

Chains, Koch Chains, and Point Sets with Many Triangulations

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We introduce the abstract notion of a chain, which is a sequence of n points in the plane, ordered by x -coordinates, so that the edge between any two consecutive points is unavoidable as far as triangulations are concerned. A general theory of the structural properties of chains is developed, alongside a general understanding of their number of triangulations.

We also describe an intriguing new and concrete configuration, which we call the Koch chain due to its similarities to the Koch curve. A specific construction based on Koch chains is then shown to have $\Omega(9.08^n)$ triangulations. This is a significant improvement over the previous and long-standing lower bound of $\Omega(8.65^n)$ for the maximum number of triangulations of planar point sets.

CCS Concepts: • **Theory of computation** → **Computational geometry**;

Additional Key Words and Phrases: Planar point set, chain, Koch chain, triangulation, maximum number of triangulations, lower bound

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1 INTRODUCTION

Let P be a set of n points in the Euclidean plane. Throughout the article, P is assumed to be in *general position*, which means for us that no two points have the same x -coordinate and that no three points are on a common line. A *geometric graph* on P is a graph with vertex set P combined with an embedding into the plane where edges are realized as straight-line segments between the corresponding endpoints. It is called *crossing-free* if the edges have no pairwise intersection, except possibly in a common endpoint.

Triangulations. Perhaps the most prominent and most studied family of crossing-free geometric graphs is the family of *triangulations*, which may be defined simply as edge-maximal crossing-free geometric graphs on P . It is easy to see that such a definition implies that the edges of any triangulation subdivide the convex hull of P into triangular regions.

Let $\text{tr}(P)$ denote the number of triangulations on a given point set P . Trying to better understand this quantity is a fundamental question in combinatorial and computational geometry. For very

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specific families of point sets, exact formulas or at least asymptotic estimates can be derived. For example, it is well known that if P is a set of n points in *convex position*, then $\text{tr}(P) = C_{n-2}$, where $C_k = \frac{1}{k+1} \binom{2k}{k} = \Theta(k^{-3/2} 4^k)$ is the k th Catalan number [13]. In general, however, this problem turns out to be much more elusive.

There is an elegant algorithm by Alvarez and Seidel [4] from 2013 that computes $\text{tr}(P)$ in exponential time $O(2^n n^2)$. It has been surpassed by Marx and Miltzow [16] in 2016, who showed how to compute $\text{tr}(P)$ in subexponential time $n^{O(\sqrt{n})}$. Moreover, Avis and Fukuda [6] showed already in 1996 how to enumerate the set of all triangulations on P (i.e., to compute an explicit representation of each element) by using a general technique called reverse search in time $\text{tr}(P) \cdot p(n)$ for some polynomial p . A particularly efficient implementation of that technique with $p(n) = O(\log \log n)$ has been described by Bespamyatnikh [7].

Extensive research has also gone into extremal upper and lower bounds in terms of the number of points. That is, if we define

$$\text{tr}_{\max}(n) = \max_{P:|P|=n} \text{tr}(P), \quad \text{tr}_{\min}(n) = \min_{P:|P|=n} \text{tr}(P)$$

to be the respectively largest and smallest numbers of triangulations attainable by a set P of n points in general position, then various authors have attempted to establish and improve upper and lower bounds on these quantities.

As far as the maximum is concerned, a seminal result by Ajtai et al. [3] from 1982 shows that the number of triangulations—and, more generally, the number of all crossing-free geometric graphs—is at most 10^{13n} . A long series of successive improvements [8, 19, 20, 22, 23] using a variety of different techniques has culminated in the currently best upper bound $\text{tr}_{\max}(n) \leq 30^n$ due to Sharir and Sheffer [21], which has remained uncontested for over a decade. Coming from the other side, attempts have been made to construct point sets with a particularly large number of triangulations. For some time, the *double chain* by García Olaverri et al. [17] with approximately $\Theta(8^n)$ triangulations was conjectured to have the largest possible number of triangulations. However, variants like the *double zig-zag chain* by Aichholzer et al. [2] with $\Theta(8.48^n)$ triangulations and a specific instance of the *generalized double zig-zag chain* by Dumitrescu et al. [9] with $\Omega(8.65^n)$ triangulations have since been discovered. But also on this front, no further progress on the lower bound $\text{tr}_{\max}(n) = \Omega(8.65^n)$ has been made for a decade.

The situation for the minimum is different insofar that the *double circle* with $\Theta(3.47^n)$ triangulations, as analyzed by Hurtado and Noy [12] in 1997, is still conjectured by many to have the smallest number of triangulations. In other words, it is believed that the resulting upper bound $\text{tr}_{\min}(n) = O(3.47^n)$ is the best possible. On the other hand, Aichholzer et al. [1] have shown that every point set has at least $\Omega(2.63^n)$ triangulations, thereby establishing the lower bound $\text{tr}_{\min}(n) = \Omega(2.63^n)$.

The focus of this article lies on $\text{tr}_{\max}(n)$ and, more specifically, on establishing an improved lower bound on that quantity. Ultimately, we show how to construct a new infinite family of point sets with $\Omega(9.08^n)$ triangulations, thereby proving $\text{tr}_{\max}(n) = \Omega(9.08^n)$.

General chains. It has occurred to us that almost all families of point sets whose numbers of triangulations have been analyzed over the years have a very special structure, which we are trying to capture in the following definition.

Definition 1. A *chain* C is a sequence of points p_0, \dots, p_n sorted by increasing x -coordinates, such that the edge $p_{i-1}p_i$ is *unavoidable* (i.e., contained in every triangulation of C) for each $i = 1, \dots, n$. These specific unavoidable edges are also referred to as *chain edges*.

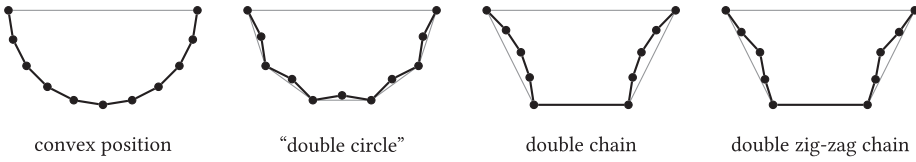


Fig. 1. Some classic point sets realized as chains. For the double circle, we need to remove one of the inner points. Chain edges are displayed in black and bold, and other unavoidable hull edges in gray.

In contrast to previous convention, we use the parameter n to denote the number of chain edges and not the number of points in C , which is $n + 1$. Also note that Definition 1 implies that the edge p_0p_n is an edge of the convex hull and, hence, is also unavoidable. Indeed, since all chain edges are unavoidable, the edge p_0p_n cannot possibly cross any of them and, hence, is either above or below all the points in between. Therefore, a chain always admits a spanning cycle of unavoidable edges with at least one hull edge. We prove in Section 2 that this is also a characterization of chains in terms of *order types* (see [11] for a definition).

THEOREM 2. *For every point set that admits a spanning cycle of unavoidable edges including at least one convex hull edge, there exists a chain with the same order type.*

All of the mentioned families of point sets (convex position, double chain, and so on) are usually neither defined nor depicted in a way that makes it clear that they may be thought of as chains as in Definition 1. Still, the premise of Theorem 2 is easily verified for all of them except for the double circle, which may, however, be transformed into a chain by removing one of the inner points. Figure 1 shows realizations of some such point sets as chains.

Imagine walking along the chain edges and recording at each point the information whether we make a left turn or a right turn. It can be noted already now that such information—while crucial—is not enough to really capture all of the relevant combinatorial structure of a given chain. Instead, the right way of looking at it turns out to be recording for each edge $p_i p_j$ whether it lies above or below all the chain edges in between.

The simple linear structure inherent to chains allows us to develop a combinatorial theory in Section 2, by which every chain admits a unique construction starting from the primitive chain with only one edge. Two types of sum operations, so-called convex and concave sums, are used to “concatenate” chains, while an inversion allows to “flip” a chain on its head. This yields for every chain a concise and unique description as an algebraic formula. Based on this, we will also see that the number of combinatorially different chains with n chain edges is equal to S_{n-1} , where $S_k = \sum_{i=0}^k \frac{1}{i+1} \binom{k}{i} \binom{k+i}{i} = \Theta(k^{-3/2}(3 + \sqrt{8})^k)$ is the k th large Schröder number [14].

Triangulations of chains. The unavoidable chain edges separate every triangulation cleanly into an *upper triangulation* of the region above the chain edges and into a *lower triangulation* of the region below. Therefore, both upper and lower triangulations may be analyzed separately. It also follows that there is no further complication due to inner vertices as one would typically encounter them in general point sets.

There is a simple cubic time dynamic programming algorithm for counting triangulations of simple polygons [10]. Such an algorithm can of course also be used to count both the upper and lower triangulations of a given chain. However, we show in Section 3 that the additional structure of chains allows us to devise an improved quadratic time algorithm, which plays a crucial role in the derivation of our main result.

