Lattice Structures from Planar Graphs

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1. Introduction

- The set of all orientations of a planar graph with prescribed outdegrees carries the structure of a distributive lattice.
- This also shows that interesting combinatorial sets related to a planar graph have lattice structure: Eulerian orientations, spanning trees and Schnyder woods.

2. Poset of fixed degree orientations

Definition

$X < Y$ if we can get from X to Y by repeatedly flipping clockwise directed cycles to counterclockwise.

The main result is the following:

Theorem

The set of fixed degree orientations make up a partially ordered set corresponding to a distributive lattice.

Reference

Felsner, Stefan. "Lattice structures from planar graphs." *the electronic journal of combinatorics* (2004): R15-R15.

3.Essential cycles

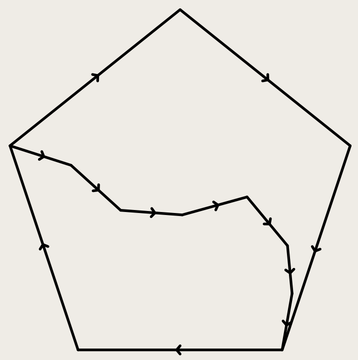
Rigid edges

There are some edges that's orientation is defined by the fixed degrees. We call these *rigid* edges.

We can "ignore" these edges, by taking them out of the graph, and modifying the fixed degrees accordingly.

Essential cycles

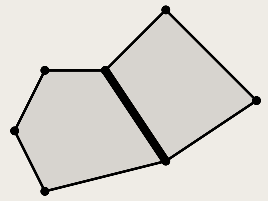
If there are no rigid edges, then we can uniquely decompose any flip into the flips of faces of the planar graph. We will call flipping a clockwise oriented face of a planar graph an essential flip.



This implies that the cover relations in the poset of fixed degree orientations must be exactly the essential flips.

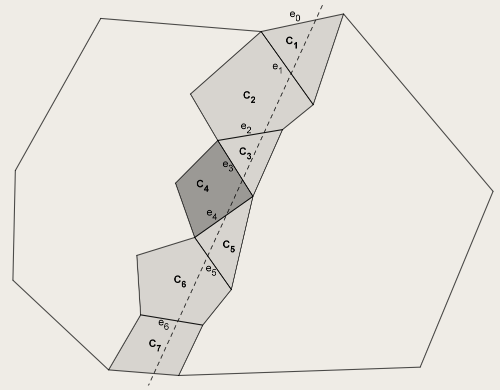
4. Flip sequences

If $X < Y$, then there's a sequence of faces $C_1, C_2, C_3, \dots, C_m$ such that we can flip them one after the other to get from X to Y. Let us call such sequences flip sequences.



The faces left and right of an edge must alternate in the sequence.

From this it is easy to show every such sequence is finite.



- There is a unique orientation such that all cycles in X_{min} are clockwise oriented.
- The poset of fixed degree orientations has a minimal element.
- Any fixed degree orientation can be achieved by a flip sequence from X_{min}.

5. Potentials

Flip sequences that transform X into Y contain the same number of flips at every face.
Let $z_X(F)$ be the number of flips at F transforming X_{min} to X.
We can characterize all functions going from the faces to $\mathbb{N}$:

Potential

A Potential is a mapping from the faces to $\mathbb{N}$ such that

- It differs by at most one in adjacent faces
- It is at most one on faces next to the border
- For every directed edge in X_{min} the function is not smaller on the face to the right

Bijection

There is a bijection between orientations and potentials via $z_X(F)$.
It also holds that

$$z_X(F) \leq z_Y(F) \forall F \iff X \leq Y$$

Lattice of potentials

The set of potentials form a distributive lattice.

- $\phi_1 \vee \phi_2 = \max\{\phi_1(C), \phi_2(C)\}$
- $\phi_1 \wedge \phi_2 = \min\{\phi_1(C), \phi_2(C)\}$

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