Discrete & Computational Geometry (2025) 73:79–91
https://doi.org/10.1007/s00454-024-00658-6

Unavoidable Patterns in Complete Simple Topological Graphs

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Received: 11 November 2022 / Revised: 3 April 2024 / Accepted: 4 May 2024 /

Abstract

We show that every complete n-vertex simple topological graph contains a topological subgraph on at least $(\log n)^{1/4-o(1)}$ vertices that is weakly isomorphic to the complete convex geometric graph or the complete twisted graph. This is the first improvement on the bound $\Omega(\log^{1/8} n)$ obtained in 2003 by Pach, Solymosi, and Tóth. We also show that every complete n-vertex simple topological graph contains a plane path of length at least $(\log n)^{1-o(1)}$.

Keywords Topological graph · Unavoidable patterns · Plane path

Mathematics Subject Classification 05C35 · 05C62 · 68R10

1 Introduction

A *topological graph* is a graph drawn in the plane or, equivalently, on the sphere, such that its vertices are represented by points and its edges are represented by non-self-intersecting arcs connecting the corresponding points. The arcs are not allowed to pass through vertices different from their endpoints, and if two edges share an interior point, then they must properly (i.e. transversally) cross at that point in common. A topological graph is *simple* if every pair of its edges intersect at most once, either at a common endpoint or at a proper crossing point. If the edges are drawn as straight-line

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Editor in Charge: János Pach

A. Suk: Supported by NSF CAREER award DMS-1800746 and NSF award DMS-1952786.
J. Zeng: Supported by NSF grant DMS-1800746.

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segments, then the graph is said to be *geometric*. If the vertices of a geometric graph are in convex position, then it is called *convex*.

Simple topological graphs have been extensively studied [11, 13, 16, 18, 22], and are sometimes referred to as *good drawings* [1, 2], or simply as *topological graphs* [14]. In this paper, we are interested in finding large unavoidable patterns in complete simple topological graphs. Two simple topological graphs G and H are *isomorphic* if there is a homeomorphism of the sphere that transforms G to H. We say that G and H are *weakly isomorphic* if there is an incidence preserving bijection between G and H such that two edges of G cross if and only if the corresponding edges in H cross as well. Clearly, any two complete convex geometric graphs on m vertices are weakly isomorphic. Hence, let C_m denote any complete convex geometric graph with m vertices.

By the famous Erdős-Szekeres convex polygon theorem [6] (see also [21]), every complete n-vertex geometric graph contains a geometric subgraph on $m = \Omega(\log n)$ vertices that is weakly isomorphic to C_m. (Note that no three vertices in a complete geometric graph are collinear.) Interestingly, the same is not true for simple topological graphs. The *complete twisted graph* T_m is a complete simple topological graph on m vertices with the property that there is an ordering on the vertex set $V(T_m) = \{v_1, v_2, \ldots, v_m\}$ such that edges v_iv_j and v_kv_ℓ cross if and only if $i < k < \ell < j$ or $k < i < j < \ell$. See Fig. 1. It was first observed by Harborth and Mengerson [10] that T_m does not contain a topological subgraph that is weakly isomorphic to C_5. However, in 2003, Pach, Solymosi, and Tóth [14] showed that it is impossible to avoid both C_m and T_m in a sufficiently large complete simple topological graph.

Theorem 1.1 (Pach–Solymosi–Tóth) *Every complete n-vertex simple topological graph contains a topological subgraph on $m \geq \Omega(\log^{1/8} n)$ vertices that is weakly isomorphic to C_m or T_m.*

The main result of this paper is the following improvement.

Theorem 1.2 *Every complete n-vertex simple topological graph has a topological subgraph on $m \geq (\log n)^{1/4-o(1)}$ vertices that is weakly isomorphic to C_m or T_m.*

In the other direction, let us consider the following construction. Let $V = \{1, 2, \ldots, n\}$ be n vertices placed on the x-axis, and for each pair $\{i, j\} \subset V$, draw a half-circle connecting i and j, with this half-circle either in the upper or lower half of the plane uniformly at random. By applying the standard probabilistic method [3], one can

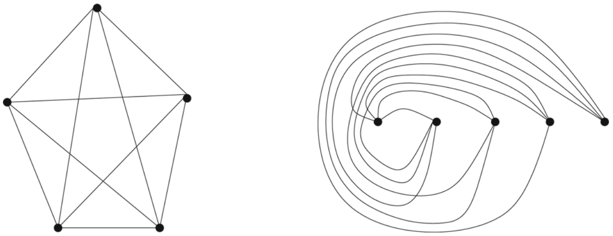

Fig. 1 C_5 and T_5

show that there is a complete n-vertex simple topological graph that does not contain a topological subgraph on $m = \lceil 8 \log n \rceil$ vertices that is weakly isomorphic to C_m or T_m. Another construction, observed by Scheucher [19], is to take n points in the plane without $m = 2\lceil \log n \rceil$ members that are in convex position, and then draw straight-line segments between all pairs of points.