THEOREM 3. *Given a chain C with n chain edges as input, it is possible to compute the number $\text{tr}(C)$ by using only $O(n^2)$ integer additions and multiplications.*

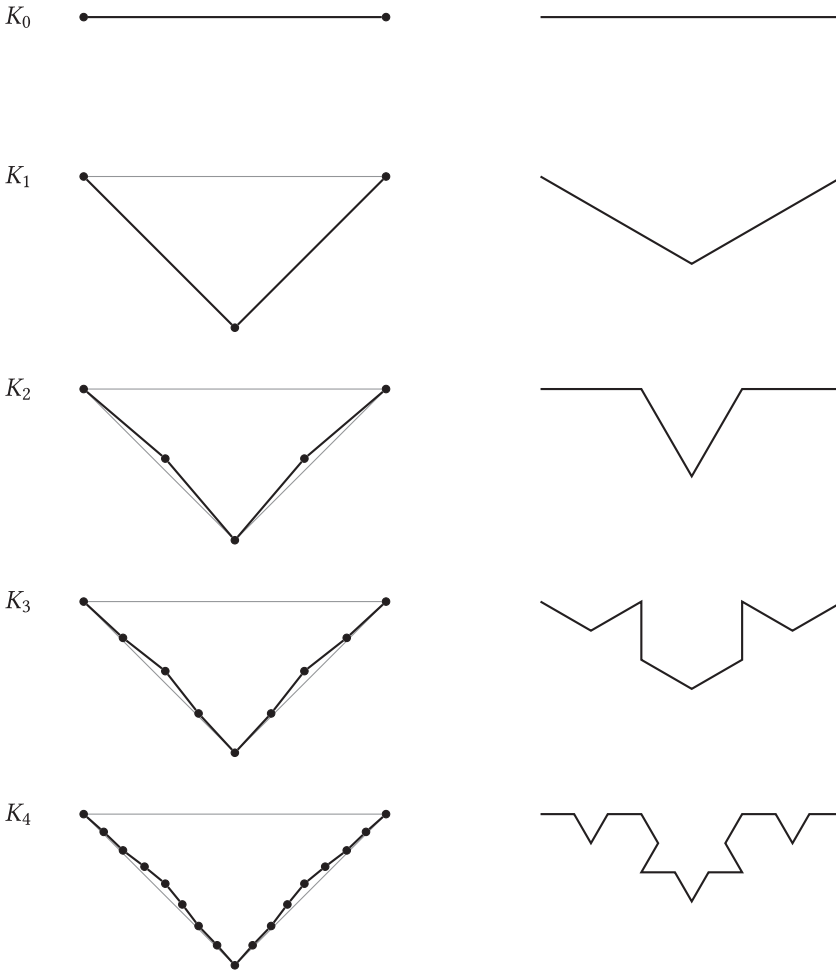


Fig. 2. The Koch chains K_s for $s = 0, \dots, 4$ and the corresponding Koch curves. Even though it is hard to recognize for larger values of s , the changes in direction along the Koch curve on the right are reflected one to one by the chain edges of the corresponding Koch chain on the left.

The Koch chain. There is a particular type of chain that has caught our interest and which, to the best of our knowledge, has not been described in the literature before. We call it the *Koch chain* due to its striking similarity in appearance and definition to the famous Koch curve. More precise definitions follow later in Definition 14; for now, suppose K_0 is a primitive chain with just one chain edge, and let the s th iteration K_s of the Koch chain be defined by concatenating two flipped and sufficiently flattened copies of K_{s-1} in such a way that the chain edges at the point of concatenation form a left turn; see Figure 2.

Koch chains turn out to have a particularly large number of triangulations, much more so than any other known point sets. For values of s up to 21, we have computed the corresponding numbers of upper and lower triangulations, as well as complete triangulations, by using our algorithm from Theorem 3. The results are displayed in Table 1.

In consequence, concatenating copies of K_{21} side by side results in an infinite family of point sets with at least 9.082799^n triangulations. This alone already establishes the improved lower bound of $\text{tr}_{\max}(n) = \Omega(9.082799^n)$.

Table 1. The Computed Numbers of Triangulations of the Koch Chain K_s for $s = 0, \dots, 21$, Where Each Entry Is Rounded Down to Six Decimal Places

s	n	$\sqrt[n]{U}$	$\sqrt[n]{L}$	$\sqrt[n]{T}$	s	n	$\sqrt[n]{U}$	$\sqrt[n]{L}$	$\sqrt[n]{T}$
0	1	1.0	1.0	1.0	11	2048	3.121029	2.858643	8.921910
1	2	1.0	1.0	1.0	12	4096	2.882177	3.121029	8.995359
2	4	1.189207	1.0	1.189207	13	8192	3.134955	2.882177	9.035496
3	8	1.791279	1.189207	2.130201	14	16384	2.889213	3.134955	9.057554
4	16	2.035453	1.791279	3.646065	15	32768	3.139056	2.889213	9.069406
5	32	2.558954	2.035453	5.208633	16	65536	2.891256	3.139056	9.075820
6	64	2.564646	2.558954	6.562814	17	131072	3.140236	2.891256	9.079229
7	128	2.935733	2.564646	7.529118	18	262144	2.891838	3.140236	9.081055
8	256	2.783587	2.935733	8.171870	19	524288	3.140569	2.891838	9.082019
9	512	3.075469	2.783587	8.560839	20	1048576	2.892001	3.140569	9.082530
10	1024	2.858643	3.075469	8.791671	21	2097152	3.140662	2.892001	9.082799

As usual, n is the number of chain edges, whereas U , L , and T stand, respectively, for the numbers of upper, lower, and complete triangulations of the corresponding Koch chain.

Poly chains and twin chains. We were unable to nail down the exact asymptotic behavior of the number of triangulations of K_s as s approaches infinity. It is also unclear how much is lost due to undercounting by not considering any interactions between the different copies of K_{21} in our simple lower bound construction from just before.

To remedy the situation somewhat, in Section 4 we define and analyze more carefully the *poly- C_0 chain* (a specific way of concatenating N copies of a fixed chain C_0) and the *twin- C_0 chain* (a construction where two copies of a poly- C_0 chain face each other, similar in spirit to the classic double chain). Based on these considerations, we get a slightly improved lower bound construction, and we are also able to conclude that the numbers in the last column of Table 1 will not grow significantly larger than what we already have.

THEOREM 4. *Let C_N be the twin- K_{21} chain that is made up of $2N$ copies of K_{21} for a total of $n = 2N \cdot 2^{21} + 1$ chain edges. Then,*

$$\lim_{N \rightarrow \infty} \sqrt[n]{\text{tr}(C_N)} = 9.083095 \dots, \quad \text{and hence } \text{tr}_{\max}(n) = \Omega(9.083095^n).$$

THEOREM 5. *For the Koch chain K_s with $n = 2^s$ chain edges, we have*

$$9.082799 \leq \lim_{s \rightarrow \infty} \sqrt[n]{\text{tr}(K_s)} \leq 9.083139.$$

2 STRUCTURAL PROPERTIES OF CHAINS

Recall Definition 1 from the introduction. Note that the unavoidable chain edges form an x -monotone curve $p_0 p_1 \dots p_n$, which we refer to as the *chain curve*. An edge $p_i p_j$ that is not a chain edge cannot cross the chain curve, and so it lies either above or below that curve.

Definition 6. To every chain C we associate a *visibility triangle* $V(C)$ with entries

$$V(C)_{i,j} = \begin{cases} +1, & \text{if } p_i p_j \text{ lies above the chain curve;} \\ -1, & \text{if } p_i p_j \text{ lies below the chain curve;} \\ 0, & \text{if } p_i p_j \text{ is a chain edge (i.e., } i + 1 = j); \end{cases} \quad (0 \leq i < j \leq n).$$

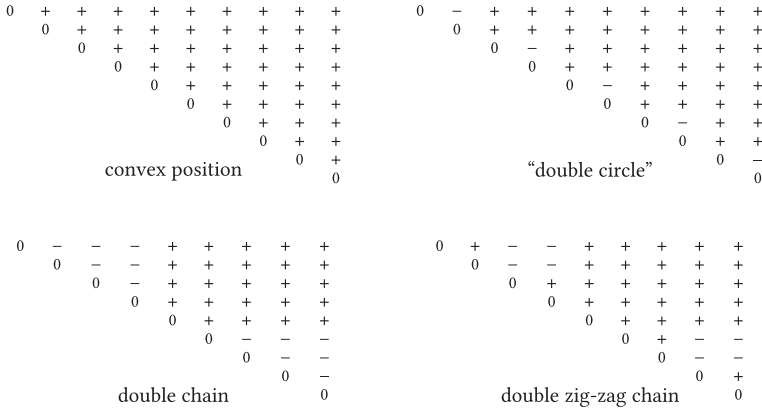


Fig. 3. The visibility triangles corresponding to the chains depicted earlier in Figure 1. For improved clarity, we only display the signs of the respective entries.

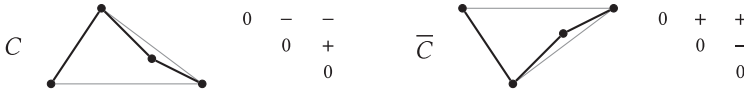


Fig. 4. A chain C and its flipped version \bar{C} with the corresponding visibility triangles.

As an example, the visibility triangles of the chains that correspond to the classic point sets from the introduction can be seen in Figure 3. In all such illustrations, the indices of $V(C)_{i,j}$ are understood in familiar matrix notation; that is, i indexes the row and j indexes the column.

For $i < j < k$, the triangle $p_i p_j p_k$ is oriented counter-clockwise if and only if p_j lies below the edge $p_i p_k$ or, equivalently, if and only if $V(C)_{i,k} = +1$. It follows that two chains with the same visibility triangle have the same order type and, therefore, the same set of crossing-free geometric graphs and triangulations. This motivates using the visibility triangle as the canonical representation of a chain. That is, we consider two chains to be equal if their visibility triangles are identical; in particular, given two chains C_1 and C_2 , we write $C_1 = C_2$ if $V(C_1) = V(C_2)$.

The edge $p_0 p_n$ plays a crucial role in determining the shape of a chain. For example, if $V(C)_{0,n} = +1$, then this edge is the only edge on the upper convex hull, and, from a global perspective, the chain curve looks like it is curving upward. Conversely, if $V(C)_{0,n} = -1$, then the chain curve looks like it is curving downward. Correspondingly, we call a chain C with $V(C)_{0,n} \geq 0$ an *upward chain*, and a chain with $V(C)_{0,n} \leq 0$ a *downward chain*. This implies that every chain is either an upward or a downward chain, and the primitive chain with only $n = 1$ chain edge is the only chain that is both.

