|A| \geq \ell|B|$ and every vertex in A has at least d_0 neighbors in B. Then there exists at least one of the following:*

(i) *a subgraph H of G with $\mathsf{v}(H) \leq 4Kd_0$ and $\mathsf{e}(H) \geq \varepsilon_{1,0} \cdot \varepsilon_{2,0} \cdot d_0^2/2$.*

(ii) *a subgraph H of G with $\mathsf{v}(H) \leq 4\ell K d_0$ and $\mathsf{e}(H) \geq \varepsilon_{1,0}^2 \cdot d_0^2/2$.*

(iii) *an $(\ell+1)$-bounded minor H of G with $\mathsf{d}(H) \geq \frac{\ell^2}{\ell+1}\left(1 - 2\varepsilon_{1,0} - 2\ell\varepsilon_{2,0} - \frac{\ell}{K}\right) d_0$.*

Theorem 3.7. *Let $K \geq k \geq 100$ be integers with $K \geq 4 \cdot k^2$. Let $\ell = \left\lceil \frac{k}{6} \right\rceil$. Let $\varepsilon_1 \in (0, \frac{1}{6})$ and $\varepsilon_2 \in (0, \frac{1}{k}]$. Let G be a graph with $d = \mathsf{d}(G) \geq \frac{k}{\min\{\varepsilon_1, \varepsilon_2\}}$. Then there exists at least one of the following:*

(i) *a subgraph H of G with $\mathsf{v}(H) \leq 3k^2Kd$ and $\mathsf{e}(H) \geq \varepsilon_1 \cdot \varepsilon_2 \cdot \frac{d^2}{2}$, or*

(ii) *a bipartite subgraph $H = (X,Y)$ of G with $|X| \geq \ell|Y|$ such that every vertex in X has at least $(1-6\varepsilon_1)d$ neighbors in Y, or*

(iii) *a k-bounded minor G' of G with $\mathsf{d}(G') \geq k \cdot \left(1 - \frac{16}{k}\right) \cdot d$.*

We prove Theorem 3.6 in Section 4 and Theorem 3.7 in Section 5. We finish this section by deriving Theorem 2.1, which we restate for convenience, from Theorems 3.6 and 3.7.

Theorem 2.1. *Let $k \geq 100$ be an integer. Let G be a graph with $d = \mathsf{d}(G) \geq k^2$. Then G contains at least one of the following:*

(i) *a subgraph H with $\mathsf{v}(H) \leq 12 \cdot k^4 \cdot d$ and $\mathsf{d}(H) \geq \frac{d}{24k^6}$, or*

(ii) *an m-bounded minor G' with $\mathsf{d}(G') \geq m \cdot \left(1 - \frac{16}{m}\right) \cdot d$ for some integer $m \in [\frac{k}{6}, k]$.*

Proof of Theorem 2.1. We apply Theorem 3.7 to G with $K = 4 \cdot k^2$ and $\varepsilon_1 = \frac{1}{k}$ and $\varepsilon_2 = \frac{1}{k}$. First suppose Theorem 3.7(i) holds. That is, G contains a subgraph H with $\mathsf{v}(H) \leq 3k^2 K d = 12k^4 d$ and $\mathsf{e}(H) \geq \varepsilon_1 \cdot \varepsilon_2 \cdot \frac{d^2}{2} = \frac{d^2}{2k}$. But then $\mathsf{d}(H) \geq \frac{d^2}{24k^6}$ and (i) holds as desired.

Next suppose Theorem 3.7(iii) holds. That is, there exists a k-bounded minor G' with $\mathsf{d}(G') \geq k \cdot \left(1 - \frac{16}{k}\right) \cdot d$. But then (ii) holds for G with $m = k$.

So we may assume that Theorem 3.7(ii) holds, that is there exists a bipartite subgraph $H = (X, Y)$ with $|X| \geq \ell |Y|$ such that every vertex in X has at least $(1 - 6\varepsilon_1)d$ neighbors in Y. We next apply Theorem 3.6to H with $d_0 = (1 - 6\varepsilon_1)d$, $\varepsilon_{1,0} = \frac{1}{\ell}$ and $\varepsilon_{2,0} = \frac{1}{\ell^2}$. Note that $d_0 \geq d/2$ since $k \geq 12$ and hence $d_0 \geq k^2 \geq \frac{1}{\varepsilon_{2,0}}$ as needed.

First suppose Theorem 3.6(i) holds for H. That is, there exists a subgraph H_0 of H with $\mathsf{v}(H_0) \leq 4K \cdot d_0$ and $\mathsf{e}(H_0) \geq \varepsilon_{1,0} \cdot \varepsilon_{2,0} \cdot \frac{d_0^2}{2}$. Note then that

$$\mathsf{v}(H_0) \leq 16 \cdot k^2 \cdot d \leq 12 \cdot k^3 \cdot d,$$

since $k \geq 2$, and

$$\mathsf{e}(H_0) \geq (1 - 6\varepsilon_1)^2 \cdot \frac{d^2}{2\ell^3} \geq \frac{12d^2}{k^3},$$

since $\ell \leq k/5$ and $6\varepsilon_1 \leq 1/2$ as $k \geq 100$. But then

$$\mathsf{d}(H_0) \geq \frac{d^2}{k^5},$$

and (i) holds as desired.

Next suppose Theorem 3.6(ii) holds for H. That is, there exists a subgraph H_0 of H with $\mathsf{v}(H_0) \leq 4\ell K \cdot d_0$ and $\mathsf{e}(H_0) \geq \varepsilon_{1,0}^2 \frac{d_0^2}{2}$. Note then that

$$\mathsf{v}(H_0) \leq 16 \cdot k^3 \cdot d \leq 16 \cdot k^3 \cdot d,$$

and

$$\mathsf{e}(H_0) \geq (1 - 6\varepsilon_1)^2 \cdot \frac{d^2}{2\ell^2} \geq \frac{8d^2}{k^3},$$

since $\ell \leq k/5$ and $6\varepsilon_1 \leq 1/3$ as $k \geq 100$. But then

$$\mathsf{d}(H_0) \geq \frac{d^2}{2k^5},$$

and (i) holds as desired.