It is not hard to see that both C_m and T_m contain a *plane* (i.e. crossing-free) subgraph isomorphic to any given tree T with at most m vertices (see, e.g., [9]). Hence, Theorem 1.1 implies that every complete n-vertex simple topological graph contains a plane subgraph isomorphic to any given tree T with at most $\Omega(\log^{1/8} n)$ vertices. (Due to an inaccuracy in the original proof, the paper [14] claimed a slightly stronger bound $\Omega(\log^{1/6} n)$, see also [15].) As a corollary of Theorem 1.2, we obtain the following improvement accordingly.

Corollary 1.3 *Every complete n-vertex simple topological graph contains a plane subgraph isomorphic to any given tree T with at most* $(\log n)^{1/4-o(1)}$ *vertices.*

In the case when T is a path, we improve this bound with the following result, which is also recently obtained in [2] independently.

Theorem 1.4 *Every complete n-vertex simple topological graph contains a plane path of length at least* $(\log n)^{1-o(1)}$.

In order to avoid confusion between topological and combinatorial edges, we write uv when referring to a topological edge in the plane, and write $\{u, v\}$ when referring to an edge (pair) in a graph. Likewise, we write $\{u_1, \dots, u_k\}$ when referring to an edge (k-tuple) in a k-uniform hypergraph. We systematically omit floors and ceilings whenever they are not crucial for the sake of clarity in our presentation. All logarithms are in base 2.

2 Monotone Paths and Online Ramsey Numbers

Before we prove Theorem 1.2, let us recall the following lemmas. Let H be a k-uniform hypergraph with vertex set $[n] = \{1, 2, \dots, n\}$. We say that H contains a *monotone k-path* of length m if there are m vertices $v_1 < v_2 < \cdots < v_m$ such that $\{v_i, v_{i+1}, \dots, v_{i+k-1}\} \in E(H)$ for $1 \leq i \leq m-k+1$. We say that the edge set $E(H)$ is *transitive* if for any $v_1 < v_2 < \cdots < v_{k+1}$ in $[n]$, the condition $\{v_1, v_2, \dots, v_k\}, \{v_2, v_3, \dots, v_{k+1}\} \in E(H)$ implies all k-element subsets of $\{v_1, \dots, v_{k+1}\}$ are in $E(H)$. We will need the following lemma due to Fox, Pach, Sudakov, and Suk.

Lemma 2.1 [7, Lem. 6.2] *Let $n > k$, and let H be a k-uniform hypergraph with vertex set $[n]$, which contains a monotone path of length n, that is, $\{i, i+1, \dots, i+k-1\} \in E(H)$ for all $1 \leq i \leq n-k+1$. If $E(H)$ is transitive, then H is the complete k-uniform hypergraph on $[n]$.*

Next, we need a lemma from Online Ramsey Theory. The *vertex online Ramsey game* is a game played by two players, *builder* and *painter*. Let $t \geq 1$ and suppose vertices $v_1, v_2, \dots, v_{t-1}$ are present. At the beginning of stage t, a new vertex v_t is added. Then for each $v_i \in \{v_1, \dots, v_{t-1}\}$, builder decides (in any order) whether to create the edge $\{v_i, v_t\}$. If builder creates the edge, then painter has to immediately color it red or blue. When builder decides not to create any more edges, stage t ends and stage $t+1$ begins by adding a new vertex. Moreover, builder must create at least one edge at every stage except for the first one. The *vertex online Ramsey number* $r(m)$ is the minimum number of edges builder has to create to guarantee a monochromatic monotone path of length m in a vertex online Ramsey game. Clearly, we have $r(m) \le O(m^4)$, which is obtained by having builder create all possible edges at each stage and applying Dilworth's theorem [5] on the m^2 vertices. Fox, Pach, Sudakov, and Suk proved the following.

Lemma 2.2 [7, Thm. 1.5] *We have* $r(m) = (1 + o(1))m^2 \log_2 m$.

3 Convex Geometric Graph Versus Twisted Graph

In this section, we prove the following theorem, from which Theorem 1.2 quickly follows.

Theorem 3.1 *Let* m_1, m_2, n *be positive integers such that*

$$9(m_1 m_2)^2 \log(m_1) \log(m_2) < \log n.$$

Then every complete n*-vertex simple topological graph contains a topological subgraph that is weakly isomorphic to* C_{m_1} *or* T_{m_2}.

Proof Let $G = (V, E)$ be a complete n-vertex simple topological graph. Notice that the edges of G divide the plane into several cells (regions), one of which is unbounded. We can assume that there is a vertex $v_0 \in V$ such that v_0 lies on the boundary of the unbounded cell. Indeed, otherwise we can project G onto a sphere, then choose an arbitrary vertex v_0, and then project G back to the plane such that v_0 lies on the boundary of the unbounded cell, moreover, the new drawing is isomorphic to the original one as topological graphs.