2.1 Flips

Chains may be flipped upside down by reflection at the x -axis, thus turning an upward chain into a downward chain, and vice versa. See Figure 4 for an example.

PROPOSITION 7. *Let C be a chain with n chain edges. Then, there is another chain (which we denote by \bar{C} and call the flipped version of C) with n chain edges and visibility triangle*

$$V(\bar{C})_{i,j} = -V(C)_{i,j} \quad (0 \leq i < j \leq n).$$

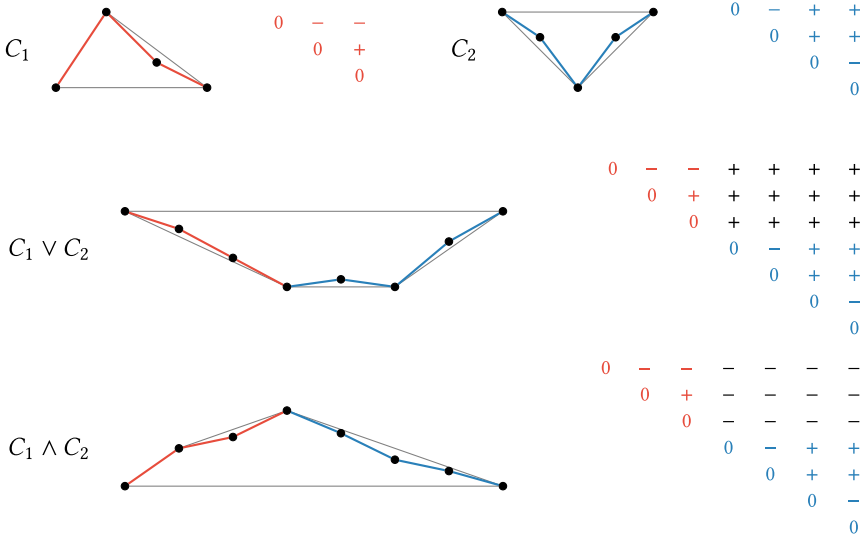


Fig. 5. In the top row, two chains C_1 and C_2 with their visibility triangles. Below, the corresponding convex and concave sums $C_1 \vee C_2$ and $C_1 \wedge C_2$. Red and blue colors are used to highlight the contained substructures and their origin.

2.2 Convex and Concave Sums

Given two chains C_1 and C_2 , we would like to concatenate them so that we get a new chain containing C_1 and C_2 as substructures. As shown in Figure 5, there are two ways to do so.

PROPOSITION 8. *Let C_1 and C_2 be chains with n_1 and n_2 chain edges, respectively. Then, there is an upward chain (which we denote by $C_1 \vee C_2$ and call the convex sum of C_1 and C_2) with $n_1 + n_2$ chain edges and visibility triangle*

$$V(C_1 \vee C_2)_{i,j} = \begin{cases} V(C_1)_{i,j}, & \text{if } i, j \in [0, n_1]; \\ V(C_2)_{i-n_1, j-n_1}, & \text{if } i, j \in [n_1, n_1 + n_2]; \\ +1, & \text{if } i < n_1 < j; \end{cases} \quad (0 \leq i < j \leq n_1 + n_2).$$

PROPOSITION 9. *Similarly, there is a downward chain (which we denote by $C_1 \wedge C_2$ and call the concave sum of C_1 and C_2) with $n_1 + n_2$ chain edges and visibility triangle*

$$V(C_1 \wedge C_2)_{i,j} = \begin{cases} V(C_1)_{i,j}, & \text{if } i, j \in [0, n_1]; \\ V(C_2)_{i-n_1, j-n_1}, & \text{if } i, j \in [n_1, n_1 + n_2]; \\ -1, & \text{if } i < n_1 < j; \end{cases} \quad (0 \leq i < j \leq n_1 + n_2).$$

PROOF OF PROPOSITION 8. We focus on the convex sum; the proof for the concave sum is analogous. We have to show that there is a point set that forms a chain with the specified visibility triangle. Intuitively speaking, this is achieved by first flattening the two given chains and then arranging them in a \vee -shape.

To be more precise, we employ vertical shearings, which are maps $(x, y) \mapsto (x, y + \lambda x)$ in \mathbb{R}^2 for some $\lambda \in \mathbb{R}$. Vertical shearings preserve signed areas and x -coordinates. Hence, if a point set realizes a specific chain, then so does its image under any vertical shearing.

With the help of an appropriate vertical shearing, we may realize C_1 as a point set in the rectangle $[-1, 0] \times [-1, 1]$ in such a way that the first point is at $(-1, 0)$ and the last point is at $(0, 0)$. Then, given any $\varepsilon \geq 0$, we may rescale vertically to get a point set $Q_1(\varepsilon)$ in the rectangle $[-1, 0] \times [-\varepsilon, \varepsilon]$.

Let now $R_1(\varepsilon)$ be the image of $Q_1(\varepsilon)$ under the vertical shearing with $\lambda = -1$. Then, the first point of $R_1(\varepsilon)$ lies at $(-1, 1)$, while the last point remains at $(0, 0)$. For $\varepsilon > 0$, since $Q_1(\varepsilon)$ is a realization of C_1 , so is $R_1(\varepsilon)$. On the other hand, for $\varepsilon = 0$, the points of $R_1(\varepsilon)$ all lie on the segment between $(-1, 1)$ and $(0, 0)$.

With C_2 we proceed similarly to get a point set $Q_2(\varepsilon)$ in the rectangle $[0, 1] \times [-\varepsilon, \varepsilon]$, but we now apply the vertical shearing with $\lambda = 1$ to get $R_2(\varepsilon)$ with the first point at $(0, 0)$ and the last point at $(1, 1)$.

Let $T(\varepsilon) = R_1(\varepsilon) \cup R_2(\varepsilon)$. We claim that for $\varepsilon > 0$ small enough, $T(\varepsilon)$ is a chain with visibility triangle $V(C_1 \vee C_2)$ as specified. Indeed, as $R_i(\varepsilon)$ is a realization of C_i , we only need to check that the edges between any point of $R_1(\varepsilon)$ and any point of $R_2(\varepsilon)$ (excluding the common point at the origin) lie above all the points in between. Since this is the case for $\varepsilon = 0$ and $T(\varepsilon)$ depends continuously on ε , the claim follows. \square

2.3 Algebraic Properties

Using the formulas for the visibility triangles from the corresponding transformations in Propositions 7, 8, and 9, it can be checked easily that the following algebraic laws hold.

LEMMA 10. *Let C_1, C_2, C_3 be arbitrary chains. Then, the following are all true:*

$$\begin{aligned} \text{Involution:} & \quad \overline{\overline{C_1}} = C_1 \\ \text{De Morgan:} & \quad \overline{C_1 \vee C_2} = \overline{C_1} \wedge \overline{C_2}, & \quad \overline{C_1 \wedge C_2} = \overline{C_1} \vee \overline{C_2} \\ \text{Associativity:} & \quad (C_1 \vee C_2) \vee C_3 = C_1 \vee (C_2 \vee C_3), & \quad (C_1 \wedge C_2) \wedge C_3 = C_1 \wedge (C_2 \wedge C_3) \end{aligned}$$

However, note that, for example, $(C_1 \wedge C_2) \vee C_3$ is not the same chain as $C_1 \wedge (C_2 \vee C_3)$.

2.4 Examples

We denote by E the primitive chain with only $n = 1$ chain edge; that is, the visibility triangle has just the entry $V(E)_{0,1} = 0$. Using this as a building block, we may define two more fundamental chains, the *convex chain* $C_{\text{cvx}}(n)$ and the *concave chain* $C_{\text{ccv}}(n)$, by setting

$$C_{\text{cvx}}(n) = \underbrace{E \vee \cdots \vee E}_{n \text{ copies}}, \quad C_{\text{ccv}}(n) = \underbrace{E \wedge \cdots \wedge E}_{n \text{ copies}}.$$

The convex chain is an upward chain, while the concave chain is a downward chain. Also, since $\overline{\overline{E}} = E$, we get $\overline{C_{\text{cvx}}(n)} = C_{\text{ccv}}(n)$ by using De Morgan's law. Finally, note that $C_{\text{cvx}}(n)$ and $C_{\text{ccv}}(n)$ are distinct as chains, even though they both describe a set of $n + 1$ points in convex position.

As already mentioned in the introduction, many previously studied point sets are in fact chains, or can be seen as such. Using flips as well as convex and concave sums, we can now describe these configurations with very concise formulas.

Example 11. For any integer $k \geq 1$, the *double chain* with $n = 2k + 1$ chain edges is the chain

$$C_{\text{dbl}}(n) = C_{\text{ccv}}(k) \vee E \vee C_{\text{cvx}}(k).$$

Example 12. For any integer $k \geq 1$, the *zig-zag chain* with $n = 2k$ chain edges (which, in essence, is a double circle with one of the inner points removed) is the chain

$$C_{\text{zz}}(n) = \underbrace{C_{\text{ccv}}(2) \vee \cdots \vee C_{\text{ccv}}(2)}_{k \text{ copies}}.$$

Example 13. For any integer $k \geq 1$, the *double zig-zag chain* with $n = 4k + 1$ chain edges is the chain

$$C_{\text{dzz}}(n) = \overline{C_{\text{zz}}(2k)} \vee E \vee \overline{C_{\text{zz}}(2k)}.$$

All these examples involve formulas of constant nesting depth only. But the tools developed up to this point allow us to also define more complicated chains via formulas of non-constant nesting depth, without having to worry about questions of realizability. One such chain with logarithmic nesting depth is indeed the Koch chain.