So we may assume that Theorem 3.6(iii) holds. That is, H contains an $(\ell+1)$-bounded minor H_0 with

$$\begin{aligned}
d(H_0) &\geq \frac{\ell^2}{\ell+1} \left(1 - 2\varepsilon_{1,0} - 2\ell\varepsilon_{2,0} - \frac{\ell}{K}\right) d_0 \\
&\geq (\ell+1) \cdot \left(1 - \frac{2}{\ell+1}\right) \left(1 - \frac{5}{\ell}\right) \cdot (1 - 6\varepsilon_1) \cdot d \\
&\geq (\ell+1) \cdot \left(1 - \frac{8}{\ell+1}\right) \cdot \left(1 - \frac{6}{k}\right) \cdot d, \\
&\geq \ell \cdot \left(1 - \frac{10}{\ell+1}\right) \cdot d,
\end{aligned}$$

since $\ell = \lceil \frac{k}{6} \rceil$. Hence (ii) holds with $G' = H_0$ and $m = \ell + 1$, as desired. $\square$

4 Dense Minors in Unbalanced Bipartite Graphs

In this section, we prove Theorem 3.6. The proof is nearly identical to that of the author in Theorem 3.4 as presented in [Pos19]. To prove Theorem 3.6, we need the following three lemmas, Lemmas 4.1, 4.2, and 4.3, from there. We include their proofs for completeness.

Lemma 4.1. *Let $\ell \geq 1$, $d_B > d_A \geq 0$ be integers. Let G be a graph and let (A, B) be a partition of $V(G)$ with $|A| \geq \ell|B|$ and B is independent. If every vertex in A has at least d_B neighbors in B and at most d_A neighbors in A, then G contains an ℓ-claw-matching F from B to A such that every vertex in $V(F) \cap A$ has at most d_A neighbors in $B \setminus V(F)$.*

Proof. Let F_0 be an ℓ^--claw-matching from B to A such that $|V(F_0) \cap A|$ is maximized. Assume first that $V(F_0) \cap A = A$. Note that $|V(F_0) \cap A| \leq \ell \cdot |V(F_0) \cap B|$. Then $V(F_0) \cap B = B$ because $|V(F_0) \cap A| = |A| \geq \ell|B|$. Hence $V(F_0) = V(G)$ and $F = F_0$ is as desired. So we may assume that $A \setminus V(F_0) \neq \emptyset$.

Let $u \in A \setminus V(F_0)$. By the maximality of F_0, $N_G(u) \cap B \subseteq V(F_0) \cap B$. For each $v \in V(G)$ with $v \neq u$, we say that a path P from u to v is a *(u, v)-F_0-alternating path* if

- P is a path in $G(A, B)$, and
- every internal vertex of P has degree exactly one in $F_0 \cap P$ (that is - informally - that every other edge of P is in F_0), and
- there does not exist a triangle of G containing an edge of F_0 and an edge of $P - E(F_0)$.

Let B_u be the set of all vertices $v \in B$ such that there exists a (u, v)-F_0-alternating path. Then $B_u \neq \emptyset$ as $d_B > d_A$.

Claim 4.1.1. *For all $v \in B_u$, we have $v \in V(F_0)$ and the component of F_0 containing v has exactly ℓ edges.*

Proof. Let $v \in B_u$. Then there exists a (u, v)-F_0-alternating path P. Let $F_0' = F_0 \triangle P$. Since P is a path in $G(A, B)$, we have $E(F_0') \subseteq E(G(A, B))$. It follows that $v \in V(F_0)$ and the component of F_0 containing v has exactly ℓ edges, else F_0' is an ℓ^--claw-matching from B to A with $|V(F_0') \cap A| > |V(F_0) \cap A|$, contrary to the choice of F_0. $\square$

Let F be the subgraph of F_0 consisting of all the components T of F_0 such that T contains a vertex in B_u. By Claim 4.1.1, F is an ℓ-claw-matching from B to A. It remains to show that every vertex in $V(F) \cap A$ has at most d_A neighbors in $B \setminus V(F)$. Let $w \in V(F) \cap A$ and let x be a neighbor of w in $B \setminus V(F)$. Then there exists $v \in B_u$ such that $vw \in E(F)$. By the definition of B_u, there exists a (u, v)-F_0-alternating path P. Then $w \notin V(P)$. Let $P' = P + vw + wx$. Then P' is a path in $G(A, B)$ from u to x such that every other edge is in F. Note that $P + vw$ is a (u, w)-F_0-alternating path. By the maximality of F_0, $x \in V(F_0)$. By the choice of F, $x \notin B_u$. Thus P' is not a (u, x)-F_0-alternating path. It follows that x is the center of a star T in $F_0 \setminus V(F)$ such that wx is contained in a triangle wxz, where $z \in A \cap V(T)$. Since w has at most d_A neighbors in A, we see that w has at most d_A neighbors in $B \setminus V(F)$, as desired. $\square$

We now apply Lemma 4.1 to obtain a mate-free ℓ-claw-matching in a dense unbalanced bipartite graph assuming that the graph itself is unmated.