Consider the topological edges emanating out from v_0, and label their endpoints $v_1, \dots, v_{n-1}$ in clockwise order. For convenience, we write $v_i \prec v_j$ if $i < j$. Given subsets $U, W \subset \{v_1, \dots, v_{n-1}\}$, we write $U \prec W$ if $u \prec w$ for all $u \in U$ and $w \in W$. Following the notation used in [14], we color the triples of $\{v_1, \dots, v_{n-1}\}$ as follows. For $v_i \prec v_j \prec v_k$, let $\chi(v_i, v_j, v_k) = xyz$, where $x, y, z \in \{0, 1\}$ such that

1. $x = 1$ if edges $v_j v_k$ and $v_0 v_i$ cross, and $x = 0$ otherwise;
2. $y = 1$ if edges $v_i v_k$ and $v_0 v_j$ cross, and $y = 0$ otherwise;
3. $z = 1$ if edges $v_i v_j$ and $v_0 v_k$ cross, and $z = 0$ otherwise.

Pach, Solymosi, and Tóth observed the following.

Observation 3.2 [14] *The only colors that appear with respect to* χ *are* 000, 001, 010, *and* 100.

See Fig. 2 for an illustration. We now make another observation.

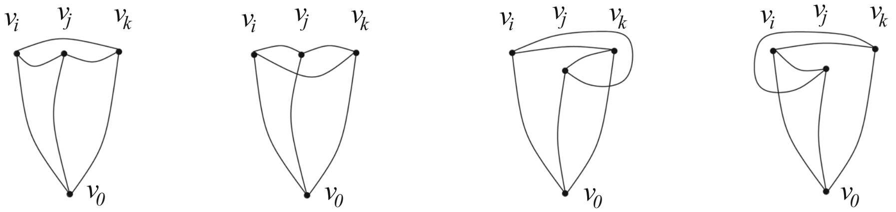

Fig. 2 Configurations for the colors 000, 010, 001, 100, respectively

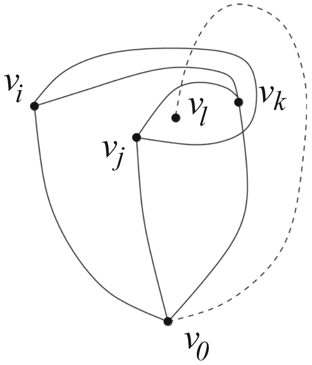

Fig. 3 The closed region bounded by edges v_jv_k, v_iv_j, and v_0v_k in Lemma 3.3

Lemma 3.3 *Colors* 001 *and* 100 *are transitive. That is, for* $v_i \prec v_j \prec v_k \prec v_\ell$,

1. *if* $\chi(v_i, v_j, v_k) = \chi(v_j, v_k, v_\ell) = 001$, *then* $\chi(v_i, v_j, v_\ell) = \chi(v_i, v_k, v_\ell) = 001$;
2. *if* $\chi(v_i, v_j, v_k) = \chi(v_j, v_k, v_\ell) = 100$, *then* $\chi(v_i, v_j, v_\ell) = \chi(v_i, v_k, v_\ell) = 100$.

Proof Suppose $\chi(v_i, v_j, v_k) = \chi(v_j, v_k, v_\ell) = 001$. Since the edges v_iv_j and v_0v_k cross, the three edges v_jv_k, v_iv_j, and v_0v_k enclose a bounded region A. Since v_0v_ℓ and v_jv_k cross, the vertex v_ℓ must lie in this region A. See Fig. 3. Since A is contained in the region bounded by the edges v_0v_i, v_iv_k, and v_0v_k, we conclude that v_0v_ℓ must cross v_iv_k. Similarly, we can argue that v_0v_ℓ must cross v_iv_j, hence we conclude $\chi(v_i, v_j, v_\ell) = \chi(v_i, v_k, v_\ell) = 001$ as wanted. If $\chi(v_i, v_j, v_k) = \chi(v_j, v_k, v_\ell) = 100$, a similar proof shows that $\chi(v_i, v_j, v_\ell) = \chi(v_i, v_k, v_\ell) = 100$. □

Based on the coloring χ, we define a coloring ϕ of the pairs of $\{v_1, v_2, \dots, v_{n-1}\}$ as follows. For $v_i \prec v_j$, let $\phi(v_i, v_j) = (a, b)$ where a is the length of the longest monotone 3-path ending at $\{v_i, v_j\}$ in color 100, and b is the length of the longest monotone 3-path ending at $\{v_i, v_j\}$ in color 001. We can assume that $a, b < m_2$. Otherwise, by Lemma 3.3 and Lemma 2.1, we would have a subset $U \subset V$ of size m_2 all of whose triples receive the same color, 100 or 001. And it is not hard to argue by induction that such a U corresponds to a topological subgraph that is weakly isomorphic to T_{m_2} as wanted.

