The Chromatic Conquest

with probability ;)

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The Journey

1. Chapter 1: A challenge of great significance
   - Tutte Tutte!
   - Shift your focus
   - A taste of topology
2. Chapter 2: Assembling the machinery
   - A neat Idea
   - A demonstration of wit
3. Chapter 3: Unforeseen consequences
   - Colors of independence
   - Expect the unexpected
   - Be careful what you wish for...
4. Chapter 4: After*math*
   - Fingers crossed
   - A glimpse of the possibilities

Chapter 1:
A challenge of great significance

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Question

Are there graphs with large chromatic number but no triangles?

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Answer

Yes and there have been many constructions of such graphs!

Tutte Tutte!

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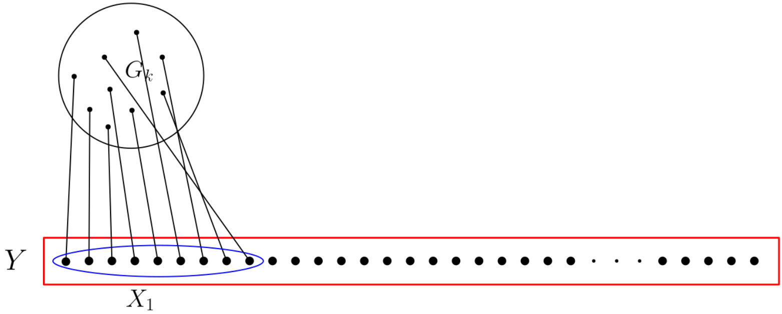

Figure: Tutte's construction

Matching between nodes of X_1 and G_k

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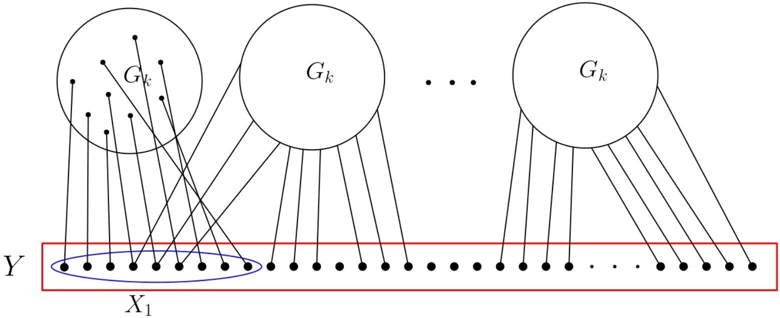

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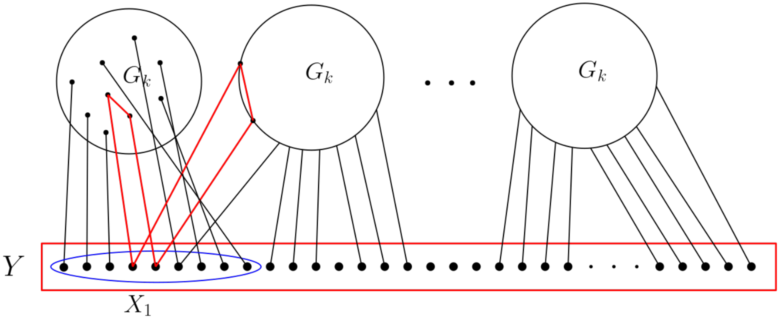

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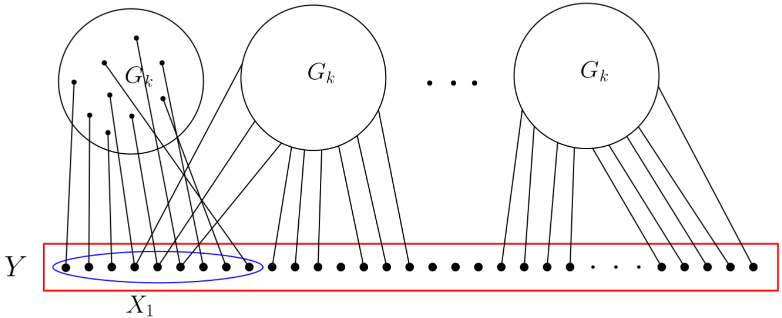

