Connectivity of Triangulation Flip Graphs in the Plane

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Setting and Definition

- Let P be a finite planar point set in general position with $|P| = n$.
- An inner edge e of an triangulation $T = (P, E)$ is *flippable in T*, if the two triangles which contain e merge to an convex quadrilateral Q. Then one can replace e by the other diagonal $\bar{e}$ of Q to obtain a triangulation $T[e] := (P, E\backslash\{e\} \cup \{\bar{e}\})$.
- The *(edge) flip graph of P* is the graph $flip(P)$ with vertices $\mathcal{T}(P)$, the set of all tiangulations of P, and edges $\{\{T, T[e]\} | T \in \mathcal{T}(P), e \in \binom{P}{2}$ flippable in $T\}$.
- A graph is *k-vertex connected*, if removing any $\leq k - 1$ vertices leaves the graph connected.

Main Statements of Wagner and Welzl

(i) The flip graph of any set P with big enough n is δ-vertex connected, with δ the minimum degree in the flip graph.

(ii) The flip graph is always $\lceil \frac{n}{2} - 2 \rceil$-vertex connected. And this is a tight bound.

Some other Statements

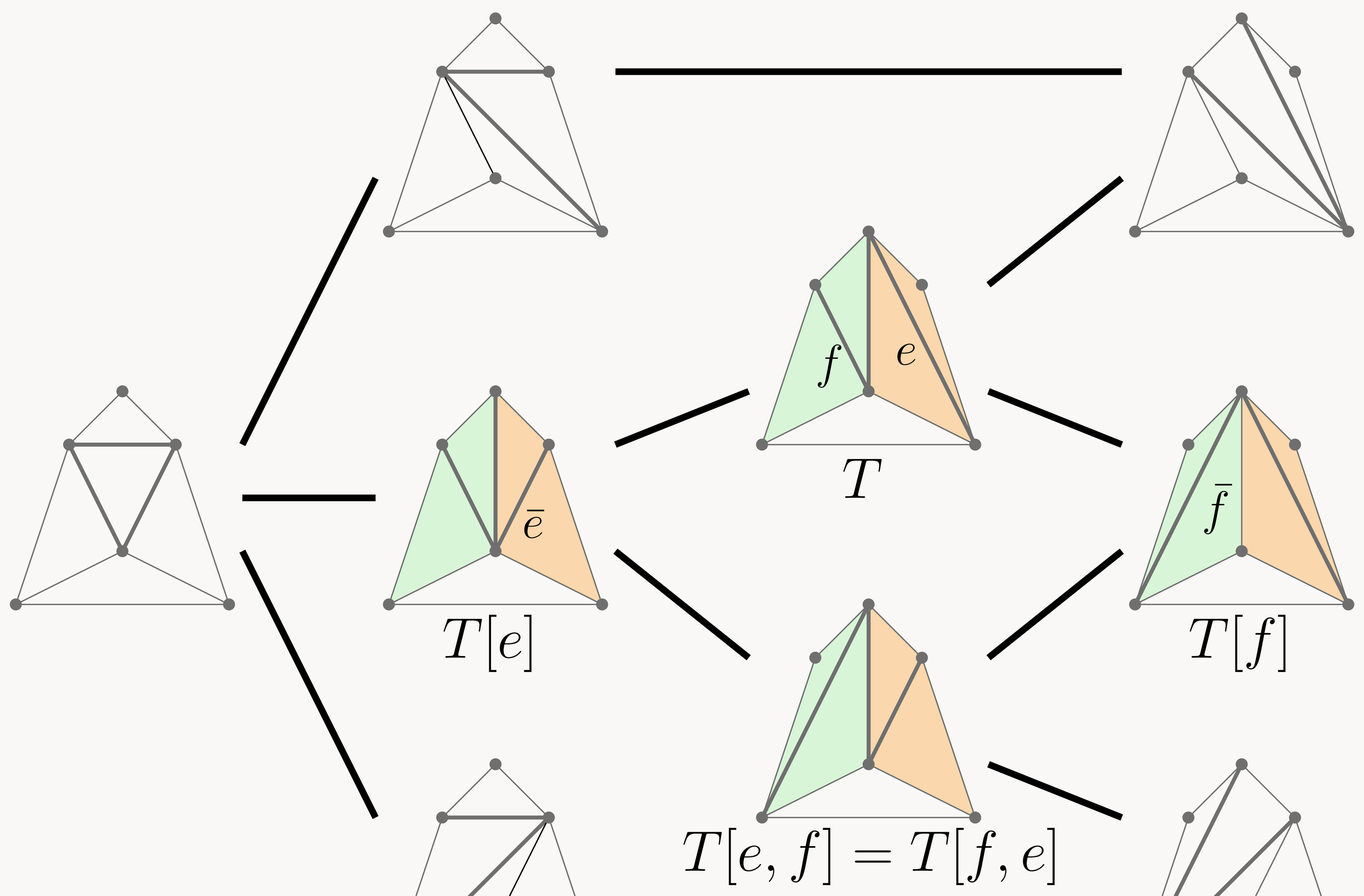
- It's known that the flip graph is connected (Lawson 1972).
- The flip graph is triangle-free.
- Local Menger: A connected graph G is k-vertex connected iff for all $u, v \in V_G$ with distance 2 $\exists k$ paarweise interior vertex-disjoint u-v-paths.