Before we continue, let us give a rough outline of the rest of the proof. In what follows, we will construct disjoint vertex subsets $V^{a,b} \subset \{v_1, \dots, v_{n-1}\}$, where $1 <$

$a, b < m_2$, such that ϕ colors every pair in $V^{a,b}$ with color (a, b). For each $V^{a,b}$, we will play the vertex online Ramsey game by letting the builder create an edge set $E^{a,b}$ and designing a painter's strategy, which gives rise to a coloring ψ on $E^{a,b}$. We then apply Lemma 2.2 to show that if n is sufficiently large, some vertex set $V^{a,b}$ will contain a monochromatic monotone 2-path of length m_1 with respect to ψ. Finally, we will show that this monochromatic monotone 2-path will correspond to a topological subgraph that is weakly isomorphic to C_{m_1}. The detailed argument follows.

For integers $t \geq 0$ and $1 < a, b < m_2$, we construct a vertex subset $V_t^{a,b} \subset \{v_1, \ldots, v_{n-1}\}$, an edge set $E_t^{a,b}$ of pairs in $V_t^{a,b}$, and a subset $S_t \subset \{v_1, \ldots, v_{n-1}\}$ such that the following holds.

1. We have $\sum_{1<a,b<m_2} |V_t^{a,b}| = t$.
2. For all $1 < a, b < m_2$, we have $V_t^{a,b} \prec S_t$.
3. For $u_1 \in V_t^{a,b}$, we have $\phi(u_1, u_2) = (a, b)$ for every $u_2 \in V_t^{a,b} \cup S_t$ with $u_1 \prec u_2$.
4. For each edge $\{u_1, u_2\} \in E_t^{a,b}$, where $u_1 \prec u_2$, we have $\chi(u_1, u_2, u_3) = \chi(u_1, u_2, u_4)$ for all $u_3, u_4 \in V_t^{a,b}$ such that $u_1 \prec u_2 \prec u_3 \prec u_4$.

We start by setting $V_0^{a,b} = \emptyset$ for all $1 < a, b < m_2$, and $S_0 = \{v_1, \ldots, v_{n-1}\}$. After stage t, we have $V_t^{a,b}$, $E_t^{a,b}$, for $1 < a, b < m_2$, and S_t as described above.

At the beginning of stage $t + 1$, let w_{t+1} be the smallest element in S_t with respect to $\prec$. By the pigeonhole principle, there exist integers $1 < \alpha, \beta < m_2$ and a subset $S_{t,0} \subset S_t \setminus \{w_{t+1}\}$ of size at least $(|S_t| - 1)/m_2^2$, such that $\phi(w_{t+1}, u) = (\alpha, \beta)$ for all $u \in S_{t,0}$. Then we set $V_{t+1}^{\alpha,\beta} := V_t^{\alpha,\beta} \cup \{w_{t+1}\}$. For all $1 < a, b < m_2$ with $(a, b) \neq (\alpha, \beta)$, we set $V_{t+1}^{a,b} := V_t^{a,b}$ and $E_{t+1}^{a,b} := E_t^{a,b}$. We shall define $E_{t+1}^{\alpha,\beta}$ and S_{t+1} after the following claim.

Claim 3.4 *For all* $u \in V_t^{\alpha,\beta}$ *and* $v \in S_{t,0}$, *we have* $\chi(u, w_{t+1}, v) \in \{000, 010\}$.

Proof For the sake of contradiction, suppose $\chi(u, w_{t+1}, v) = 100$, where $u \in V_t^{\alpha,\beta}$ and $v \in S_{t,0}$. Since $\phi(u, w_{t+1}) = (\alpha, \beta)$, the longest monotone 3-path in color 100 ending at $\{u, w_{t+1}\}$ has length α. Hence, the longest monotone 3-path in color 100 ending at $\{w_{t+1}, v\}$ has length at least $\alpha+1$. This contradicts the fact that $\phi(w_{t+1}, v) = (\alpha, \beta)$. A similar argument follows if $\chi(u, w_{t+1}, v) = 001$. □

Now that we have constructed $V_{t+1}^{\alpha,\beta}$ by adding w_{t+1} to $V_t^{\alpha,\beta}$, we play the vertex online Ramsey game so that builder chooses and creates edges of the form $\{u, w_{t+1}\}$, where $u \in V_t^{\alpha,\beta}$, according to his strategy. After each edge $\{u, w_{t+1}\}$ is created, painter immediately colors it $\psi(u, w_{t+1}) \in \{000, 010\}$ as follows. In painter's strategy, after the j-th edge $\{u_j, w_{t+1}\}$ is created and colored, a set $S_{t,j} \subset S_{t,0}$ will be constructed such that all triples $\{u_j, w_{t+1}, v\}$ with $v \in S_{t,j}$ are colored by χ with the same color in $\{000, 010\}$. After the $(j+1)$-th edge $\{u_{j+1}, w_{t+1}\}$ is created, painter looks at all triples of the form $\{u_{j+1}, w_{t+1}, v\}$ with $v \in S_{t,j}$. Since $\chi(u_{j+1}, w_{t+1}, v) \in \{000, 010\}$ by Claim 3.4, the pigeonhole principle implies that there exists a subset $S_{t,j+1} \subset S_{t,j}$ with size at least $|S_{t,j}|/2$ such that all triples $\{u_{j+1}, w_{t+1}, v\}$ with $v \in S_{t,j+1}$ are colored by χ with the same color $xyz \in \{000, 010\}$. Then painter sets $\psi(u_{j+1}, w_{t+1}) = xyz$.