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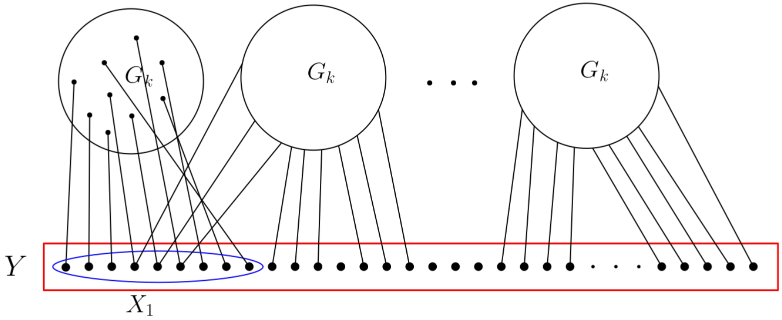

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$$\chi(G_k) \geq k$$

Shift your focus

$$\text{Let } n > 2k > 2,$$
$$V(G_{n,k}) = \{(a_1, \ldots, a_k) : 1 \leq a_1 < a_2 < \ldots < a_k \leq n\}$$

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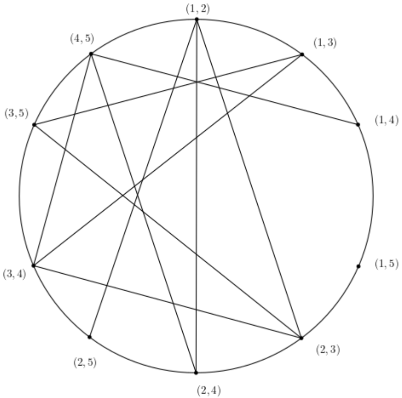

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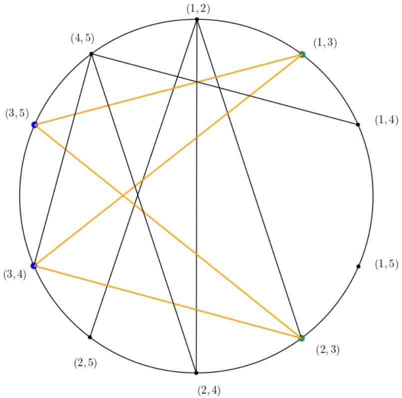

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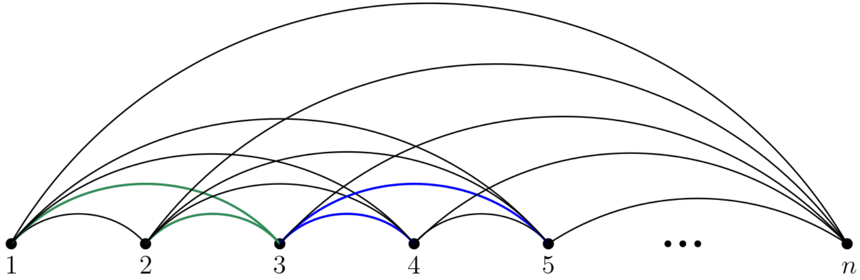

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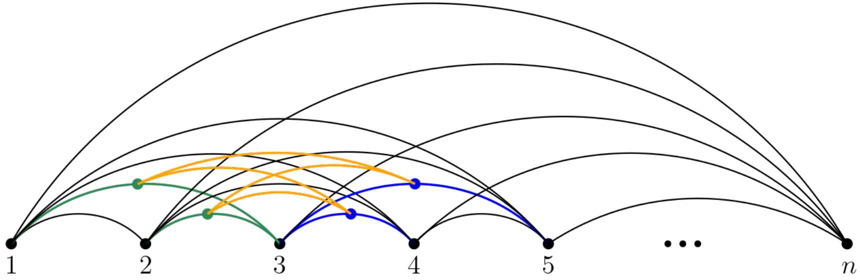