Proof Idea for Statement *(i)*

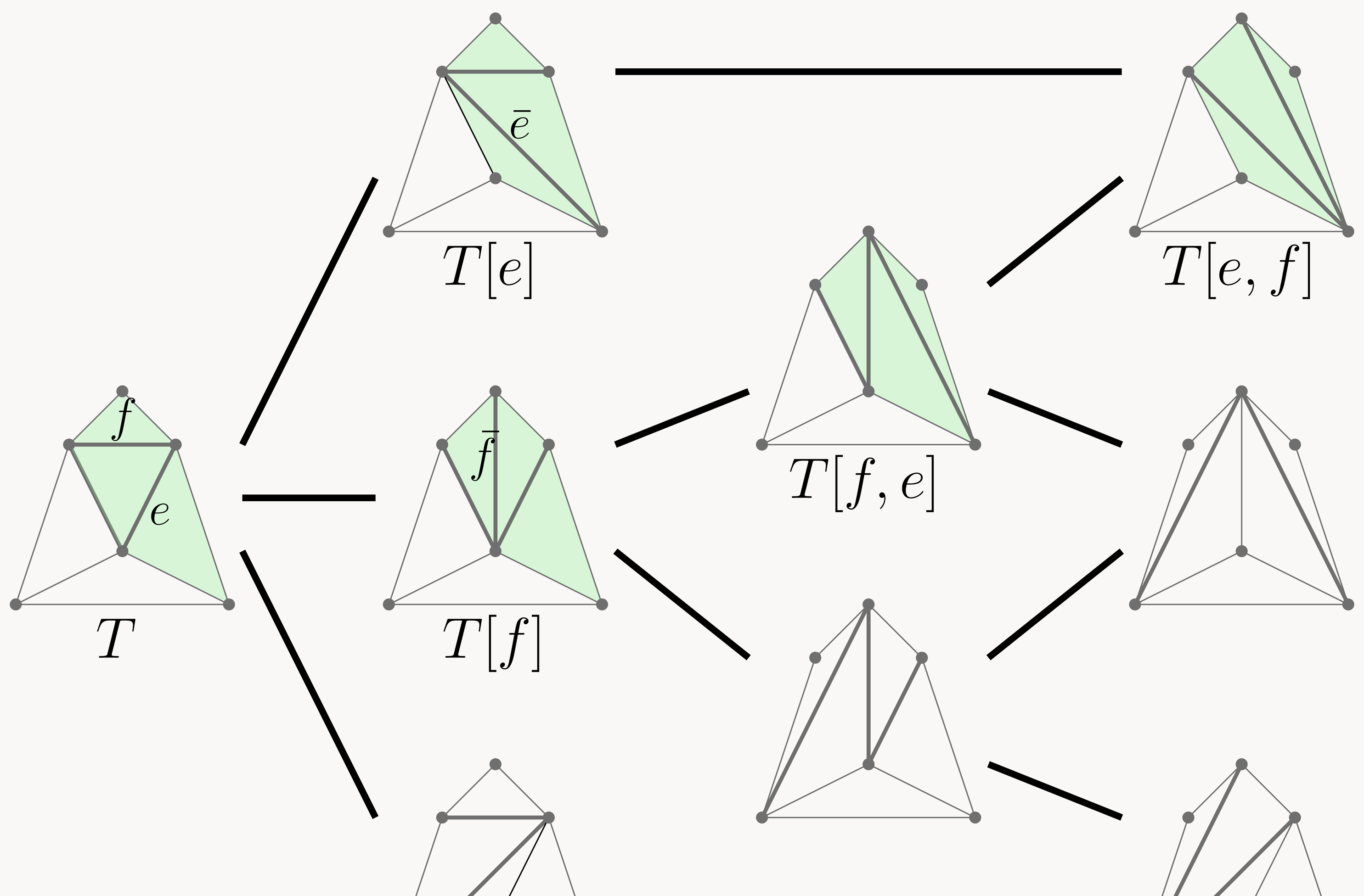
- (a): For $\{T, T[e]\} \in E_{flip(P)}$ there are δ' edge- disjoint T-$T[e]$-paths of length ≤ 9 with δ' the minimum degree of T and $T[e]$.
- To proof (a): Construct path $(T, T[f], ..., T[e])$ for every f flippable in T. Make case distinctions (see below) and use the 4- and 5-cycles. For the at most 2 with e dependetly flippable edges in T construct other paths by hand (possible if n is big enough).
- (b): Similar there are δ interior vertex-disjoint paths between $T[e]$ and $T[f]$.
- (b) together with the *Local Menger* Lemma will give the δ-vertex connectivity of $flip(P)$.