Definition 14. The *Koch chain* K_s is an upward chain with $n = 2^s$ chain edges, defined recursively via $K_0 = E$ and $K_s = \overline{K_{s-1}} \vee \overline{K_{s-1}}$ for all integers $s \geq 1$.

Indeed, after expanding the recursive definition twice and using De Morgan's law on both sides, we see that the formula $K_s = (K_{s-2} \wedge K_{s-2}) \vee (K_{s-2} \wedge K_{s-2})$ has a complete binary parse tree with alternating convex and concave sums on any path from the root to a leaf.

2.5 Unique Construction

We want to prove the following result. In essence, it states that every chain can be constructed in a unique way by using only convex and concave sums.

THEOREM 15. *Every chain can be expressed as a formula involving convex sums, concave sums, parentheses, and copies of the primitive chain with only one chain edge. This formula is unique up to redundant parentheses (redundant due to associativity as in Lemma 10).*

In particular, the above theorem allows us to encode a chain with $O(n)$ bits (as opposed to the $O(n^2)$ bits required for the visibility triangle) and to easily enumerate all chains of a fixed size. We further see by means of a straightforward bijection to Schröder trees (these are ordered trees that have no vertex of outdegree one; as such, they describe precisely the parse trees of the unique formulas of upward chains) that the number of upward chains is given by the little Schröder numbers [15] and the number of all chains is given by the large Schröder numbers [14].

The theorem follows by induction from the following proposition (and from an analogous proposition that expresses downward chains as a unique concave sum of upward chains).

PROPOSITION 16. *Let C be an upward chain with $n > 1$ chain edges. Suppose that the lower convex hull of C is $p_{i_0}p_{i_1} \dots p_{i_k}$ with $0 = i_0 < \dots < i_k = n$. For $j = 1, \dots, k$, let C_j be the chain with points $p_{i_{j-1}}, \dots, p_{i_j}$. Then, each C_j is a downward chain. Moreover, $C = C_1 \vee \dots \vee C_k$ and any formula that evaluates to C has the same top-level structure.*

PROOF. As $p_{i_{j-1}}p_{i_j}$ is an edge of the lower convex hull of C , it is below all the points in between. Hence, each C_j is indeed a downward chain.

To prove $C = C_1 \vee \dots \vee C_k$, we have to show that both chains have the same visibility triangle. By definition of the C_j , the visibility triangles clearly agree on all entries that stem from an edge $p_a p_b$, where p_a and p_b are both part of the same C_j . On the other hand, if p_a and p_b are not part of the same C_j , then there is a j with $a < i_j < b$. As p_{i_j} is a vertex of the lower convex hull, it lies below the edge $p_a p_b$ and hence $V(C)_{a,b} = +1$. But this is precisely what we also get for the visibility triangle of the convex sum $C_1 \vee \dots \vee C_k$.

For uniqueness, suppose we are given any formula for C . Since C is assumed to be an upward chain and since any concave sum is a downward chain, the formula must be of the form $C'_1 \vee \dots \vee C'_k$. We may further assume that each C'_j is a downward chain by omitting redundant parentheses. Observe now that in any such convex sum of downward chains, the resulting lower convex hull

is determined by the points that are shared by any two consecutive chains C'_j . Since the given formula evaluates to C , we must have $k' = k$ and $C'_j = C_j$ for all j . \square

2.6 Geometric Characterization

As already mentioned in the introduction, the chain edges together with the hull edge p_0p_n form a spanning cycle of unavoidable edges. We are now ready to prove that this property characterizes chains geometrically in terms of order types.

Suppose we are given a set of $n + 1$ points p_0, p_1, \dots, p_n as in Theorem 2. Without loss of generality, let $SC = p_0p_1 \dots p_n$ be the spanning cycle of unavoidable edges in counter-clockwise order, with p_0p_n an edge of the convex hull, which we call the *base edge*. As SC consists of unavoidable edges only, it cannot be crossed by any edge that is not part of SC . Hence, we can associate a visibility triangle with the given point set, similar to the visibility triangle of a chain, by setting

$$V_{i,j} = \begin{cases} +1, & \text{if } p_i p_j \text{ is inside } SC \text{ or the base edge;} \\ -1, & \text{if } p_i p_j \text{ is outside } SC; \\ 0, & \text{if } p_i p_j \text{ is part of } SC \text{ (i.e., } i + 1 = j); \end{cases} \quad (0 \leq i < j \leq n).$$

PROPOSITION 17. *For $i < j < k$, the triangle $p_i p_j p_k$ is oriented counter-clockwise if and only if we have $V_{i,k} = +1$.*

PROOF. We will make use of the following key fact about simple polygons. Consider two simple polygons that share an edge e but do not touch or intersect in any other way. Then, exactly one of the following is true. Either one polygon is contained in the other, or the two polygons touch the edge e from different sides.

We split the proof of the proposition into the following three cases:

- Suppose that $p_i p_k$ is the base edge (i.e., $i = 0$ and $k = n$). Then, the triangle $p_i p_j p_k$ is always oriented counter-clockwise since $p_i p_k = p_0 p_n$ is an edge of the convex hull. Also, we have $V_{i,k} = V_{0,n} = +1$ by definition.
- Suppose that $p_i p_k$ is not the base edge (i.e., $i \neq 0$ or $k \neq n$) and inside SC (i.e., $V_{i,k} = +1$). Let SC' denote the simple polygon $p_0, p_1, \dots, p_i, p_k, p_{k+1}, \dots, p_n$. Let us now apply our key fact about simple polygons to SC' and the triangle $p_i p_j p_k$. These two polygons share the edge $p_i p_k$ but do not touch or intersect in any other way since all other edges of SC' are unavoidable. Moreover, since p_j is outside of SC' , the triangle is not contained in SC' . On the other hand, SC' contains the hull vertices p_0 and p_n , and so SC' is also not contained in the triangle. Therefore, since SC' touches the edge $p_i p_k$ from the left side, p_j has to be to the right of $p_i p_k$, and so the triangle $p_i p_j p_k$ is indeed oriented counter-clockwise.
- Suppose that $p_i p_k$ is not the base edge (i.e., $i \neq 0$ or $k \neq n$) and outside SC (i.e., $V_{i,k} = -1$). Let again SC' denote the simple polygon $p_0, p_1, \dots, p_i, p_k, p_{k+1}, \dots, p_n$, and let us again apply our key fact to SC' and $p_i p_j p_k$. In this case, since p_j lies inside of SC' , the triangle is contained in SC' . Therefore, since SC' touches the edge $p_i p_k$ from the left side, p_j has to be to the left of $p_i p_k$, and so the triangle $p_i p_j p_k$ is indeed oriented clockwise. \square

Having Proposition 17 at hand, it will now suffice to construct a chain whose visibility triangle agrees with V in order to finish the proof of Theorem 2.

Let thus $p_{i_0}, p_{i_1}, \dots, p_{i_k}$ be the vertices of the convex hull with $0 = i_0 < \dots < i_k = n$. For $1 \leq j \leq k$, let $P_j = \{p_{i_{j-1}}, \dots, p_{i_j}\}$. We now see that either P_j consists of only two points or that it admits a spanning cycle of unavoidable edges, namely $SC_j = p_{i_{j-1}} p_{i_{j-1}+1} \dots p_{i_j}$ with base edge $p_{i_{j-1}} p_{i_j}$. The situation is depicted in Figure 6. Note that the inside of SC_j is outside of SC . In fact, SC_j forms

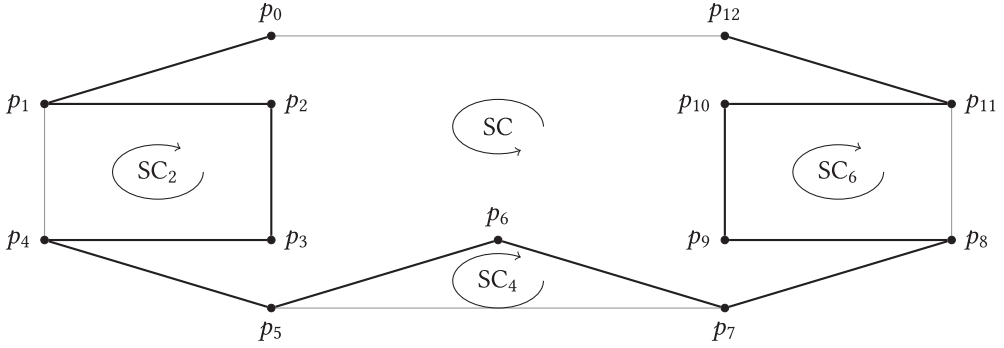


Fig. 6. The situation in the proof of Theorem 2. Beware that this is just a sketch; in reality, the pockets would need to be much more narrow in order to make all edges of SC unavoidable.

a so-called pocket of SC, which means that all edges of the cycle SC_j except for $p_{i_{j-1}}p_{i_j}$ are also edges of SC.