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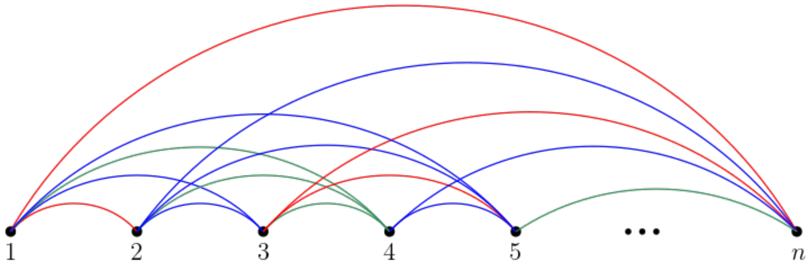

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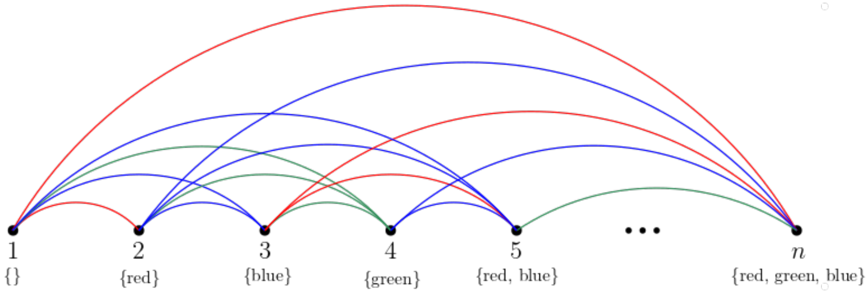

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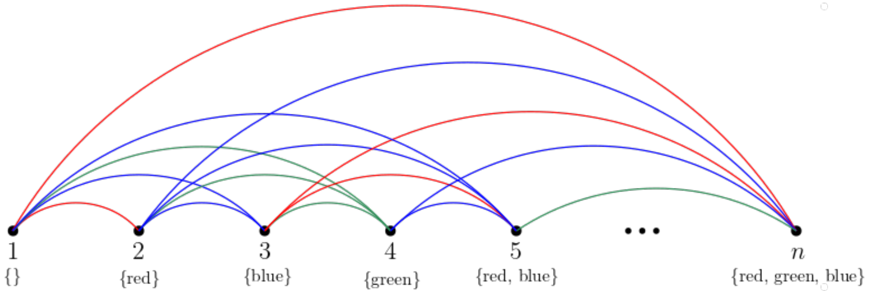

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$$\chi(G_{n,2}) \geq \lceil \log_2 n \rceil \xrightarrow{n\to\infty} \infty$$

A taste of topology

The Borsuk-Ulam theorem

If $f : S^n \to \mathbb{R}^n$ is continuous then there exists an $x \in S^n$ such that $f(x) = f(-x)$.

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The general Lusternik-Schnirelmann theorem

If $S^n \subseteq \mathbb{R}^{n+1}$ is covered by $n+1$ sets $A_1, A_2, \ldots, A_{n+1}$ such that each A_i is either open or closed, then there exist i and $x \in S^n$ such that $-x, x \in A_i$.

The Kneser graphs

Definition

Let $KG_{n,k}$ denote the Kneser graph with vertices $\mathcal{F} := \binom{[n]}{k}$ and edges $E := \{\{F_1, F_2\} : F_1, F_2 \in \mathcal{F}, F_1 \cap F_2 = \emptyset\}$

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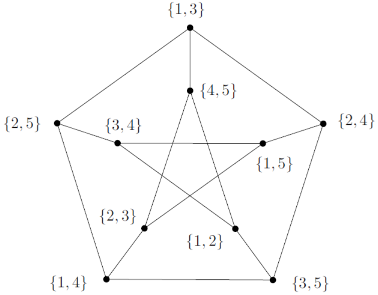

Figure: The $KG_{5,2}$ is the Peterson Graph

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Lovász–Kneser theorem

$$\chi(KG_{n,k}) = n - 2k + 2 \quad \forall\ k > 0,\ n \geq 2k - 1$$

All around the world

Proof: $\chi(KG_{n,k}) \leq n - 2k + 2$:

Set

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If $c(F_1) = c(F_2) = n - 2k + 2$:

$$F_1, F_2 \subseteq \{n - 2k + 2, \ldots, n\}$$

but $|\{n - 2k + 2, \ldots, n\}| = 2k - 1$ so $F_1 \cap F_2 \neq \emptyset$

$\square$

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Proof: $\chi(KG_{n,k}) > n - 2k + 1$:

Suppose $KG_{n,k}$ is $d \coloneqq n - 2k + 1$ colorable.