Lemma 4.2. *Let $K \geq \ell \geq 1$ and $d_0 \geq 1$ be integers, and let $\varepsilon_{1,0}, \varepsilon_{2,0} \in (0,1)$ be real. Let $G = (A, B)$ be a bipartite graph such that $|A| \geq \ell|B|$ and every vertex in A has at least d_0 neighbors in B. If G is $(K, \varepsilon_{1,0}, \varepsilon_{2,0}, d_0)$-unmated, then G contains an $(\varepsilon_{2,0}, d_0)$-mate-free ℓ-claw-matching F from B to A such that every vertex in $V(F) \cap A$ has at most $\varepsilon_{1,0} \cdot d_0$ neighbors in $B \setminus V(F)$.*

Proof. Since $K \geq 1$ and G is $(K, \varepsilon_{1,0}, \varepsilon_{2,0}, d_0)$-unmated, we see that every vertex of A is (K, d_0)-small, and has at most $\varepsilon_{1,0} \cdot d_0$ many $(\varepsilon_{2,0}, d_0)$-mates in G. Let G' be obtained from G by adding all possible edges uv, where $u, v \in A$ are $(\varepsilon_{2,0}, d_0)$-mates in G. Then in G', every vertex of A has at least $d_B = d_0$ neighbors in B and at most $d_A = \varepsilon_{1,0} \cdot d_0 < d_B$ neighbors in A. By Lemma 4.1, G' contains an ℓ-claw-matching F from B to A such that every vertex in $V(F) \cap A$ has at most d_A neighbors in $B \setminus V(F)$. It remains to show that every component T of F is $(\varepsilon_{2,0}, d_0)$-mate-free.

Let $x, y \in V(T)$ be distinct. We may assume that $x \in A$. Assume first that $y \in B$. Then $xy \in E(G)$, and so x and y are not $(\varepsilon_{2,0}, d_0)$-mates in G, because G is bipartite. So we may assume that $y \in A$. Since T is an ℓ-claw in G', we see that $xy \notin E(G')$. By the choice of G', x and y are not $(\varepsilon_{2,0}, d_0)$-mates in G, as desired. □

Next we clean the ℓ-claw-matching obtained from Lemma 4.2. To do this, we have to remove components whose centers are big in $G[V(F)]$ and then switch edges as necessary.

Lemma 4.3. *Let $K \geq \ell \geq 1$ be integers. Let $\varepsilon'_{1,0}, \varepsilon'_{2,0} \in (0,1)$, and let $d'_0 \geq \frac{1}{\varepsilon'_{2,0}}$ be an integer. Let $G = (A, B)$ be a bipartite graph such that $|A| = \ell|B|$ and every vertex in A has exactly d'_0 neighbors in B. Suppose G is $(K, \varepsilon'_{1,0}, \varepsilon'_{2,0}, d'_0)$-unmated, and has an $(\varepsilon'_{2,0}, d'_0)$-mate-free ℓ-claw-matching F_1 from B to A with $V(F_1) = V(G)$. Then G contains at least one of the following:*

(i) *a subgraph H of G with $\mathsf{v}(H) \leq (\ell+1)(K+1)d'_0$ and $\mathsf{e}(H) \geq (\varepsilon'_{1,0})^2 \cdot (d'_0)^2/2$, or*

(ii) *a (K, d'_0)-small, $(\varepsilon_{2,0}, d'_0)$-mate-free, $(\varepsilon'_{1,0} + \ell \cdot \varepsilon'_{2,0}, d'_0)$-clean ℓ-claw-matching F from B to A such that $\mathsf{v}(F) \geq \mathsf{v}(G)\left(1 - \frac{\ell}{K}\right)$.*

Proof. Suppose not. Since G is bipartite and $K \geq 1$, we see that every vertex in A is (K, d'_0)-small in G. Note that $\mathsf{e}(G) = d'_0|A| = d'_0\ell \cdot |B|$. Hence the number of (K, d'_0)-big vertices in G is at most $\frac{\ell}{K}|B| \leq \frac{\ell}{K} \cdot \frac{\mathsf{v}(G)}{\ell+1}$. Let F^* be the subgraph of F_1 consisting of all the components T of F_1 such that each T contains only (K, d'_0)-small vertices of G. Then F^* is a (K, d'_0)-small, $(\varepsilon'_{2,0}, d'_0)$-mate-free ℓ-claw-matching from B to A in G and

$$\mathsf{v}(F^*) \geq \mathsf{v}(G) - \left(\frac{\ell}{K(\ell+1)} \cdot \mathsf{v}(G)\right)(\ell+1) = \mathsf{v}(G)\left(1 - \frac{\ell}{K}\right).$$

Given distinct components T_1, T_2 of an ℓ-claw-matching F from B to A and edges $u_1v_1 \in E(T_1)$ and $u_2v_2 \in E(T_2)$ with $v_1, v_2 \in B$, we say that $\{u_1v_1, u_2v_2\}$ is a *bad pair of F* if

$E(G(V(T_1), V(T_2)) = \{u_1v_2, u_2v_1\}$. We say the bad pair is *mated* if there exists $w_1 \in T_1, w_2 \in T_2$ that are $(\varepsilon'_{2,0}, d'_0)$-mates and *unmated* otherwise.

Now let F be an $(\varepsilon'_{2,0}, d'_0)$-mate-free ℓ-claw-matching from B to A with $V(F) = V(F^*)$ such that F has the minimum number of bad pairs. Then F is (K, d'_0)-small.

We now upper bound how many unmated bad pairs of F containing a given edge of F.

Claim 4.3.1. *If $uv \in E(F)$, then uv is in at most $\varepsilon'_{1,0}d'_0$ unmated bad pairs of F.*

Proof. Suppose not. Let $b = \lceil \varepsilon'_{1,0}d'_0 \rceil$. Thus there exist b unmated bad pairs of F, $\{(uv, u_iv_i)\}_{i=1}^b$, where $u_1v_1 \in E(T_1), u_2v_2 \in E(T_2), \ldots, u_bv_b \in E(T_b)$, and $T_1, T_2, \ldots, T_b$ are distinct components of $F \setminus V(T)$. We may further assume without loss of generality that $v, v_1, \ldots, v_b \in B$. For each such i, let T'_i be obtained from T by deleting u and adding the edge vu_i, and let T''_i be obtained from T_i by deleting u_i and adding the edge v_iu. Let F'_i be obtained from $F \setminus V(T \cup T_i)$ by adding T'_i and T''_i.