Using the statement of Theorem 2 inductively for the strictly smaller set of points P_j , we now see that there is a chain C_j with the same order type as P_j , that is, with

$$V(C_j)_{a,b} = \begin{cases} +1, & \text{if } p_{i_{j-1}+a}p_{i_{j-1}+b} \text{ is inside } SC_j \text{ or the base edge;} \\ -1, & \text{if } p_{i_{j-1}+a}p_{i_{j-1}+b} \text{ is outside } SC_j; \\ 0, & \text{if } p_{i_{j-1}+a}p_{i_{j-1}+b} \text{ is part of } SC_j \text{ (i.e., } a+1=b); \end{cases} \quad (0 \leq a < b \leq i_j - i_{j-1}).$$

Let us consider the flipped version \overline{C}_j . As noted before, the inside of SC_j is outside of SC. As SC_j moreover forms a pocket of SC, any edge outside of SC_j is inside SC. Hence,

$$V(\overline{C}_j)_{a,b} = \begin{cases} +1, & \text{if } p_{i_{j-1}+a}p_{i_{j-1}+b} \text{ is inside SC;} \\ -1, & \text{if } p_{i_{j-1}+a}p_{i_{j-1}+b} \text{ is outside SC;} \\ 0, & \text{if } p_{i_{j-1}+a}p_{i_{j-1}+b} \text{ is part of SC (i.e., } a+1=b); \end{cases} \quad (0 \leq a < b \leq i_j - i_{j-1}).$$

We claim that $C = \overline{C}_1 \vee \dots \vee \overline{C}_k$ has the desired visibility triangle V . We have just seen that the entries stemming from the individual \overline{C}_j are correct. So, all that is left to observe is that edges between different pockets lie inside of SC, which is indeed the case.

2.7 Point Sets That Are Almost Chains

Recall the assumption in Theorem 2 that says that at least one of the edges of the unavoidable spanning cycle must be a convex hull edge. This assumption is crucial since, for example, the classic double circle cannot be realized as a chain even though it admits an unavoidable spanning cycle. Nevertheless, we can show that the corresponding order types are reasonably close to being a chain, in the sense that a single point may be replaced by two new points in order to obtain a chain. In addition, this transformation does not change the number of triangulations by much.

THEOREM 18. *Let P be a set of n points that admits an unavoidable spanning cycle p_1, \dots, p_n , where p_n is a vertex of the convex hull. Then, there exists a chain $C = q_0, q_1, \dots, q_n$ such that p_1, \dots, p_{n-1} has the same order type as q_1, \dots, q_{n-1} . Moreover, we have*

$$2 \cdot \text{tr}(P) \leq \text{tr}(C) \leq (n-1) \cdot \text{tr}(P).$$

PROOF. We mimic the proof of Theorem 2 from the previous subsection. That is, consider the augmented sequence p_0, p_1, \dots, p_n with $p_0 = p_n$ and let p_{i_0}, \dots, p_{i_k} be the vertices of the convex hull with $0 = i_0 < \dots < i_k = n$ and $k \geq 3$. For $1 \leq j \leq k$, let $P_j = \{p_{i_{j-1}}, \dots, p_{i_j}\}$. Each such P_j again

forms a so-called pocket, which may be realized as a chain C_j . The construction is concluded by defining $C = C_1 \vee \dots \vee C_k$. Observe by using a statement analogous to Proposition 17 that all involved triangles maintain their orientation, except that the duplicated points q_0 and q_n that came from $p_0 = p_n$ behave antagonistically with respect to points in C_k and C_1 , respectively. For example, the triangle $q_i q_j q_n$ for $q_i, q_j \in C_1$ with $i < j$ is always oriented counter-clockwise, even if originally $p_i p_j p_n$ was oriented clockwise.

Let us fix any triangulation of P . We show how to map it injectively to two distinct triangulations of C , thus proving the first inequality. First of all, note that the triangulations of the individual pockets of P directly correspond to a lower triangulation of C , which means that we may limit our attention to the upper triangulation of C . There, we start by adding either the triangle $q_0 q_1 q_n$ or the triangle $q_0 q_{n-1} q_n$. In the first case, we are left to triangulate the simple polygon q_1, \dots, q_n , whereas in the second case, the respective polygon is q_0, \dots, q_{n-1} . In the first case, any remaining edge $p_i p_j$ of the fixed triangulation of P (i.e., any edge that is not contained inside of a pocket) is simply mapped to the edge $q_i q_j$ in C . In the second case, we do the same except that the point q_0 takes the role of q_n .

Let us fix any triangulation of C . To prove the second inequality, we construct a corresponding triangulation of P in such a way that at most $n - 1$ triangulations of C have the same correspondence. First of all, we may again map the lower triangulation of C to triangulations of the individual pockets of P . As far as the upper triangulation of C is concerned, there is exactly one triangle of the form $q_0 q_m q_n$, where $1 \leq m \leq n - 1$. This triangle is mapped to the edge $p_n p_m$ in the corresponding triangulation of P . This has the effect of splitting the inside of the unavoidable spanning cycle of P into two separate simple polygons that are left to be triangulated. They are triangulated simply by mapping all the remaining edges $q_i q_j$ of the upper triangulation of C to edges $p_i p_j$, where any appearance of q_0 is mapped to p_n . \square

3 TRIANGULATIONS OF CHAINS

In the previous section, we have seen that any chain can be expressed as a formula involving only convex and concave sums. Our goal here is to understand how triangulations behave with respect to such convex and concave sums. In order for this to work out, we have to consider not just triangulations, but a more general notion of partial triangulations.

We start by decomposing triangulations of a chain C into an upper and a lower part. An edge $p_i p_j$ is an *upper edge* if $V(C)_{i,j} = +1$, a *chain edge* if $V(C)_{i,j} = 0$, and a *lower edge* if $V(C)_{i,j} = -1$. That is, upper edges lie above the chain curve, while lower edges lie below.

Definition 19. An *upper (lower) triangulation* of a given chain C is a crossing-free geometric graph on C that is edge-maximal subject to only containing chain edges and upper (lower) edges. We denote the number of upper and lower triangulations by $U(C)$ and $L(C)$, respectively, and as always the number of (complete) triangulations by $\text{tr}(C)$.

Note that the chain edges are contained in every upper and lower triangulation. Moreover, every triangulation is the union of a unique upper and a unique lower triangulation, which implies $\text{tr}(C) = U(C) \cdot L(C)$. A lower triangulation of a chain C is an upper triangulation of the flipped version \bar{C} , and therefore $L(C) = U(\bar{C})$. For this reason, we may restrict our attention to studying only upper triangulations.

Intuitively speaking, we can create a partial upper triangulation by combining all the chain edges with some upper edges in such a way that all bounded faces are triangles. Note that then, only some of the used edges are visible from above.

Definition 20. Let C be any chain with n chain edges, and let $V = p_{i_0} p_{i_1} \dots p_{i_v}$ with $0 = i_0 < i_1 < \dots < i_v = n$ be an (x -monotone) curve composed of chain edges and upper edges only. A *partial*

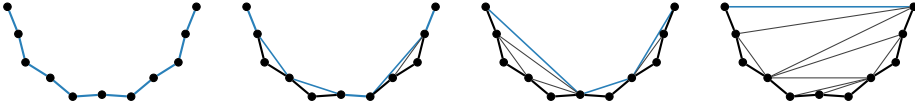


Fig. 7. Four partial upper triangulations of the “double circle” with 10, 6, 3, and 1 visible edge, respectively. As usual, chain edges are in bold, while visible edges are in blue.

upper triangulation of C (with visible edges V) consists of all chain edges, all edges in V , and a triangulation of the areas between the two.

Figure 7 depicts some partial upper triangulations and their visible edges. We are interested in counting such triangulations parameterized by the number of triangles. It can be noted that a partial upper triangulation with k triangles has $n - k$ visible edges.

Definition 21. Let C be any chain with n chain edges. For $k = 0, \dots, n - 1$, let $t_k(C)$ be the number of partial upper triangulations of C with k triangles (i.e., with $n - k$ visible edges). The *upper triangulation polynomial of C* is the corresponding generating function

$$T_C(x) = \sum_{k=0}^{n-1} t_k(C)x^k.$$

As an example, enumerating all partial upper triangulations of the convex chain $C_{\text{cvx}}(4)$ shows that $T_{C_{\text{cvx}}(4)}(x) = 1 + 3x + 5x^2 + 5x^3$. In general, note that for every chain C we have $t_0(C) = 1$ and that the leading coefficient of $T_C(x)$ is equal to $U(C)$. Moreover, we may again think of $T_{\overline{C}}(x)$ as the “lower triangulation polynomial” of C .

3.1 Convex and Concave Sums

Let us start with the easy case. For concave sums, we can establish the following relation.

LEMMA 22. A partial upper triangulation of $C_1 \wedge C_2$ is the union of a unique partial upper triangulation of C_1 and a unique partial upper triangulation of C_2 . Hence,

$$T_{C_1 \wedge C_2}(x) = T_{C_1}(x) \cdot T_{C_2}(x), \quad U(C_1 \wedge C_2) = U(C_1) \cdot U(C_2).$$

Convex sums are more tricky. The main insight is that every partial upper triangulation of $C_1 \vee C_2$ consists of a partial upper triangulation of C_1 , a partial upper triangulation of C_2 , and some edges between C_1 and C_2 . More precisely:

PROPOSITION 23. There is a triangle-preserving bijection between

- all triples (T_1, T_2, T_3) , where T_1 is a partial upper triangulation of C_1 (with v_1 visible edges), T_2 is a partial upper triangulation of C_2 (with v_2 visible edges), and T_3 is a partial upper triangulation of the convex sum $C_{\text{ccv}}(v_1) \vee C_{\text{ccv}}(v_2)$, and
- all partial upper triangulations of $C_1 \vee C_2$.

This bijection is defined by taking the union of all triangles; see Figure 8. The proposition then directly implies the following equation for the upper triangulation polynomial.

LEMMA 24. Let C_1 and C_2 be chains with n_1 and n_2 chain edges, respectively. Then,

$$T_{C_1 \vee C_2}(x) = \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2-1} t_{k_1}(C_1) \cdot t_{k_2}(C_2) \cdot x^{k_1+k_2} \cdot T_{C_{\text{ccv}}(n_1-k_1) \vee C_{\text{ccv}}(n_2-k_2)}(x).$$

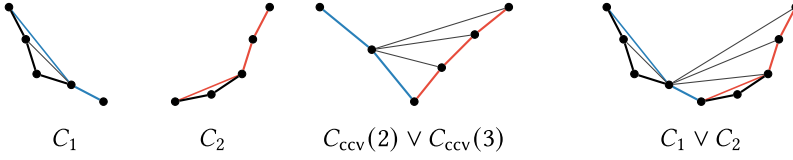


Fig. 8. From left to right, the respective partial upper triangulations T_1 of C_1 , T_2 of C_2 , T_3 of $C_{\text{ccv}}(2) \vee C_{\text{ccv}}(3)$, and the resulting partial upper triangulation of $C_1 \vee C_2$ as in Proposition 23.