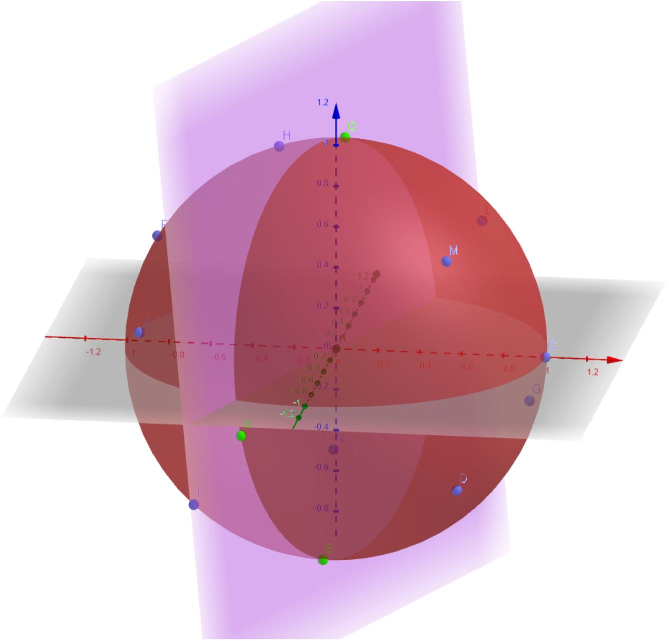

Figure: Example with $n = 13,\ k = 6$ in $\mathbb{R}^{d+1}$

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Suppose $KG_{n,k}$ is $d := n - 2k + 1$ colorable.
Define $A_1, \ldots, A_d \subseteq S^d$ by $x \in A_i$ iff there is an i-colored k-tuple in $H(x) := \{y \in S^d : \langle x, y \rangle > 0\}$.

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If $i = d + 1$ then $|H(x)| \leq k - 1$, so $|S^d \setminus \big(H(x) \cup H(-x)\big)| \geq d + 1$.
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□

Now $\chi(KG_{3k-1,k}) = k + 1$, and $KG_{3k-1,k}$ is triangle free.

Composition

Chapter 2:

Assembling the machinery

A neat Idea

The Probabilistic Method

If, in a given set of objects, the probability that an object does not have a certain property $\mathcal{P}$, is less than 1, then there must exist an object with property $\mathcal{P}$.

A demonstration of wit

Definition

Consider for $d, n \in \mathbb{N}$, $d \geq 2$, a finite set X and a family $\mathcal{F} = \{A_1, \ldots, A_n\}$ of subsets of X, each of which having cardinality d. We say this family $\mathcal{F}$ of d-sets is 2-colorable if there exists a coloring of X with 2 colors, such that no set of $\mathcal{F}$ is monochromatic.

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Example:

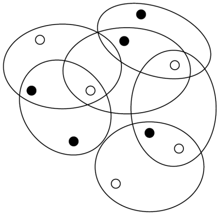

Figure: A 2-colored family of 3-sets

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$$\mathbb{P}(\bigcup_{A \in \mathcal{F}} E_A) < \sum_{A \in \mathcal{F}} \mathbb{P}(E_A) = 2^{-(d-1)} |\mathcal{F}| \leq 1,$$

if $|\mathcal{F}| \leq 2^{d-1}$. $\square$

Chapter 3:
Unforeseen consequences

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Main Theorem

For every $k \geq 2$, there exists a graph G with chromatic number $\chi(G) > k$ and girth $\gamma(G) > k$.

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The Erdős–Rényi model

Look at family $\mathcal{G}(n, p)$ of n-vertex graphs that include any edge independently with probability p.