If builder decides to stop creating edges from w_{t+1} to $V_t^{\alpha,\beta}$ after j edges are created and colored, then the stage ends and we set $S_{t+1} = S_{t,j}$ and let $E_{t+1}^{\alpha,\beta}$ be the union of $E_t^{\alpha,\beta}$ and all edges built during this stage. Let e_{t+1} denote the total number of edges builder creates in stage $t+1$. Recall that $e_{t+1} \geq 1$ unless $V_t^{\alpha,\beta} = \emptyset$. As long as $|S_{t+1}| > 0$, we continue this construction process by starting the next stage. Clearly, $V_{t+1}^{a,b}$, $E_{t+1}^{a,b}$, for all $1 < a, b < m_2$, and S_{t+1} have the four properties described above. For example, let u_1, u_2, u_3, u_4 be as in the fourth property. There exists some stage $t' < t$ when $u_2 = w_{t'}$ and the edge $\{u_1, u_2\}$ is created by builder. Since $u_3, u_4 \in S_{t'}$, we have $\chi(u_1, u_2, u_3) = \chi(u_1, u_2, u_4)$ as wanted.

Claim 3.5 *For $t \geq 1$, we have*

$$|S_t| \geq \frac{n-1}{m_2^{2t} \cdot 2^{\sum_{i=2}^t e_i}} - \sum_{i=2}^{t} \frac{1}{m_2^{2(t+1-i)} \cdot 2^{\sum_{j=i}^t e_j}}.$$

Proof We proceed by induction on t. For the base case $t = 1$, there is no edge for the builder to build in the first stage, so $|S_1| = |S_{0,0}| \geq (n-1)/m_2^2$ as desired. For the inductive step, assume the statement holds for $t \geq 1$. When we start stage $t+1$ and introduce vertex w_{t+1}, the set S_t shrinks to $S_{t,0}$ whose size is guaranteed to be at least $(|S_t| - 1)/m_2^2$, and each time builder creates an edge from w_{t+1} to $V_t^{\alpha,\beta}$, our set decreases by a factor of two. Since builder creates e_{t+1} edges during stage $t+1$, we have

$$\begin{aligned} |S_{t+1}| &\geq \frac{|S_t| - 1}{m_2^2 2^{e_{t+1}}} \geq \frac{n-1}{m_2^{2(t+1)} \cdot 2^{\sum_{i=2}^{t+1} e_i}} - \sum_{i=2}^{t} \frac{1}{m_2^{2((t+1)+1-i)} \cdot 2^{\sum_{j=i}^{t+1} e_j}} - \frac{1}{m_2^2 2^{e_{t+1}}} \\ &= \frac{n-1}{m_2^{2(t+1)} \cdot 2^{\sum_{i=2}^{t+1} e_i}} - \sum_{i=2}^{t+1} \frac{1}{m_2^{2((t+1)+1-i)} \cdot 2^{\sum_{j=i}^{t+1} e_j}}, \end{aligned}$$

which is what we want. □

After t stages, builder has created a total of $\sum_{i=1}^t e_i$ edges, such that each edge has color 000 or 010 with respect to ψ. If there is no monochromatic 2-path of length m_1 with respect to ψ on any $(V_t^{a,b}, E_t^{a,b})$, then Lemma 2.2 implies that

$$\sum_{i=1}^t e_i < m_2^2 r(m_1) \leq 2(m_1 m_2)^2 \log m_1.$$

Also, since $e_i \geq 1$ for all but m_2^2 many indices $1 \leq i \leq t$, we have

$$t \leq m_2^2 + \sum_{i=1}^t e_i < 3(m_1 m_2)^2 \log m_1.$$

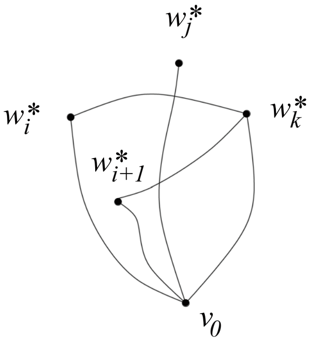

Fig. 4 A figure illustrating Claim 3.6

Then it follows from the assumption $\log n > 9(m_1m_2)^2\log(m_1)\log(m_2)$ that

$$\begin{aligned}|S_t| &\geq \frac{n-1}{m_2^{2t}\cdot 2^{\sum_{i=2}^{t}e_i}} - \sum_{i=2}^{t}\frac{1}{m_2^{2(t+1-i)}\cdot 2^{\sum_{j=i}^{t}e_j}}\\ &\geq \frac{n-1}{2^{8(m_1m_2)^2\log(m_1)\log(m_2)}} - \sum_{i=2}^{t}\frac{1}{2^{t-i+1}} > 1.\end{aligned}$$

Hence, we can continue to the next stage and introduce a new vertex w_{t+1}. Therefore, when this process stops, say at stage s, we must have a monochromatic monotone 2-path of length m_1 with respect to ψ on some $(V_s^{a,b}, E_s^{a,b})$.