Since $\{uv, u_iv_i\}$ is an unmated bad pair of F for each $i \in [b]$, we have that F'_i is an $(\varepsilon'_{2,0}, d'_0)$-mate-free ℓ-claw-matching from A to B with $V(F'_i) = V(F) = V(F^*)$ for all $i \in [b]$. Let X be the union of the vertex sets of all components of $F \setminus V(T)$ containing a neighbor of u or v. Then $|X| \le (\ell+1)(K+1)d'_0$. Let $H = G[X]$. It follows from the choice of F that for every $i \in [b]$ there are at least b bad pairs of F'_i, which are not bad pairs of F. Each such pair must contain one of the edges vu_i and v_iu. It follows that $\deg_H(v_i) + \deg_H(u_i) \ge b$, and consequently $\mathsf{e}(H) \ge b^2/2 \ge (\varepsilon'_{1,0})^2 \cdot (d'_0)^2/2$. Hence (i) holds, a contradiction. $\square$

Let T be a component of F. Since G is $(K, \varepsilon'_{1,0}, \varepsilon'_{2,0}, d'_0)$-unmated, every vertex in $V(F)$ has at most $\varepsilon'_{1,0}d'_0$ $(\varepsilon'_{2,0}, d'_0)$-mates in G. Hence there are most $(\ell+1)\varepsilon'_{1,0}d'_0$ mated bad pairs of F containing an edge of T. Yet by Claim 4.3.1, it follows that there are at most $\ell\varepsilon'_{1,0}d'_0$ unmated bad pairs of F containing an edge of T. Combining these facts, we find that there are at most $2(\ell+1)\varepsilon'_{1,0}d'_0$ bad pairs of F containing an edge of T.

It follows that there are at most $2(\ell+1)\varepsilon'_{1,0}d'_0 \cdot \frac{\mathsf{v}(F)}{\ell+1} \cdot \frac{1}{2} = \varepsilon'_{1,0}d'_0 \cdot \mathsf{v}(F)$ bad pairs of F in total. Note that every pair of edges of G that become parallel in $G/E(F)$ corresponds to a bad pair or a common neighbor of two leaves of some component in F. Note also that $\mathsf{e}(F) \le \frac{\ell}{\ell+1} \cdot \mathsf{v}(F)$. Since F is $(\varepsilon'_{2,0}, d'_0)$-mate-free, it follows that

$$\begin{aligned} \mathsf{e}(G) - \mathsf{e}(G/E(F)) &\le \mathsf{e}(F) + \binom{\ell}{2}\varepsilon'_{2,0}d'_0 \cdot \frac{\mathsf{v}(F)}{\ell+1} + \varepsilon'_{1,0}d'_0 \cdot \mathsf{v}(F) \\ &\le \frac{\ell}{\ell+1} \cdot \mathsf{v}(F) + \frac{\ell}{2} \cdot \varepsilon'_{2,0}d'_0 \cdot \mathsf{v}(F) + \varepsilon'_{1,0}d'_0 \cdot \mathsf{v}(F) \\ &\le (\ell \cdot \varepsilon'_{2,0} + \varepsilon'_{1,0})d'_0 \cdot \mathsf{v}(F), \end{aligned}$$

since $\ell \ge 1$, $\varepsilon'_{2,0}d'_0 \ge 1$ and $\mathsf{e}(F) \le \mathsf{v}(F)$. Hence F is $(\varepsilon'_{1,0} + \ell \cdot \varepsilon'_{2,0}, d'_0)$-clean and (ii) holds, a contradiction. $\square$

We finish this section by proving Theorem 3.6, which we restate below for convenience. The bound in outcome (ii) has been improved over the previous version in [Pos19] namely by a factor of 2 to be nearly optimal. This is accomplished by adding the assumption that $K \ge \ell(\ell+1)$.

Theorem 3.6. *Let $K, \ell \geq 2$ be integers with $K \geq \ell(\ell+1)$, and let $\varepsilon_{1,0} \in \left(0, \frac{1}{\ell}\right], \varepsilon_{2,0} \in \left(0, \frac{1}{\ell^2}\right]$ and $d_0 \geq 1/\varepsilon_{2,0}$ be real. Let $G = (A, B)$ be a bipartite graph such that $|A| \geq \ell|B|$ and every vertex in A has at least d_0 neighbors in B. Then there exists at least one of the following:*

(i) *a subgraph H of G with $\mathsf{v}(H) \leq 4Kd_0$ and $\mathsf{e}(H) \geq \varepsilon_{1,0} \cdot \varepsilon_{2,0} \cdot d_0^2/2$.*

(ii) *a subgraph H of G with $\mathsf{v}(H) \leq 4\ell K d_0$ and $\mathsf{e}(H) \geq \varepsilon_{1,0}^2 \cdot d_0^2/2$.*

(iii) *an $(\ell+1)$-bounded minor H of G with $\mathsf{d}(H) \geq \frac{\ell^2}{\ell+1}\left(1 - 2\varepsilon_{1,0} - 2\ell\varepsilon_{2,0} - \frac{\ell}{K}\right) d_0$.*

Proof of Theorem 3.6. Note we may assume without loss of generality that every vertex in A has exactly d_0 neighbors in B. Assume first that G is not $(K, \varepsilon_{1,0}, \varepsilon_{2,0}, d_0)$-unmated. By Proposition 3.2(i), there exists a subgraph H of G with $\mathsf{v}(H) \leq 3Kd_0$ and $\mathsf{e}(H) \geq \varepsilon_{1,0} \cdot \varepsilon_{2,0} \cdot d_0^2/2$. Hence (i) holds as desired.