Let us consider the special case of a convex sum of two concave chains with n_1 and n_2 chain edges, respectively. Note that any partial upper triangulation of such a chain has at most one upper edge that is visible. Summing over all possibilities for that edge and using the formula from the analysis of the classic double chain from [17], we get

$$T_{C_{\text{ccv}}(n_1) \vee C_{\text{ccv}}(n_2)}(x) = 1 + \sum_{l=1}^{n_1} \sum_{r=1}^{n_2} \binom{l+r-2}{l-1} x^{l+r-1}.$$

Combining the above equation with Lemma 24 allows us to compute $T_{C_1 \vee C_2}(x)$ from $T_{C_1}(x)$ and $T_{C_2}(x)$. Furthermore, by comparing the coefficients in the formulas from Lemmas 22 and 24, we get the following obvious but important fact.

COROLLARY 25. $C_1 \vee C_2$ has at least as many (partial) upper triangulations as $C_1 \wedge C_2$. That is,

$$\forall k : t_k(C_1 \vee C_2) \geq t_k(C_1 \wedge C_2), \quad U(C_1 \vee C_2) \geq U(C_1 \wedge C_2).$$

Finally, note that the two chains $C_1 \vee C_2$ and $C_2 \vee C_1$ can be quite different from a geometric point of view. But in terms of the number of triangulations, they are the same.

COROLLARY 26. For any two chains C_1 and C_2 , we have

$$T_{C_1 \vee C_2}(x) = T_{C_2 \vee C_1}(x), \quad T_{C_1 \wedge C_2}(x) = T_{C_2 \wedge C_1}(x).$$

3.2 Dynamic Programming

In this subsection, we show how to use dynamic programming in order to speed up the computations for a convex sum. To simplify the analysis, we assume a computational model where all additions and multiplications take only constant time.

PROPOSITION 27. Let C_1 and C_2 be chains with n_1 and n_2 chain edges, respectively. Given the coefficients of $T_{C_1}(x)$ and $T_{C_2}(x)$, we can compute $T_{C_1 \vee C_2}(x)$ in $O(n_1 n_2)$ time.

Recall that by Theorem 15, we can write any chain C as a formula involving only convex sums, concave sums, and primitive chains with only one chain edge. Therefore, using Proposition 27 for convex sums and Lemma 22 for concave sums, we are able to compute $T_C(x)$ in quadratic time. Clearly, this proves Theorem 3 from the introduction.

PROOF OF PROPOSITION 27. Observe that every partial upper triangulation of $C_1 \vee C_2$ either corresponds to a partial upper triangulation of $C_1 \wedge C_2$ or has a unique visible upper edge that connects a vertex of C_1 with a vertex of C_2 . Let us call this edge the *bridge*. Let further $\text{DP}[l][r]$ be the number of partial upper triangulations whose visible edges consist of l visible edges in C_1 , followed by the bridge, followed by r visible edges in C_2 . Then,

$$T_{C_1 \vee C_2}(x) = T_{C_1 \wedge C_2}(x) + \sum_{l=0}^{n_1-1} \sum_{r=0}^{n_2-1} \text{DP}[l][r] \cdot x^{n_1+n_2-l-r-1}.$$

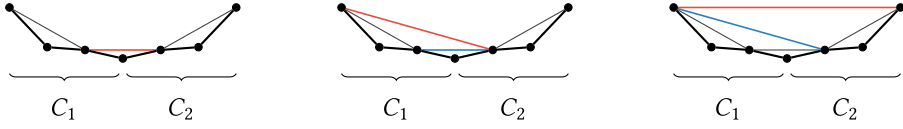


Fig. 9. The three cases when removing the bridge from a partial upper triangulation of $C_1 \vee C_2$ in the proof of Proposition 27. On the left, both C_1 and C_2 gain a visible edge. In the middle, only C_1 gains a visible edge. On the right, only C_2 gains a visible edge. The current bridge is red, and the edge that becomes the new bridge is blue.

To compute the table DP, let us see what happens when we remove the bridge. We either end up with a partial upper triangulation of $C_1 \wedge C_2$ with $l + 1$ and $r + 1$ visible edges in C_1 and C_2 , respectively, or get a new bridge, which used to be an edge of the triangle below the old bridge. In the latter case, depending on which of the two possible edges this is, we end up with one more visible edge in either C_1 or C_2 . Figure 9 depicts these three cases. To summarize, for all l and r ($0 \leq l < n_1, 0 \leq r < n_2$),

$$\text{DP}[l][r] = t_{n_1-l-1}(C_1) \cdot t_{n_2-r-1}(C_2) + \text{DP}[l+1][r] + \text{DP}[l][r+1],$$

with the base case $\text{DP}[n_1][r] = \text{DP}[l][n_2] = 0$. Therefore, filling up the table DP takes $O(n_1 n_2)$ time, as desired. \square

3.3 Koch Chains

Recall Definition 14 and that the formula for Koch chains expands to the nested expression

$$K_s = (K_{s-2} \wedge K_{s-2}) \vee (K_{s-2} \wedge K_{s-2})$$

with alternating convex and concave sums. This repeated mixing of the two types of sums appears to make an exact analysis of the number of triangulations of K_s very difficult.

Instead, we have implemented the quadratic time algorithm from the previous subsection and used it to compute $T_{K_s}(x)$ and $T_{\overline{K_s}}(x)$ for all $s \leq 21$. To deal with the exponentially growing coefficients, we rely on a custom floating point type with a 64-bit mantissa and a 32-bit exponent from the boost multiprecision library. As only additions and multiplications are involved, we do not have to deal with numerical issues; in fact, the rounding errors grow at most linearly. In addition, we make use of multi-threading and take advantage of symmetries of K_s for a constant factor speed-up. This allows us to compute $T_{K_{21}}(x)$ in around a day on a regular workstation (Intel i7-6700HQ, 2.6GHz).

Table 1 from the introduction lists the resulting numbers. For example, K_{21} has approximately 9.082799^n triangulations, where $n = 2^{21}$. In the next section, we show how the computed coefficients of $T_{K_{21}}(x)$ can be used to give bounds on $\text{tr}(K_s)$ as $s \rightarrow \infty$.

4 POLY CHAINS AND TWIN CHAINS

Let C_0 be a fixed chain with m chain edges. We want to define two particular families of chains that can be built from many copies of C_0 via concave and convex sums.

Definition 28. For integers $N \geq 1$, the *poly- C_0 chains (of length $n = Nm$)* are the chains

$$C_{\text{poly}}(C_0, N) = \underbrace{\overline{C_0} \vee \cdots \vee \overline{C_0}}_{N \text{ copies}}.$$

Definition 29. For integers $N \geq 1$, the *twin- C_0 chains (of length $n = 2Nm + 1$)* are the chains

$$C_{\text{twin}}(C_0, N) = \overline{C_{\text{poly}}(C_0, N)} \vee E \vee \overline{C_{\text{poly}}(C_0, N)}.$$

Table 2. The Computed Values of λ and τ as in Theorem 30 for the Special Case $C_0 = K_s$,
Where Each Entry Is Rounded Down to Six Decimal Places

s	m	λ	τ	$\lambda\tau$	s	m	λ	τ	$\lambda\tau$
0	1	4.0	2.0	8.0	11	2048	2.894607	3.137597	9.082113
1	2	3.464101	2.449489	8.485281	12	4096	3.142184	2.890518	9.082542
2	4	3.534118	2.449489	8.656787	13	8192	2.892806	3.139805	9.082850
3	8	3.124013	2.841004	8.875335	14	16384	3.141127	2.891623	9.082957
4	16	3.290140	2.721989	8.955727	15	32768	2.892276	3.140445	9.083034
5	32	2.974654	3.033787	9.024469	16	65536	3.140819	2.891940	9.083061
6	64	3.191872	2.835019	9.049019	17	131072	2.892123	3.140627	9.083081
7	128	2.919234	3.106209	9.067752	18	262144	3.140731	2.892029	9.083087
8	256	3.157095	2.874272	9.074351	19	524288	2.892080	3.140677	9.083092
9	512	2.900536	3.130176	9.079191	20	1048576	3.140706	2.892054	9.083094
10	1024	3.145716	2.886746	9.080887	21	2097152	2.892068	3.140692	9.083095

Note that both resulting chains are upward chains, as long as $N > 1$. For example, the poly- E chains are the convex chains, the twin- E chains are the classic double chains, and the twin- $(E \vee E)$ chains are the double zig-zag chains.

We are interested in the asymptotic behavior of the number of triangulations of these constructions as N goes to infinity. Lemma 22 allows us to express the number of lower triangulations in terms of upper triangulations.

$$L(C_{\text{poly}}(C_0, N)) = U(C_0 \wedge \cdots \wedge C_0) = U(C_0)^N$$

$$L(C_{\text{twin}}(C_0, N)) = U(C_{\text{poly}}(C_0, N) \wedge E \wedge C_{\text{poly}}(C_0, N)) = U(C_{\text{poly}}(C_0, N))^2$$

For the upper triangulations, we make use of the following general result, whose proof can be found in the following subsections.