Now let $W^* = \{w_1^*, \dots, w_{m_1}^*\}$, where $w_1^* \prec \cdots \prec w_{m_1}^*$, be the vertex set that induces a monochromatic monotone 2-path of length m_1 with respect to ψ on $(V_s^{a,b}, E_s^{a,b})$. Since ϕ colors every pair in W^* with the color (a, b), by following the proof of Claim 3.4, we have $\chi(w_i^*, w_j^*, w_k^*) \in \{000, 010\}$ for every $i < j < k$. Hence, the following argument due to Pach, Solymosi, and Tóth [14] shows that W^* induces a topological subgraph that is weakly isomorphic to C_{m_1}. For the sake of completeness, we include the proof.

Claim 3.6 *Let $W^* = \{w_1^*, \dots, w_{m_1}^*\}$ be as described above. Then W^* induces a topological subgraph that is weakly isomorphic to C_{m_1}.*

Proof Suppose $\psi(w_i^*, w_{i+1}^*) = 000$ for all i. It suffices to show that every triple in W^* has color 000 with respect to χ. For the sake of contradiction, suppose we have $w_i^* \prec w_j^* \prec w_k^*$ such that $\chi(w_i^*, w_j^*, w_k^*) = 010$, and let us assume that $j - i$ is minimized among all such examples. Since $\{w_i^*, w_{i+1}^*\} \in E_s^{a,b}$, we have $\chi(w_i^*, w_{i+1}^*, w_k^*) = \psi(w_i^*, w_{i+1}^*) = 000$. This implies that $j > i + 1$ and the edge $w_{i+1}^*w_k^*$ crosses $v_0w_j^*$ (see Fig. 4), which contradicts the minimality condition. A similar argument follows if $\psi(w_i^*, w_{i+1}^*) = 010$ for all i. □

This completes the proof of Theorem 3.1. □

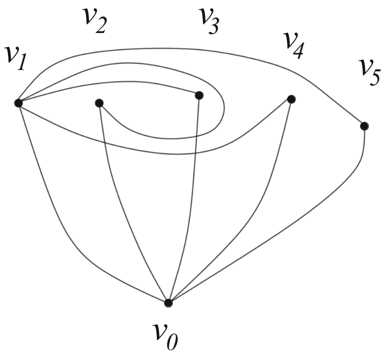

Fig. 5 An example with $\theta(v_1) = (v_4, v_3, v_2, v_5)$

4 Plane Path

In this section, we prove Theorem 1.4. We will need the following lemma, which was observed by Fulek and Ruiz-Vargas in [8].

Lemma 4.1 *If a complete simple topological graph G contains a topological subgraph that is isomorphic to a plane K_{2,m^2}, then G contains a plane path of length $\Omega(m)$.*

Let us briefly explain how to establish this lemma, as it is not explicitly stated in [8]. In [23], Tóth proved that every n-vertex geometric graph with more than 2^9k^2n edges contains k pairwise disjoint edges. His proof easily generalizes to simple topological graphs whose edges are drawn as x-monotone curves, and, in fact, establishes the existence of a plane path of length $2k$.

Given a plane topological subgraph K_{2,m^2} inside a complete simple topological graph G, Fulek and Ruiz-Vargas [8] showed that there exists a topological subgraph $G' \subset G$, with m^2 vertices and $\Omega(m^4)$ edges, that is weakly-isomorphic to an x-monotone simple topological graph G''. Hence, we can conclude Lemma 4.1 by applying Tóth's result stated above with $k = \Omega(m)$.

Proof of Theorem 1.4 First, we keep the following notations from the proof of Theorem 1.2. Let $G = (V, E)$ be a complete n-vertex simple topological graph. We can assume that there is a vertex $v_0 \in V$ such that v_0 lies on the boundary of the unbounded cell. We label the other vertices by $v_1, \ldots, v_{n-1}$ such that the edges v_0v_i, for $1 \le i < n$, emanate out from v_0 in clockwise order. We write $v_i \prec v_j$ if $i < j$, and color every triple $v_i \prec v_j \prec v_k$ by $\chi(v_i, v_j, v_k) \in \{000, 010, 100, 001\}$.

For each v_i, we arrange the vertices $\{v_{i+1}, \ldots, v_{n-1}\}$ into a sequence $\theta(v_i) = (v_{j_1}, \ldots, v_{j_{n-1-i}})$ such that the topological edges $v_iv_0, v_iv_{j_1}, v_iv_{j_2}, \ldots, v_iv_{j_{n-1-i}}$ emanate out from v_i in counterclockwise order. See Fig. 5. We call a sequence of vertices $S = (v_{i_1}, \ldots, v_{i_k})$ increasing (or decreasing) if $v_{i_1} \prec v_{i_2} \prec \cdots \prec v_{i_k}$ (or $v_{i_1} \succ v_{i_2} \succ \cdots \succ v_{i_k}$).