Assume next that G is $(K, \varepsilon_{1,0}, \varepsilon_{2,0}, d_0)$-unmated. By Lemma 4.2, G contains an $(\varepsilon_{2,0}, d_0)$-mate-free ℓ-claw-matching F_1 from B to A such that every vertex in $V(F_1) \cap A$ has at most $\varepsilon_{1,0} d_0$ neighbors in $B \setminus V(F_1)$. Let $d_0' = \lceil d_0(1 - \varepsilon_{1,0}) \rceil$ and for each $i \in \{1, 2\}$, let $\varepsilon_{i,0}' = \frac{\varepsilon_{i,0} d_0}{d_0'}$. Then for each $i \in \{1, 2\}$, we have that $\varepsilon_{i,0}' \in (0, 1)$ and $d_0' \varepsilon_{i,0}' = d_0 \varepsilon_{i,0} \geq 1$. Let G' be obtained from G with $V(G') = V(F_1)$ and $E(F_1) \subseteq E(G')$ such that every vertex in $V(F_1) \cap A$ has exactly d_0' neighbors in $V(F_1) \cap B$ in G'. Since G is $(K, \varepsilon_{1,0}, \varepsilon_{2,0}, d_0)$-unmated, we see that G' is $(K, \varepsilon_{1,0}', \varepsilon_{2,0}', d_0')$-unmated. Furthermore, F_1 is an $(\varepsilon_{2,0}', d_0')$-mate-free ℓ-claw-matching in G' from $V(F_1) \cap B$ to $V(F_1) \cap A$ where $V(F_1) = V(G')$. By Lemma 4.3 applied to G' with parameters $K, \ell, \varepsilon_{1,0}', \varepsilon_{2,0}', d_0'$, at least one of Lemma 4.3(i) or (ii) holds for G'.

First suppose that Lemma 4.3(i) holds for G'. That is, G' has a subgraph H with $\mathsf{v}(H) \leq (\ell+1)(K+1)d_0'$ and $\mathsf{e}(H) \geq \varepsilon_{1,0}' \cdot \varepsilon_{2,0}' \cdot (d_0')^2/2$. Since $K \geq \ell \geq 1$, we find that $\mathsf{v}(H) \leq 4\ell K d_0$. Since $d_0' \varepsilon_{i,0}' = d_0 \varepsilon_{i,0}$ for each $i \in \{1, 2\}$, we find that $\mathsf{e}(H) \leq \varepsilon_{1,0} \cdot \varepsilon_{2,0} \cdot (d_0)^2/2$. But then (ii) holds for G as desired.

So we may assume that Lemma 4.3(ii) holds for G'. That is, there exists a (K, d_1)-small, $(\varepsilon_{2,0}', d_0')$-mate-free $(\varepsilon_{1,0}' + \ell \cdot \varepsilon_{2,0}', d_0')$-clean ℓ-claw-matching F from $V(F_1) \cap B$ to $V(F_1) \cap A$ in G' such that $\mathsf{v}(F) \geq \mathsf{v}(G')\left(1 - \frac{\ell}{K}\right)$. Now let $H = (G'/E(F)) \setminus (A \setminus V(F))$. Note that $|A \setminus V(F)| = \ell|B \setminus V(F)|$ and hence $|B \setminus V(F)| \leq \frac{\mathsf{v}(G')}{K}$. But then H is an $(\ell+1)$-bounded minor of G with

$$\mathsf{v}(H) \leq \left(\frac{1}{\ell+1} + \frac{1}{K}\right) \cdot \mathsf{v}(G') \leq \frac{1}{\ell} \cdot \mathsf{v}(G'),$$

because $K \geq \ell(\ell+1)$. Since F is $(\varepsilon_{1,0}' + \ell \cdot \varepsilon_{2,0}', d_0')$-clean and

$$e(G'((A \cap V(G')) \setminus V(F), B)) \leq d_0' \cdot |(A \cap V(G')) \setminus V(F)| \leq d_0' \cdot \frac{\ell}{K} \cdot \mathsf{v}(G'),$$

we have that

$$\begin{aligned}
\mathsf{e}(H) &\geq \mathsf{e}(G') - (\varepsilon'_{1,0} + \ell \cdot \varepsilon'_{2,0})d'_0 \cdot \mathsf{v}(F) - d_0 \cdot \frac{\ell}{K} \cdot \mathsf{v}(G') \\
&\geq d'_0 \cdot \frac{\ell}{\ell+1} \cdot \mathsf{v}(G') - (\varepsilon'_{1,0} + \ell \cdot \varepsilon'_{2,0})d'_0 \cdot \mathsf{v}(G') - d_0 \frac{\ell}{K} \cdot \mathsf{v}(G') \\
&\geq \left(d_0 \cdot (1 - \varepsilon_{1,0}) \cdot \frac{\ell}{\ell+1} - (\varepsilon_{1,0} + \ell \cdot \varepsilon_{2,0}) \cdot d_0 - \frac{\ell}{K} \cdot d_0 \right) \cdot \mathsf{v}(G') \\
&\geq \frac{\ell}{\ell+1} \cdot \left(1 - 2\varepsilon_{1,0} - 2\ell\varepsilon_{2,0} - \frac{\ell}{K} \right) d_0 \cdot \mathsf{v}(G'),
\end{aligned}$$

where we use the fact that $\ell \geq 1$. Hence

$$\mathsf{d}(H) = \frac{\mathsf{e}(H)}{\mathsf{v}(H)} \geq \frac{\ell^2}{\ell+1} \cdot \left(1 - 2\varepsilon_{1,0} - 2\ell\varepsilon_{2,0} - \frac{\ell}{K} \right) \cdot d_0,$$

and (iii) holds, as desired. $\square$

5 Dense Minors in General Graphs

In this section we prove Theorem 3.7, which we restate for convenience.