THEOREM 30. *The poly- C_0 chains of length n have $\tilde{\Theta}(\lambda^n)$ upper triangulations (where the $\tilde{\Theta}$ -notation suppresses potential additional factors that are polynomial in n), while the twin- C_0 chains of length n have $\tilde{\Theta}(\tau^n)$ upper triangulations, where*

$$\lambda = \sqrt[m]{\sum_{k=1}^m 2^k (k+1) \cdot t_{m-k}(\overline{C_0})}, \quad \tau = \sqrt[m]{\sum_{k=1}^m 2^k \cdot t_{m-k}(C_0)}.$$

It follows that the twin- C_0 chains of length n have $\tilde{\Theta}((\lambda\tau)^n)$ complete triangulations.

Example 31. Let us analyze the poly- $C_{\text{cvx}}(4)$ chains and twin- $C_{\text{cvx}}(4)$ chains. We have

$$T_{\overline{C_{\text{cvx}}(4)}}(x) = 1, \quad T_{C_{\text{cvx}}(4)}(x) = 1 + 3x + 5x^2 + 5x^3,$$

which yields $\lambda = \sqrt[4]{80}$ and $\tau = \sqrt[4]{70}$. Therefore, the twin- $C_{\text{cvx}}(4)$ chains have $\tilde{\Theta}(\sqrt[4]{5600}^n)$ triangulations, where $\sqrt[4]{5600} \approx 8.6506154$. Note that these chains are the generalized double zig-zag chains from [9]. By comparison, the numerical bound there was $\Omega(8.6504^n)$.

In the same way, by using the many coefficients of $T_{K_s}(x)$ and $T_{\overline{K_s}}(x)$ that we found for all $s \leq 21$ with our algorithm from the previous section, we can also compute the corresponding values of λ and τ for poly-Koch chains and twin-Koch chains. The results can be found in Table 2. The last row in particular allows us to analyze twin- K_{21} chains and, therefore, prove Theorem 4 from the introduction.

Table 3. The Computed Values of λ for Poly- K_s Chains, the Corresponding $\bar{\lambda}$ for Poly- \bar{K}_s Chains, and the Product $\lambda\bar{\lambda}$ for the Resulting Upper Bound Given by Lemma 34, Where Each Entry Is Rounded Down to Six Decimal Places

s	m	λ	$\bar{\lambda}$	$\lambda\bar{\lambda}$	s	m	λ	$\bar{\lambda}$	$\lambda\bar{\lambda}$
0	1	4.0	4.0	16.0	11	2048	2.894607	3.145716	9.105615
1	2	3.464101	4.0	13.856406	12	4096	3.142184	2.894607	9.095390
2	4	3.534118	3.464101	12.242546	13	8192	2.892806	3.142184	9.089731
3	8	3.124013	3.534118	11.040634	14	16384	3.141127	2.892806	9.086674
4	16	3.290140	3.124013	10.278444	15	32768	2.892276	3.141127	9.085007
5	32	2.974654	3.290140	9.787032	16	65536	3.140819	2.892276	9.084117
6	64	3.191872	2.974654	9.494717	17	131072	2.892123	3.140819	9.083637
7	128	2.919234	3.191872	9.317823	18	262144	3.140731	2.892123	9.083383
8	256	3.157095	2.919234	9.216301	19	524288	2.892080	3.140731	9.083247
9	512	2.900536	3.157095	9.157271	20	1048576	3.140706	2.892080	9.083176
10	1024	3.145716	2.900536	9.124266	21	2097152	2.892068	3.140706	9.083138

COROLLARY 32. The twin- K_{21} chains of length n have $\tilde{\Theta}(\tau^n)$ upper triangulations, $\tilde{\Theta}(\lambda^n)$ lower triangulations, and hence $\tilde{\Theta}((\lambda\tau)^n)$ complete triangulations, where $\tau \approx 3.140692$, $\lambda \approx 2.892068$, and $\lambda\tau \approx 9.083095$.

The next two lemmas, when combined with the first part of Theorem 30, can further be used to prove asymptotic bounds for families of chains that are built from the same C_0 .

LEMMA 33. Let C be any chain that can be written as a formula involving convex sums, concave sums, and exactly N copies of C_0 . Then,

$$U(C_0)^N \leq U(C) \leq U(C_{\text{poly}}(\bar{C}_0, N)).$$

PROOF. We get $U(C_0 \wedge \dots \wedge C_0) \leq U(C) \leq U(C_0 \vee \dots \vee C_0)$ by using induction on N combined with Corollary 25. Lemma 22 and the definition of poly- \bar{C}_0 chains then concludes the proof. \square

LEMMA 34. In the same setting, we have

$$\text{tr}(C_0)^N \leq \text{tr}(C) \leq U(C_{\text{poly}}(C_0, N)) \cdot U(C_{\text{poly}}(\bar{C}_0, N)).$$

PROOF. Apply Lemma 33 twice. First to C with C_0 , then to \bar{C} with \bar{C}_0 . \square

The Koch chains K_s with $s \geq 21$ can be written as formulas involving only copies of K_{21} (or only copies of \bar{K}_{21} , depending on the parity of s), and so Lemma 34 applies to them. For the lower bound, we obtain $9.082799^n \leq \text{tr}(K_s)$ for all $s \geq 21$ from the last entry in Table 1. For the upper bound, we make use of Table 3, which contains once more the computed values λ for poly- K_s chains, but it also includes the corresponding values (denoted by $\bar{\lambda}$) for poly- \bar{K}_s chains. Note that poly- \bar{K}_s chains are in fact nothing else than poly- K_{s-1} chains, and so the values λ and $\bar{\lambda}$ are simply shifted by one index. For all $s \geq 21$, we obtain $\text{tr}(K_s) \leq U(C_{\text{poly}}(K_{21}, N)) \cdot U(C_{\text{poly}}(\bar{K}_{21}, N)) = \tilde{\Theta}((\lambda\bar{\lambda})^n)$, where $\lambda\bar{\lambda} \approx 9.083138$. Clearly, this proves Theorem 5 from the introduction.

4.1 Tools for the Proof of Theorem 30

We use similar ideas as in the analysis of the generalized double zig-zag chain by Dumitrescu et al. (Section 2 of [9]), with three key improvements that allow us to get an exact $\tilde{\Theta}$ instead of just a numerical lower bound.

First improvement. Our first improvement is that our chain framework allows us to analyze even more general “double circles,” defined as follows.

Definition 35. Let $N_1, \dots, N_m \geq 0$ be integers, at least one non-zero. Then, the corresponding *generalized double circle* (with $n = 1N_1 + 2N_2 + \dots + mN_m$ chain edges) is the chain

$$C_{\text{gdc}}(N_1, \dots, N_m) = C_{\text{poly}}(C_{\text{cvx}}(1), N_1) \vee C_{\text{poly}}(C_{\text{cvx}}(2), N_2) \vee \dots \vee C_{\text{poly}}(C_{\text{cvx}}(m), N_m),$$

where we simply omit the poly- $C_{\text{cvx}}(k)$ chains with $N_k = 0$.

THEOREM 36. *Let $N = N_1 + \dots + N_m$. There is a constant c_m depending only on m such that*

$$U(C_{\text{gdc}}(N_1, \dots, N_m)) \geq \frac{1}{(N+1)^{c_m}} \prod_{k=1}^m \left(2^k(k+1)\right)^{N_k}.$$

PROOF. By Corollary 25 and Lemma 22, we get

$$U(C_{\text{gdc}}(N_1, \dots, N_m)) \geq \prod_{k=1}^m U(C_{\text{poly}}(C_{\text{cvx}}(k), N_k)).$$

Asinowski et al. [5] show for any fixed k that

$$U(C_{\text{poly}}(C_{\text{cvx}}(k), M)) = \tilde{\Omega}((2^k(k+1))^M) \quad (\text{as } M \rightarrow \infty),$$

implying the existence of an absolute constant d_k such that

$$U(C_{\text{poly}}(C_{\text{cvx}}(k), M)) \geq \frac{1}{(M+1)^{d_k}} \cdot (2^k(k+1))^M \quad (\text{for all } M \geq 1).$$

Substituting the respective N_k for M and defining $c_m = d_1 + \dots + d_m$ concludes the proof. \square

Second improvement. The authors of [9] use numerical optimization in the final step of analyzing generalized double zig-zag chains. We solve the corresponding optimization problem algebraically, via Lagrange multipliers. This is best captured by the following lemma.

LEMMA 37. *Let $u_1, \dots, u_m \geq 0$ be given and let $H(\alpha_1, \dots, \alpha_m) = -\sum_{k=1}^m \alpha_k \cdot \ln(\alpha_k)$ be the entropy function. Then,*

$$\max_{\substack{0 \leq \alpha_1, \dots, \alpha_m \leq 1 \\ \alpha_1 + \dots + \alpha_m = 1}} e^{H(\alpha_1, \dots, \alpha_m)} \cdot \prod_{k=1}^m u_k^{\alpha_k} = \sum_{k=1}^m u_k.$$

PROOF. Without loss of generality, assume that $u_k > 0$ for all k (otherwise put $\alpha_k = 0$, after which we are left with the same proof for a smaller number m of variables). To simplify notation, define

$$f(\alpha_1, \dots, \alpha_m) := e^{H(\alpha_1, \dots, \alpha_m)} \cdot \prod_{k=1}^m u_k^{\alpha_k}.$$

The proof is by induction on m . If $m = 1$, then $\alpha_1 = 1$ and the statement is trivial. For $m \geq 2$, note that the maximum of f always exists since we are maximizing over a compact region.