Lemma 4.2 *If there exists a vertex u such that $\theta(u)$ contains an increasing subsequence $(u_1, \ldots, u_{m^2})$, then the edges v_0u_i and uu_i, for all $1 \le i \le m^2$, form a plane subgraph K_{2,m^2}.*

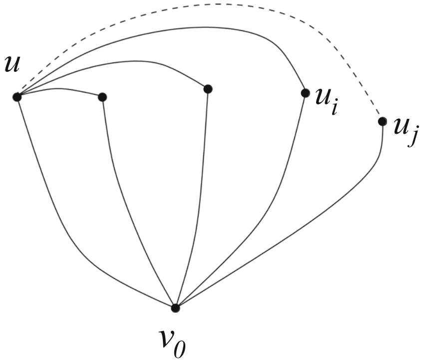

Fig. 6 An increasing subsequence of $\theta(u)$ induce a plane K_{2,m^2}

Proof It suffices to show that v_0u_i and uu_j do not cross each other for every $1 \le i, j \le k$. When $i = j$, this follows from G being simple. When $i < j$, we have $u \prec u_i \prec u_j$ by the increasing subsequence assumption. Observe that this condition forces u_j to be outside the region $\Delta_{v_0uu_i}$ bounded by the topological edges v_0u, uu_i, and u_iv_0. Otherwise, we have $\chi(u, u_i, u_j) = 001$ and the edges uv_0, uu_i, uu_j will not emanate out from u in counterclockwise order. Now the Jordan arc uu_j starting at u, initially outside $\Delta_{v_0uu_i}$, cannot enter $\Delta_{v_0uu_i}$ then leave again to end at u_j. In particular, uu_j does not cross v_0u_i. See Fig. 6 for an illustration. A similar argument follows if $j < i$. □

We set $m = \left\lfloor \frac{\log n}{2\log\log n} \right\rfloor$ and prove that G contains a plane path of length $\Omega(m)$. We can assume $m > 1$, otherwise there is nothing to prove. If some sequence $\theta(v_i)$ contains an increasing subsequence of length m^2, then by Lemma 4.2 and Lemma 4.1, we are done. Therefore, we assume that $\theta(v_i)$ does not contain an increasing subsequence of length m^2 for every i.

For integer $t \ge 1$, we inductively construct subsets $U_t, S_t \subset \{v_1, \dots, v_{n-1}\}$ with $U_t = \{u_1, \dots, u_t\}$, where $u_1 \prec \dots \prec u_t$, and $U_t \prec S_t$. Initially, we set $U_1 = \{u_1 := v_1\}$ and $S_1 = \{v_2, \dots, v_{n-1}\}$. Suppose that for some t, we have already constructed U_t and S_t. If $|S_t| \le m^2$, we stop this construction process, otherwise we continue to construct U_{t+1} and S_{t+1} as follows: Let θ' be the subsequence of $\theta(u_t)$ that contains exactly those vertices in S_t. Note that the length of θ' equals to $|S_t|$. According to our assumption, the length of the longest increasing subsequence in θ' is less than m^2. Hence, by Dilworth's theorem [5], θ' contains a decreasing subsequence of length at least $|S_t|/m^2$. Let S'_{t+1} be the set of vertices that appear in this decreasing subsequence of θ'. Next, we take u_{t+1} to be the smallest element of S'_{t+1} with respect to $\prec$ and let $U_{t+1} := U_t \cup \{u_{t+1}\}$. Consider the region $\Delta_{v_0u_tu_{t+1}}$ bounded by the topological edges v_0u_t, u_tu_{t+1}, and $u_{t+1}v_0$. Each vertex in $S'_{t+1} \setminus \{u_{t+1}\}$ is either inside or outside $\Delta_{v_0u_tu_{t+1}}$. So, by the pigeonhole principle, there exists a subset $S_{t+1} \subset S'_{t+1} \setminus \{u_{t+1}\}$ with $|S_{t+1}| \ge |S'_{t+1} \setminus \{u_{t+1}\}|/2$ such that the whole set S_{t+1} is either inside or outside

$\Delta_{v_0 u_t u_{t+1}}$. Clearly, we have $U_{t+1} \prec S_{t+1}$ and

$$|S_{t+1}| \geq \frac{|S'_{t+1}| - 1}{2} \geq \frac{|S_t|/m^2 - 1}{2} \geq \frac{|S_t|}{(2m)^2}.$$

Using the inequality above and the fact that $|S_1| = n - 2$, we can inductively prove $|S_t| \geq \frac{n}{(2m)^{2t}}$. When $t = m - 1$, this gives us

$$|S_{m-1}| \geq \frac{n}{(2m)^{2(m-1)}} > \frac{n}{(2m)^{\log n/\log\log n - 2}} > m^2 \cdot \frac{n}{(\log n)^{\log n/\log\log n}} = m^2.$$

Hence, the construction process ends at a certain $t > m - 1$, and we will always construct $U_m = \{u_1, \ldots, u_m\}$.