Theorem 3.7. *Let $K \geq k \geq 100$ be integers with $K \geq 4 \cdot k^2$. Let $\ell = \lceil \frac{k}{6} \rceil$. Let $\varepsilon_1 \in (0, \frac{1}{6})$ and $\varepsilon_2 \in (0, \frac{1}{k}]$. Let G be a graph with $d = \mathsf{d}(G) \geq \frac{k}{\min\{\varepsilon_1, \varepsilon_2\}}$. Then there exists at least one of the following:*

(i) *a subgraph H of G with $\mathsf{v}(H) \leq 3k^2Kd$ and $\mathsf{e}(H) \geq \varepsilon_1 \cdot \varepsilon_2 \cdot \frac{d^2}{2}$, or*

(ii) *a bipartite subgraph $H = (X, Y)$ of G with $|X| \geq \ell|Y|$ such that every vertex in X has at least $(1 - 6\varepsilon_1)d$ neighbors in Y, or*

(iii) *a k-bounded minor G' of G with $\mathsf{d}(G') \geq k \cdot \left(1 - \frac{16}{k}\right) \cdot d$.*

Proof of Theorem 3.7. Suppose not. Let G be a counterexample with $\mathsf{v}(G)$ minimized. Thus we may assume that $\mathsf{d}(H) < \mathsf{d}(G)$ for every proper subgraph H of G, and hence $\delta(G) \geq d$. Since (i) does not hold for G, we have by Proposition 3.2 with $K_0 = K$ that G is $(K, \varepsilon_1, \varepsilon_2, d)$-unmated.

Let A denote the set of all (K, d)-small vertices of G, and let $B = V(G) \setminus A$ be the set of (K, d)-big vertices. Then $K \cdot d \cdot |B| \leq 2\mathsf{e}(G) = 2d \cdot \mathsf{v}(G)$. Hence $|B| \leq \frac{2}{K} \cdot \mathsf{v}(G) \leq \frac{1}{2 \cdot k^2} \cdot \mathsf{v}(G)$ since $K \geq 4k^2$. Then

$$|A| \geq \left(1 - \frac{1}{2k} \right) \cdot \mathsf{v}(G) \geq \frac{4\ell}{K} \cdot \mathsf{v}(G) \geq 2\ell|B|,$$

as $k \geq 27$. Since G is $(K, \varepsilon_1, \varepsilon_2, d)$-unmated, we see that every vertex in A has fewer than $\varepsilon_1 d$ (ε_2, d)-mates in G.

Let $c = 4\varepsilon_2$. Let F be a (K, d)-small, (c, d)-clean, star forest F where every component of F has size exactly k, and subject to that, $\mathsf{v}(F)$ is maximized. Note that $V(F) \subseteq A$ as F is (K, d)-small.

Claim 5.0.1. *Every k-bounded minor of G is $(Kk, \varepsilon_1, \varepsilon_2, d)$-unmated.*

Proof. Let G' be a k-bounded minor of G. By Corollary 3.3 applied to G and G' with $K_0 = Kk$, it follows that either G has a subgraph H with $\mathsf{v}(H) \leq 3k^2Kd$ and $\mathsf{e}(H) \geq k \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \frac{d^2}{2}$, or G' is $(Kk, \varepsilon_1, \varepsilon_2, d)$-unmated. In the first case, (i) holds, a contradiction. Hence G' is $(Kk, \varepsilon_1, \varepsilon_2, d)$-unmated. $\square$

Claim 5.0.2. *If F_0 is a star forest in G and $v \in V(G) \setminus V(F_0)$ has at least $2\varepsilon_1 d$ neighbors in $A \setminus V(F_0)$, then there exists a k-star T in $A \setminus V(F_0)$ with center v such that*

$$\mathsf{e}(G/E(F_0)) - \mathsf{e}(G/(E(F_0) \cup E(T))) \leq 2k \cdot \varepsilon_2 \cdot d.$$

Proof. Suppose not. Let S be a star in $A \setminus V(F_0)$ with center v such that

$$\mathsf{e}(G/E(F_0)) - \mathsf{e}(G/(E(F_0) \cup E(S))) \leq 2\mathsf{v}(S) \cdot \varepsilon_2 \cdot d,$$

and subject to that, $\mathsf{v}(S)$ is maximized. Since the claim does not hold, we have that $\mathsf{v}(S) < k$. Note that S exists since $S = \{v\}$ satisfies the conditions above. Let $G' = G/(E(F_0) \cup E(S))$. Note that G' is a k-bounded minor of G. Hence by Claim 5.0.1 G is $(Kk, \varepsilon_1, \varepsilon_2, d)$-unmated.

Let v_S be the vertex of G' corresponding to S. Since $S \subseteq A$ and every vertex in A is (K, d)-small, it follows that $\deg_{G'}(v_S) \leq kKd$ and hence v_S is (Kk, d)-small in G'. Since G is $(Kk, \varepsilon_1, \varepsilon_2, d)$-unmated, we have that v_S has at most $\varepsilon_1 \cdot d$ (ε_2, d)-mates in G'. Since $d \geq \frac{k}{\varepsilon_1}$, we have that $\varepsilon_1 d \geq k > \mathsf{v}(S)$. Since v has at least $2\varepsilon_1 d$ neighbors in $A \setminus V(F_0)$, we find that there exists a neighbor u of v in $A \setminus (V(F_0) \cup V(S))$ such that u is not a (ε_2, d)-mate of v_S in G'.