Two in T flippable edges e and f

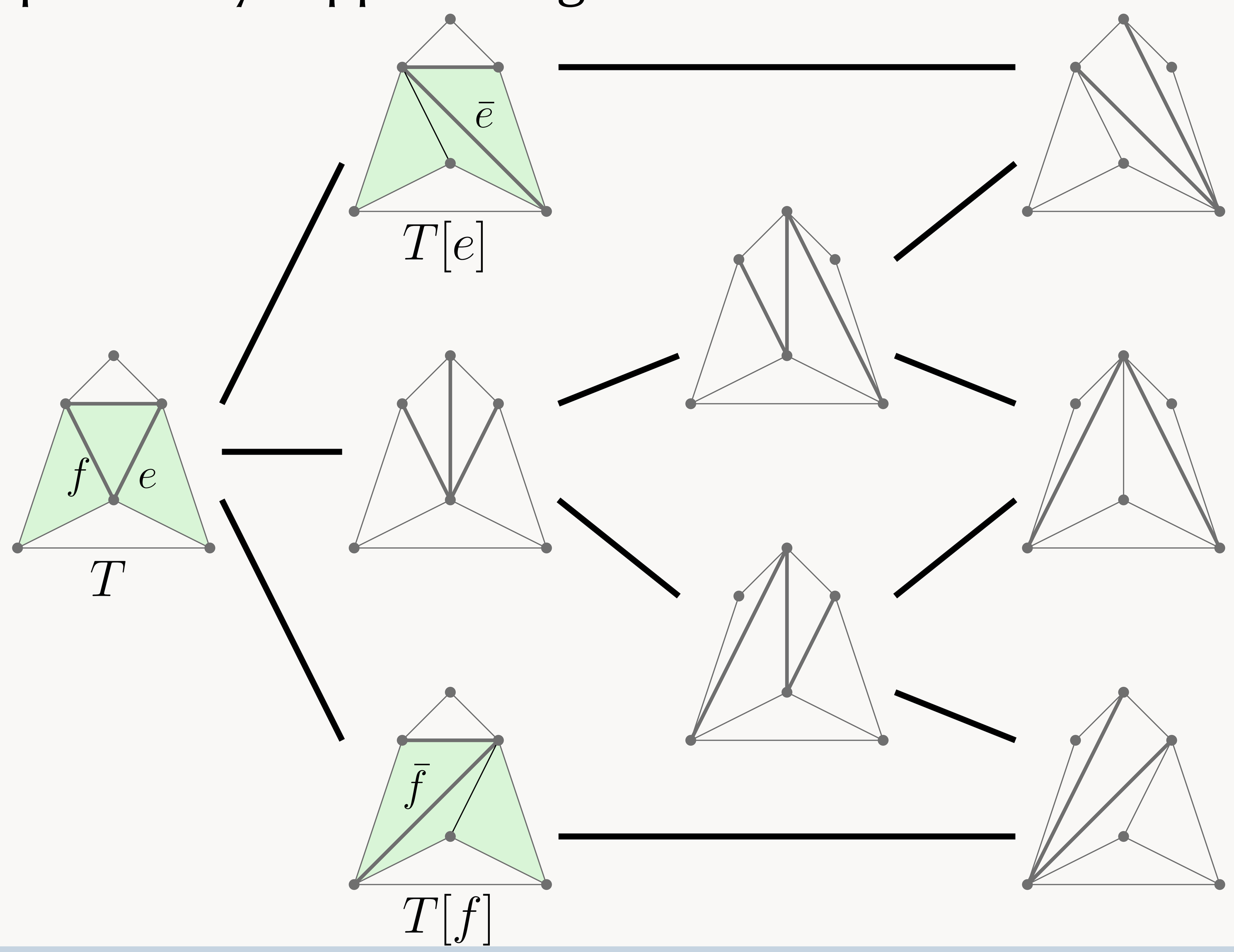
- e and f are *independently flippable* if it holds that $T[e, f] = T[f, e]$.
- $\Leftrightarrow$ $(T, T[e], T[e, f], T[f], T)$ induced 4-cycle.



- e and f are *weakly independently flippable* if it holds that $T[e, f] \neq T[f, e]$.
- $\Leftrightarrow$ $(T, T[e], T[e, f], T[f, e], T[f], T)$ induced 5-cycle.



- e and f are *dependently flippable* if f is not flippable in $T[e]$ (and e not flippable in $T[f]$).
- $e \in E_T$, $deg(T) \leq deg(T[e]) \Rightarrow$ there are ≤ 2 with e dependently flippable edges



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