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The Erdős–Rényi model

Look at family $\mathcal{G}(n, p)$ of n-vertex graphs that include any edge independently with probability p. Show:

$$\mathbb{P}\Big((\chi \leq k) \cup (\gamma \leq k)\Big) \leq \mathbb{P}\Big(\chi \leq k\Big) + \mathbb{P}\Big(\gamma \leq k\Big) < \frac{1}{2} + \frac{1}{2}$$

◂ Idea

Colors of independence

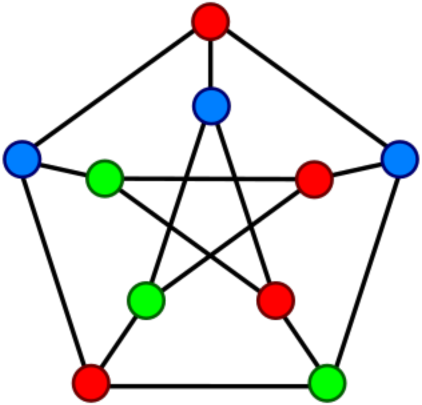

Figure: 3-coloring of the Petersen Graph

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$$\mathbb{P}(\alpha \geq r) \leq \binom{n}{r}(1-p)^{\binom{r}{2}} \leq \left(n(1-p)^{(r-1)/2}\right)^r \leq \left(ne^{-p(r-1)/2}\right)^r,$$

as $1-p \leq e^{-p}$.

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as $1-p \leq e^{-p}$. With $p := n^{-\frac{k}{k+1}}$ and $r := \lceil \frac{n}{2k} \rceil$ and for n large enough we can get

$$\mathbb{P}\left(\alpha \geq \frac{n}{2k}\right) \leq \ldots \leq \left(\frac{e}{n}\right)^{\frac{r}{2}} \xrightarrow{n\to\infty} 0.$$

Expect the unexpected

Proof (part 2)

Let X be the number of cycles of length $\leq k$ and X_C the indicator of the cycle C. $\mathbb{E}(X_C) = p^i$ for an i-cycle C.

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as $np = n^{\frac{1}{k+1}} \geq 1$ since $p = n^{-\frac{k}{k+1}}$. With Markov's inequality

$$\mathbb{P}\left(X \geq \frac{n}{2}\right) \leq \frac{\mathbb{E}(X)}{n/2} \leq (k-2)\frac{(np)^k}{n} = (k-2) n^{-\frac{1}{k+1}} \xrightarrow{n\to\infty} 0.$$

The puzzle completed

Proof (part 3)

For large enough n there is an n-vertex graph H with $\alpha(H) < \frac{n}{2k}$ and less than $\frac{n}{2}$ cycles of length $\leq k$.

Idea

The puzzle completed

Proof (part 3)

For large enough n there is an n-vertex graph H with $\alpha(H) < \frac{n}{2k}$ and less than $\frac{n}{2}$ cycles of length $\leq k$. Let G result by deleting one vertex from each such cycle. Then $\gamma(G) > k$,

The puzzle completed

Proof (part 3)

For large enough n there is an n-vertex graph H with $\alpha(H) < \frac{n}{2k}$ and less than $\frac{n}{2}$ cycles of length $\leq k$. Let G result by deleting one vertex from each such cycle. Then $\gamma(G) > k$, and $\alpha(G) \leq \alpha(H) < \frac{n}{2k}$, so we obtain

$$\chi(G) \geq \frac{n/2}{\alpha(G)} \geq \frac{n}{2\alpha(H)} > \frac{n}{n/k} = k$$

$\square$

Chapter 4:
After*math*

Fingers crossed

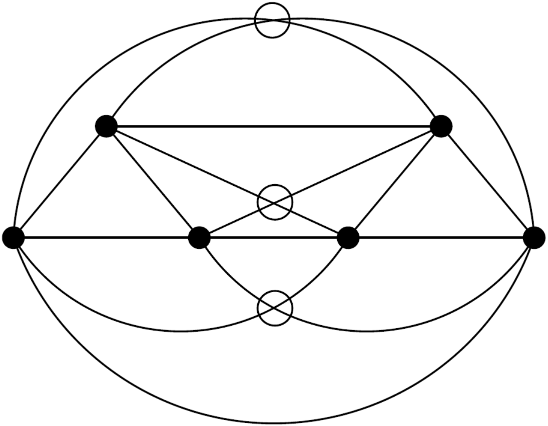
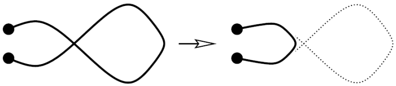


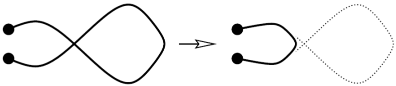
Figure: Minimal Drawing of the K_6

Fingers crossed

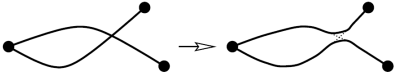


No edge can cross itself.