Let $S' = S + uv$. Note that S' is a star with center v and $\mathsf{v}(S') > \mathsf{v}(S)$. Since u is not a (ε_2, d)-mate of v_S in G', we have that

$$\mathsf{e}(G/(E(F_0) \cup E(S))) - \mathsf{e}(G/(E(F_0) \cup E(S'))) \leq 1 + \varepsilon_2 \cdot d \leq 2\varepsilon_2 \cdot d,$$

since $\varepsilon_2 d \geq 1$. But then

$$\mathsf{e}(G/E(F_0)) - \mathsf{e}(G/(E(F_0) \cup E(S'))) \leq 2\mathsf{v}(S) \cdot \varepsilon_2 \cdot d + 2\varepsilon_2 \cdot d = 2\mathsf{v}(S') \cdot \varepsilon_2 \cdot d,$$

contradicting the maximality of S. $\square$

Let $G_0 = G/E(F)$. Then G_0 is a k-bounded minor of G. Let $A' = A \setminus V(F)$.

Claim 5.0.3. *If T is a component of F, then there exists at most one vertex of T with at least $3\varepsilon_1 d$ neighbors in A'.*

Proof. Suppose not. That is, there exists distinct $u_1, u_2 \in V(T)$ such that $|N_G(u_i) \cap A'| \geq 3\varepsilon_1 d$. Let $F_0 = F \setminus V(T)$. Note that since $E(F_0) \subseteq E(F)$, we have that

$$\mathsf{e}(G/e(F_0)) \geq \mathsf{e}(G/E(F)).$$

Since F is (c, d)-clean, we find that

$$\mathsf{e}(G) - \mathsf{e}(G/E(F_0)) \leq \mathsf{e}(G) - \mathsf{e}(G/E(F)) \leq c \cdot d \cdot \mathsf{v}(F).$$

Since u_1 has at least $3\varepsilon_1 d$ neighbors in A', we have by Claim 5.0.2 that there exists a k-star T_1 in $A \setminus V(F_0)$ with center u_1 such that

$$\mathsf{e}(G/E(F_0)) - \mathsf{e}(G/(E(F_0) \cup E(T_1))) \leq 2k \cdot \varepsilon_2 \cdot d.$$

Let $F_1 = F_0 + T_1$. As u_2 has at least $3\varepsilon_1 d$ neighbors in A', we find that u_2 has at least $2\varepsilon_1 d$ neighbors in $A' \setminus V(T_1)$ since $\varepsilon_1 d \geq 1$. Hence by Claim 5.0.2 that there exists a k-star T_2 in $A \setminus V(F_1)$ with center u_2 such that

$$\mathsf{e}(G/E(F_1)) - \mathsf{e}(G/(E(F_1) \cup E(T_2))) \leq 2k \cdot \varepsilon_2 \cdot d.$$

Let $F_2 = F_1 + T_2$. Note that $\mathsf{v}(F_2) = \mathsf{v}(F_0) + 2k = \mathsf{v}(F) + k$. Yet

$$\mathsf{e}(G/E(F)) - \mathsf{e}(G/E(F_2)) \leq 4k \cdot \varepsilon_2 \cdot d = k \cdot c \cdot d,$$

and hence F_2 is also (c, d)-clean. But F_2 is a star forest where every component has size exactly k and F_2 is also (K, d)-small, contradicting the maximality of F. □

Claim 5.0.4. *If $v \in A'$, then v has at most $2\varepsilon_1 d$ neighbors in A' in G.*

Proof. Suppose not. Hence v has at least $2\varepsilon_1 d$ neighbors in A' in G. By Claim 5.0.2 that there exists a k-star T in $A \setminus V(F_0)$ with center v such that

$$\mathsf{e}(G/E(F)) - \mathsf{e}(G/(E(F) \cup E(T))) \leq 2k \cdot \varepsilon_2 \cdot d \leq k \cdot c \cdot d.$$

Let $F' = F + T$. Then F' is a (K, d)-small star forest where every component has size exactly k. Moreover, $\mathsf{v}(F') = \mathsf{v}(F) + k$. Since F is (c, d)-clean, we find that

$$\mathsf{e}(G) - \mathsf{e}(G/E(F)) \leq c \cdot d \cdot \mathsf{v}(F).$$

Hence

$$\mathsf{e}(G) - \mathsf{e}(G/E(F')) \leq c \cdot d \cdot \mathsf{v}(F) + k \cdot c \cdot d = \mathsf{v}(F') \cdot c \cdot d,$$

and so F' is (c, d)-clean, contradicting the maximality of F. □

Let C be the set of vertices in F with at least $3k\varepsilon_1 d$ neighbors in A'. By Claim 5.0.3, every component of F has at most one vertex in C. Hence $|C| \leq \frac{1}{k} \cdot \mathsf{v}(G)$.

Claim 5.0.5. $|A'| \leq \frac{\mathsf{v}(G)}{2}$.

Proof. Suppose not. Let $A_1 = \{v \in A', |N(v) \cap (B \cup C)| \geq (1 - 6\varepsilon_1)d\}$ and let $A_2 = A' \setminus A_1$. First suppose that $|A_1| \geq \left(\frac{1}{3} + \frac{2}{k}\right) \cdot \mathsf{v}(G)$. Since $|B \cup C| \leq \frac{2}{k} \cdot \mathsf{v}(G)$, we have that $|A| \geq \left(\frac{k}{6} + 1\right) \cdot \mathsf{v}(G) \geq \ell |B|$ and hence (ii) holds, a contradiction.

