Diagonal Flips in Hamiltonian Triangulations

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Main Theorem

The diameter of the flipgraph of a triangulation (maximal planar simple graph) T on $n \geq 19$ vertices is upper bounded by $5.2n - 33.6$.

Making T 4-connected

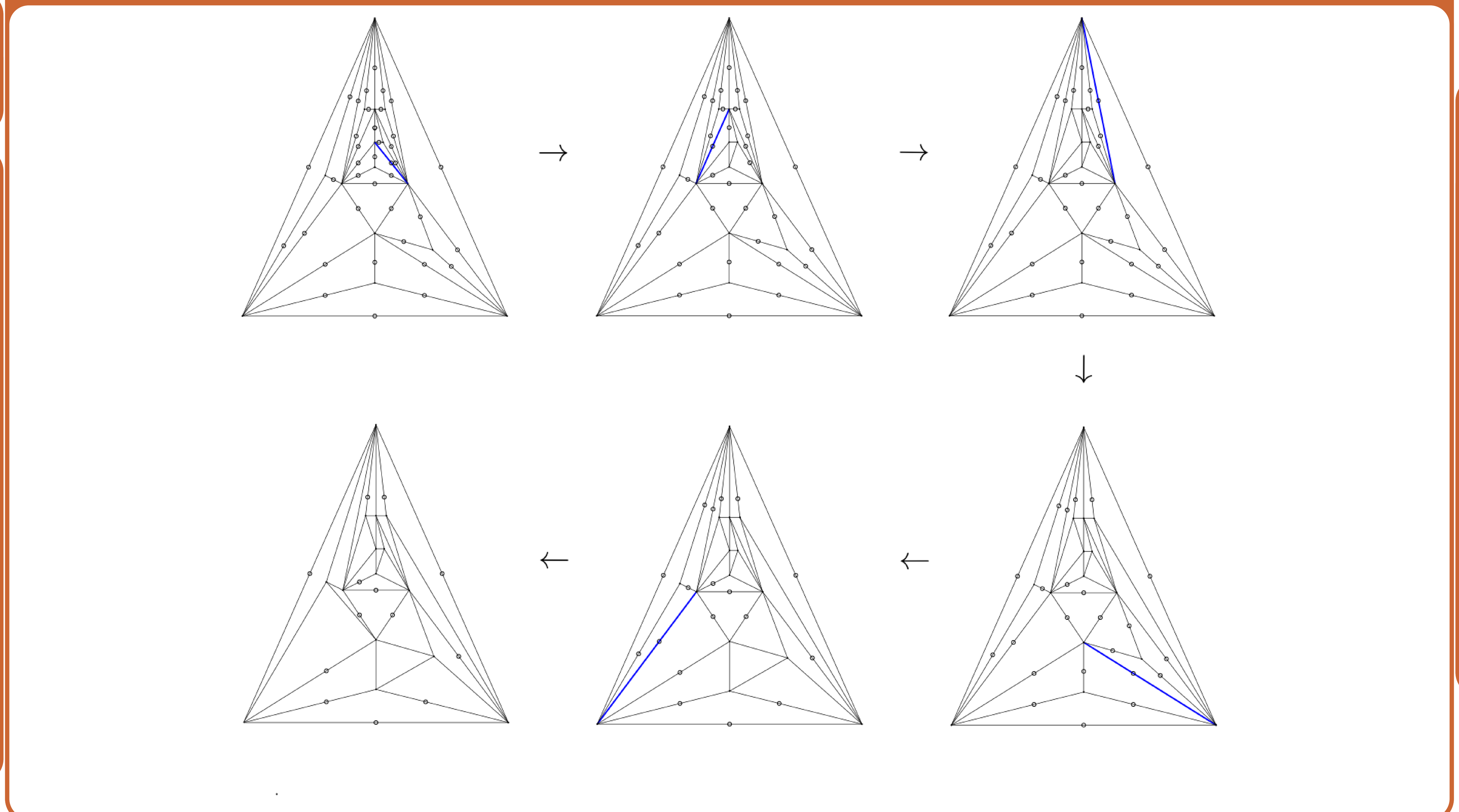
Theorem 1

It takes at most $(3n-9)/5$ flips to make T on $n \geq 6$ vertices 4-connected.