Fingers crossed

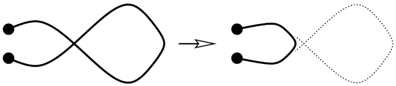


No edge can cross itself.



Edges with common endvertex cannot cross.

Fingers crossed



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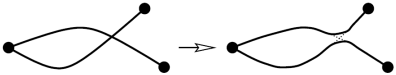

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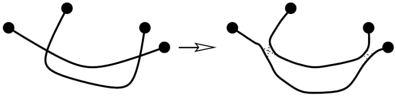
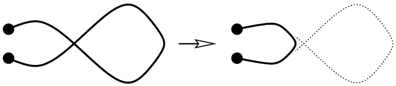


No two edges cross twice.

Fingers crossed



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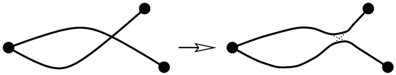

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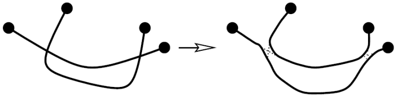
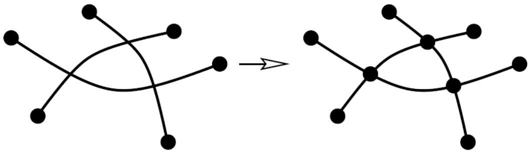


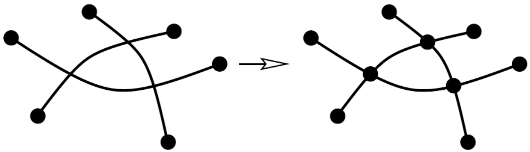
No two edges cross twice.

Every crossing uses two distinct edges and four distinct vertices.

Fingers crossed



Fingers crossed



Together with the bound on $|E|$ for planar graphs $|E| \leq 3|V| - 6$, we get:

$$|E| + 2\mathrm{cr}(G) \leq 3(|V| + \mathrm{cr}(G)) - 6 \iff \mathrm{cr}(G) \geq |E| - 3|V| + 6$$

A glimpse of the possibilities

Crossing Lemma

Let G be a simple graph with n vertices and m edges, where $m \geq 4n$. Then

$$\mathrm{cr}(G) \geq \frac{1}{64}\frac{m^3}{n^2}$$

A glimpse of the possibilities

Proof

For a minimal drawing of G and $p \in [0, 1]$ let G_p be a random subgraph of G where each vertex is picked with probability p.

A glimpse of the possibilities

Proof

For a minimal drawing of G and $p \in [0, 1]$ let G_p be a random subgraph of G where each vertex is picked with probability p. Let n_p, m_p, X_p be random variables counting the number of vertices, edges and crossings in G_p. We have

$$\mathbb{E}(n_p) = np, \qquad \mathbb{E}(m_p) = mp^2, \qquad \mathbb{E}(X_p) = \mathrm{cr}(G)p^4.$$

A glimpse of the possibilities

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Combining this with the previous result: $\mathrm{cr}(G) \geq m - 3n + 6$, we get

$$0 \leq 6 \leq \mathbb{E}(X_p - m_p + 3n_p) = p^4\mathrm{cr}(G) - p^2 m + 3pn$$

A glimpse of the possibilities

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By setting $p \coloneqq \frac{4n}{m} \leq 1$ we get the desired:

$$cr(G) \geq \frac{m}{p^2} - \frac{3n}{p^3} = \frac{1}{64}\left[\frac{4m}{(n/m)^2} - \frac{3n}{(n/m)^3}\right] = \frac{1}{64}\frac{m^3}{n^2}$$

$\square$

Adiós Amigos