So we may assume that $|A_1| \leq \left(\frac{1}{3} + \frac{2}{k}\right) \mathsf{v}(G)$. By Claim 5.0.4, every vertex in A' has at most $3\varepsilon_1 d$ neighbors in A'. Since $\delta(G) \geq d$, it follows that every vertex in A_2 has at least $3\varepsilon_1 d$ neighbors in $V(F) \setminus C$. Hence

$$\mathsf{e}(G(A_2, V(F) \setminus C)) \geq 3\varepsilon_1 d \cdot |A_2|.$$

Yet by definition of C, we find that

$$\mathsf{e}(G(A_2, V(F) \setminus C)) \leq 3\varepsilon_1 d \cdot |C|.$$

Hence $|A_2| \leq |C| \leq \frac{1}{k} \cdot \mathsf{v}(G)$. Since $k \geq 18$, we find that

$$|A'| = |A_1| + |A_2| \leq \left(\frac{1}{3} + \frac{3}{k}\right) \cdot \mathsf{v}(G) \leq \frac{\mathsf{v}(G)}{2}.$$

$\square$

Note that by Claim 5.0.5, we have that

$$|A' \cup B \cup C| \leq \frac{3}{4} \cdot \mathsf{v}(G),$$

since $k \geq 8$. Hence F is nonempty. In addition, we find that $G[A' \cup B \cup C]$ is a proper subgraph of G. So by the minimality of G, we have that

$$\mathsf{e}(G[A' \cup B \cup C]) \leq d \cdot |A' \cup B \cup C| \leq \left(\frac{2}{k}\mathsf{v}(G) + |A'|\right) d \leq \frac{3}{4} \cdot d \cdot \mathsf{v}(G),$$

since $k \geq 8$. Moreover by Claim 5.0.4, we have that

$$\mathsf{e}(G(A', V(F) \setminus C)) \leq 3\varepsilon_1 d \cdot |C| \leq \frac{3\varepsilon_1}{k} \cdot d \cdot \mathsf{v}(G).$$

Let $a' = \frac{|A'|}{\mathsf{v}(G)}$. Hence

$$\mathsf{e}(G) - \mathsf{e}(G \setminus A') \leq \left(\frac{2 + 3\varepsilon_1}{k} + a'\right) \cdot d \cdot \mathsf{v}(G).$$

Let $G' = G_0 \setminus A'$. Since F is (c, d)-clean, we have that

$$\mathsf{e}(G) - \mathsf{e}(G_0) \leq c \cdot d \cdot \mathsf{v}(F) \leq 2\varepsilon_2 \cdot d \cdot \mathsf{v}(G).$$

Hence

$$\mathsf{e}(G \setminus A') - \mathsf{e}(G') \leq 2\varepsilon_2 \cdot d \cdot \mathsf{v}(G),$$

and so

$$\mathsf{e}(G) - \mathsf{e}(G') \leq \left(\frac{2 + 3\varepsilon_1}{k} + 2\varepsilon_2 + a'\right) \cdot d \cdot \mathsf{v}(G).$$

Yet

$$\mathsf{v}(G') = |B| + |C| \leq \frac{1}{k^2} \cdot \mathsf{v}(G) + \frac{\mathsf{v}(G) - |A'|}{k} = \frac{\mathsf{v}(G)}{k}\left(\frac{k+1}{k} - a'\right).$$

Thus

$$\mathsf{d}(G') \geq \frac{\mathsf{e}(G) - \left(\frac{2+3\varepsilon_1}{k} + 2\varepsilon_2 + a'\right) \cdot d \cdot \mathsf{v}(G)}{\frac{\mathsf{v}(G)}{k}\left(\frac{k+1}{k} - a'\right)} = k \cdot d \cdot \frac{1 - \left(\frac{2+3\varepsilon_1}{k} + 2\varepsilon_2\right) - a'}{\frac{k+1}{k} - a'}.$$

Let $c_1 = 1 - (\frac{2+3\varepsilon_1}{k})$ and $c_2 = \frac{k+1}{k}$. Since $\varepsilon_1 \leq 1$ and $k \geq 10$, we find that $c_2 > c_1 > \frac{1}{2} > 0$ and hence the function $\frac{c_1 - a'}{c_2 - a'}$ is decreasing in a' for $a' \in [0, c_1)$. Since $a' \leq \frac{1}{2}$, it follows that

$$\begin{aligned}
\mathsf{d}(G') &\geq k \cdot d \cdot \frac{1 - \left(\frac{2+3\varepsilon_1}{k} + 2\varepsilon_2\right) - \frac{1}{2}}{\frac{k+1}{k} - \frac{1}{2}} \\
&= k \cdot d \cdot \frac{1 - 2\left(\frac{2+3\varepsilon_1}{k} + 2\varepsilon_2\right)}{\frac{k+2}{k}} \\
&= k \cdot d \cdot \left(1 - 2\left(\frac{2+3\varepsilon_1}{k} + 2\varepsilon_2\right)\right) \cdot \left(1 - \frac{2}{k+2}\right) \\
&\leq k \cdot d \cdot \left(1 - \frac{16}{k}\right),
\end{aligned}$$

since $\varepsilon_1 \leq 1$ and $\varepsilon_2 \leq \frac{1}{k}$. But now (iii) holds, a contradiction. □

6 Concluding Remarks

The main obstacle now to improving the bound in Theorem 1.6 using this approach is remains improving the function $g(s) = O((1 + \log s)^6)$ in Theorem 1.7. The author has now made various attempts to optimize this value, successively improving the constant until this new value of 6. We note that the bipartite case only requires an exponent of 5. Further bottlenecks beyond improving $g(s)$ exist for certain better bounds. For a bound of $O(t(\log\log t)^2)$, there is a bottleneck in Theorem 1.8 caused by a division into two cases, the inseparable and separable cases. This bottleneck can be overcome by a more technical handling of the two cases together. Beyond that point, Hadwiger's conjecture seems to become quite difficult as there would then be two new distinct bottlenecks: the $O(t \log\log t)$ factor from the use of recursion in the separable case; and the $O(t \log\log t)$ bound for the chromatic number of small graphs with no K_t minor which is used in the inseparable case.

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