Algorithm

1. find deepest separating triangle (after removal T is split into 2 non-empty connected components) D, preferring ones that do not use an edge of the outer face
   (a) if D does not share any edge with other sep. triangles, flip the edge that is not on the outer face
   (b) if D shares exactly one edge with another sep. triangle, flip this edge
   (c) if D shares multiple edges with other sep. triangles, flip one of the shared edges that is not shared with cont. triangle
2. repeat until T is 4-connected

Example on 15 vertices



Proof via charging scheme

- put a coin on each edge
- use the algorithm and charge 5 coins for each flip
- keep following invariants during each step:
  - every edge of a sep. triangle has a coin
  - every vertex of a sep. triangle has an incident free edge that has a coin

Lower bound to make T 4-connected

Theorem 3

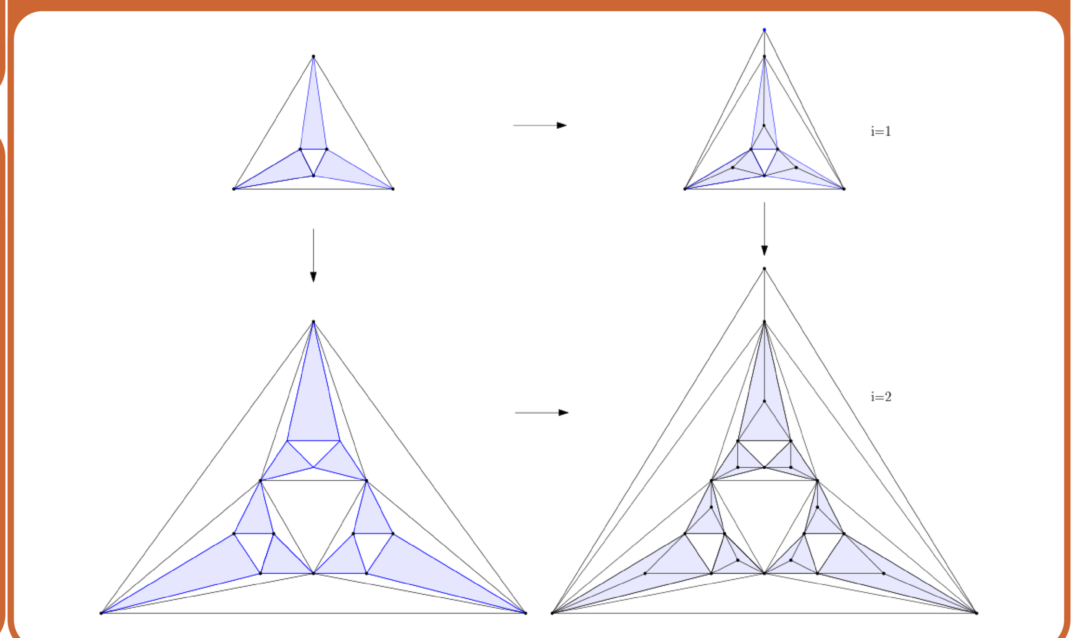
There are triangulations that require $(3n-10)/5$ flips to make them 4-connected.

Proof

- L_i: number of triangles to iterate on in step i
- V_i: number of vertices of construction in step i
- S_i: number of separating triangles of construction in step i
- $V_i = V_{i-1} + 5L_{i-1} = 10 + 5\sum_{k=2}^{i-1} L_k$
- $S_i = S_{i-1} + 3L_{i-1} = 4 + 3\sum_{k=2}^{i-1} L_k$

$\Rightarrow (V_i - 10)/5 = \sum_{k=2}^{i-1} L_k \Rightarrow S_i = 4 + 3(V_i - 10)/5 = (3V_i - 10)/5$

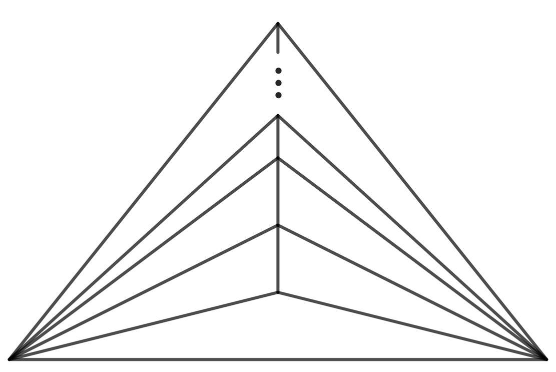
Construction



Transforming 4-connected T to the canonical triangulation

Theorem 2

It takes at most $2n - 15$ flips to transform T on $n \geq 19$ vertices to the canonical triangulation.



Lemma

Let (u, v) be in 4-connected T. There is a Hamiltonian cycle that uses (u, v) such that all non-cycle edges incident to u are on one side of the cycle and all non-cycle edges incident to v are on the other.

Example on 15 vertices