Now we show that $u_1, u_2, \ldots, u_m$ form a plane path. Our argument is based on the following two claims.

Claim 4.3 *For any vertices $u_i \prec u_{i+1} \prec u_j \prec u_k$, we have that u_j and u_k are either both inside or both outside the region $\Delta_{v_0 u_i u_{i+1}}$.*

It can be checked that Claim 4.3 is guaranteed by the construction process of U_m.

Claim 4.4 *For any vertices $u_i \prec u_j \prec u_k$, the topological edges $v_0 u_i$ and $u_j u_k$ do not cross each other.*

Proof Consider the region $\Delta_{v_0 u_i u_j}$ bounded by the topological edges $v_0 u_i$, $u_i u_j$, and $u_j v_0$, then u_k is either inside or outside $\Delta_{v_0 u_i u_j}$. If u_k is inside $\Delta_{v_0 u_i u_j}$, then $v_0 u_k$ must cross $u_i u_j$. By Observation 3.2, we have $\chi(u_i, u_j, u_k) = 001$, which implies $v_0 u_i$ and $u_j u_k$ do not cross. See the third configuration in Fig. 2.

Suppose u_k is outside $\Delta_{v_0 u_i u_j}$. By the construction process of U_m, u_j and u_k belong to a decreasing subsequence of $\theta(u_i)$, hence the edges $u_i v_0$, $u_i u_k$ and $u_i u_j$ must emanate from u_i in counterclockwise order, which implies that $u_i u_k$ crosses $v_0 u_j$. Then, by Observation 3.2, $\chi(u_i, u_j, u_k) = 010$ and $u_j u_k$ does not cross $v_0 u_i$. See the second configuration in Fig. 2. □

Finally, we argue that the edges $u_i u_{i+1}$ and $u_j u_{j+1}$ do not cross for any $i < j$. When $j = i + 1$, this follows from G being simple. When $j > i + 1$, by Claim 4.3, the vertices u_j and u_{j+1} are either both inside or both outside the region $\Delta_{v_0 u_i u_{i+1}}$. So, the edge $u_j u_{j+1}$ crosses the boundary of $\Delta_{v_0 u_i u_{i+1}}$ an even number of times. On the other hand, by Claim 4.4, $u_j u_{j+1}$ does not cross $v_0 u_i$ or $v_0 u_{i+1}$. So $u_j u_{j+1}$ does not cross $u_i u_{i+1}$. See Fig. 7 for an illustration. This concludes the proof of Theorem 1.4. □

5 Concluding Remarks

Answering a question of Pach and Tóth [16], Suk showed that every complete n-vertex simple topological graph contains $\Omega(n^{1/3})$ pairwise disjoint edges [20] (see also [8]). This bound was later improved to $n^{1/2-o(1)}$ by Ruiz-Vargas in [17]. Hence, we conjecture a similar bound for plane paths.

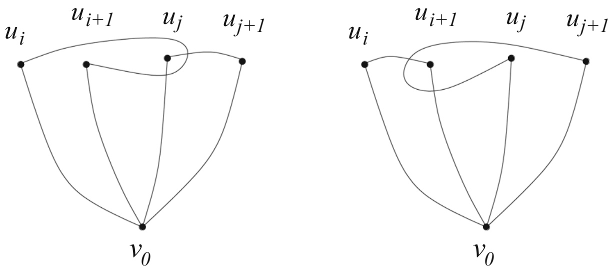

Fig. 7 For u_iu_{i+1} and u_ju_{j+1} with $i+1<j$ to cross each other, either u_j and u_{j+1} are not both inside or both outside $\Delta_{v_0u_iu_{i+1}}$ (left graph), or the topological edge u_ju_{j+1} crosses one edge in $\{v_0u_i, v_0u_{i+1}\}$ (right graph)

Conjecture 5.1 *There is an absolute constant* $\varepsilon > 0$*, such that every complete* n*-vertex simple topological graph contains a plane path of length* n^ε*.*

Let $h = h(n)$ be the smallest integer such that every complete n-vertex simple topological graph contains an edge crossing at most h other edges. A construction due to Valtr (see page 398 in [4]) shows that $h(n) \geq \Omega(n^{3/2})$. In the other direction, Kynčl and Valtr [12] used an asymmetric version of Theorem 1.1 to show that $h(n) = O(n^2/\log^{1/4} n)$. By using Theorem 3.1 instead, their arguments show that $h(n) \leq n^2/(\log n)^{1/2-o(1)}$. We conjecture the following.

Conjecture 5.2 *There is an absolute constant* $\varepsilon > 0$ *such that* $h(n) \leq n^{2-\varepsilon}$*.*

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