Let us now consider the boundary first. If $\alpha_i = 0$ for some index i , then by induction

$$\max_{\substack{0 \leq \alpha_1, \dots, \alpha_m \leq 1 \\ \alpha_1 + \dots + \alpha_m = 1 \\ \alpha_i = 0}} f(\alpha_1, \dots, \alpha_m) = \sum_{k=1}^m u_k - u_i \leq \sum_{k=1}^m u_k.$$

On the other hand, if $\alpha_i = 1$ for some i , then $\alpha_j = 0$ for all other indices j and the same argument applies. To summarize, if the maximum is on the boundary, then it is at most $u_1 + \dots + u_m$.

Let us now consider the interior. After taking logs, we are left to show that

$$\max_{\substack{0 < \alpha_1, \dots, \alpha_m < 1 \\ \alpha_1 + \dots + \alpha_m = 1}} \ln(f(\alpha_1, \dots, \alpha_m)) = \ln\left(\sum_{k=1}^m u_k\right).$$

By making use of the definitions

$$g(\alpha_1, \dots, \alpha_m) := \ln(f(\alpha_1, \dots, \alpha_m)) = H(\alpha_1, \dots, \alpha_m) + \sum_{k=1}^m \alpha_k \cdot \ln(u_k),$$

$$h(\alpha_1, \dots, \alpha_m) := \alpha_1 + \dots + \alpha_m - 1,$$

we see that, equivalently, we are maximizing g over the open set $0 < \alpha_1, \dots, \alpha_m < 1$ under the constraint $h(\alpha_1, \dots, \alpha_m) = 0$. By Lagrange multipliers, any such maximum $(\alpha_1, \dots, \alpha_m)$, if it exists, has to solve

$$\frac{\partial g}{\partial \alpha_k} = \lambda \cdot \frac{\partial h}{\partial \alpha_k}$$

for some $\lambda \in \mathbb{R}$ and all $k = 1, \dots, m$. For our concrete functions g and h , these equations are

$$-\ln(\alpha_k) - 1 + \ln(u_k) = \lambda$$

or, equivalently,

$$\alpha_k = e^{-\lambda-1} \cdot u_k.$$

Plugging this into the constraint $\alpha_1 + \dots + \alpha_m = 1$ yields

$$e^{-\lambda-1} \cdot (u_1 + \dots + u_m) = 1,$$

which then allows us to eliminate λ and we arrive at

$$\alpha_k = \frac{1}{u_1 + \dots + u_m} \cdot u_k.$$

For these particular values α_k , we now calculate

$$\begin{aligned} g(\alpha_1, \dots, \alpha_m) &= H(\alpha_1, \dots, \alpha_m) + \sum_{k=1}^m \alpha_k \cdot \ln(u_k) \\ &= -\sum_{k=1}^m \frac{u_k}{u_1 + \dots + u_m} \cdot \ln\left(\frac{u_k}{u_1 + \dots + u_m}\right) + \sum_{k=1}^m \frac{u_k}{u_1 + \dots + u_m} \cdot \ln(u_k) \\ &= \sum_{k=1}^m \frac{u_k}{u_1 + \dots + u_m} \cdot \left(-\ln\left(\frac{u_k}{u_1 + \dots + u_m}\right) + \ln(u_k)\right) \\ &= \sum_{k=1}^m \frac{u_k}{u_1 + \dots + u_m} \cdot \ln(u_1 + \dots + u_m) \\ &= \ln(u_1 + \dots + u_m) \end{aligned}$$

and, hence, $f(\alpha_1, \dots, \alpha_m) = u_1 + \dots + u_m$. Given that this value is at least as large as what we got for the boundary, there has to be a maximum in the interior. By Lagrange multipliers, that maximum has to be at the values α_k we just considered. \square

Third improvement. Finally, we define a special generating function that behaves well with regard to convex sums. This will allow us to prove a matching upper bound for Theorem 36.

Definition 38. Let C be any chain with n chain edges. The *upper triangulation generating function* of C is defined as

$$\phi_C(x) = T_C(x) - \left(\frac{x}{1-x}\right)^{n+1} T_C(1-x).$$

Note that $\phi_C(x)$ is a rational function. As a formal power series, it satisfies $\phi_C(x) = T_C(x) + O(x^{n+1})$. In other words, for $k = 0, \dots, n$, the coefficient of x^k in the Taylor expansion of $\phi_C(x)$ at $x = 0$ is equal to the triangulation number $t_k(C)$.

LEMMA 39. For any two chains C_1 and C_2 , we have

$$\phi_{C_1 \vee C_2}(x) = \frac{1-x}{1-2x} \cdot \phi_{C_1}(x) \cdot \phi_{C_2}(x).$$

The proof of Lemma 39 is deferred to the next subsection since it involves some tedious computations. Here, we focus on the following implication.

THEOREM 40. For the generalized double circle, we have

$$U(C_{\text{gdc}}(N_1, \dots, N_m)) \leq \prod_{k=1}^m (2^k(k+1))^{N_k}.$$

PROOF. First, by repeated invocation of Lemma 39, we get

$$\phi_{C_{\text{gdc}}(N_1, \dots, N_m)}(x) = \left(\frac{1-x}{1-2x}\right)^{N_1 + \dots + N_{m-1}} \cdot \prod_{k=1}^m \left(\phi_{C_{\text{ccv}}(k)}(x)\right)^{N_k}.$$

Second, recall that for concave chains we have $T_{C_{\text{ccv}}(k)}(x) = 1$. Therefore, by definition of ϕ and by the geometric series,

$$\phi_{C_{\text{ccv}}(k)}(x) = 1 - \left(\frac{x}{1-x}\right)^{k+1} = \frac{1-2x}{1-x} \cdot \sum_{i=0}^k \left(\frac{x}{1-x}\right)^i.$$

Third, for all $n \geq 2$ and $k \geq 0$ we have the combinatorial identity

$$[x^{n-1}] \left(\frac{x}{1-x}\right)^k = \binom{n-2}{k-1} < 2^n.$$

With $n = 1N_1 + 2N_2 + \dots + mN_m$ being the number of chain edges, we can now combine these three ingredients and get

$$\begin{aligned} U(C_{\text{gdc}}(N_1, \dots, N_m)) &= [x^{n-1}] \phi_{C_{\text{gdc}}(N_1, \dots, N_m)}(x) \\ &= [x^{n-1}] \left(\frac{1-x}{1-2x}\right)^{N_1 + \dots + N_{m-1}} \cdot \prod_{k=1}^m \left(\phi_{C_{\text{ccv}}(k)}(x)\right)^{N_k} \\ &= [x^{n-1}] \frac{1-2x}{1-x} \prod_{k=1}^m \left(\sum_{i=0}^k \left(\frac{x}{1-x}\right)^i\right)^{N_k} \\ &\leq [x^{n-1}] \prod_{k=1}^m \left(\sum_{i=0}^k \left(\frac{x}{1-x}\right)^i\right)^{N_k} \leq 2^n \prod_{k=1}^m (k+1)^{N_k}, \end{aligned}$$

where expanding the second-to-last term yields $\prod_k (k+1)^{N_k}$ summands, each some power of $\frac{x}{1-x}$, for which the coefficient of x^{n-1} is always less than 2^n . \square

4.2 Proof of Lemma 39

First, let us consider the simpler case where $C_1 = C_{\text{ccv}}(n_1)$ and $C_2 = C_{\text{ccv}}(n_2)$ for arbitrary integers $n_1, n_2 \geq 1$. As already seen in the previous subsection, in such a case we have

$$\phi_{C_{\text{ccv}}(n_i)}(x) = 1 - \left(\frac{x}{1-x}\right)^{n_i+1}.$$

In the paragraph after Lemma 24, we have further seen that

$$T_{C_{\text{ccv}}(n_1) \vee C_{\text{ccv}}(n_2)}(x) = 1 + \sum_{l=1}^{n_1} \sum_{r=1}^{n_2} \binom{l+r-2}{l-1} x^{l+r-1}.$$

Therefore, in order to prove Lemma 39 for this specific choice of C_1 and C_2 , it suffices to show the following identity.

PROPOSITION 41. *Let $n_1, n_2 \geq 0$. Define the formal power series*

$$\begin{aligned} T_{n_1, n_2}(x) &:= 1 + \sum_{l=1}^{n_1} \sum_{r=1}^{n_2} \binom{l+r-2}{l-1} x^{l+r-1} = 1 + \sum_{l=0}^{n_1-1} \sum_{r=0}^{n_2-1} \binom{l+r}{l} x^{l+r+1}, \\ L_{n_1, n_2}(x) &:= T_{n_1, n_2}(x) - \left(\frac{x}{1-x}\right)^{n_1+n_2+1} T_{n_1, n_2}(1-x), \\ R_{n_1, n_2}(x) &:= \frac{1-x}{1-2x} \cdot \left(1 - \left(\frac{x}{1-x}\right)^{n_1+1}\right) \cdot \left(1 - \left(\frac{x}{1-x}\right)^{n_2+1}\right). \end{aligned}$$

Then, we have that $L_{n_1, n_2}(x) = R_{n_1, n_2}(x)$.

PROOF. We use induction on (n_1, n_2) . If at least one of n_1 and n_2 is zero, then the proposition follows easily. We may hence assume that $n_1, n_2 \geq 1$.

First, we do some computations with $T_{n_1, n_2}(x)$. We use the letter k to denote the sum $l+r$, and we also use Iverson brackets to simplify notation:

$$\begin{aligned} T_{n_1, n_2}(x) &= 1 + \sum_{k=0}^{\infty} \sum_{l+r=k} [0 \leq l < n_1, 0 \leq r < n_2] \binom{k}{l} x^{k+1} \\ &= 1 + \sum_{k=0}^{\infty} 2^k x^{k+1} - \sum_{k=n_1}^{\infty} \sum_{l=n_1}^k \binom{k}{l} x^{k+1} - \sum_{k=n_2}^{\infty} \sum_{r=n_2}^k \binom{k}{r} x^{k+1} + \sum_{l=n_1}^{\infty} \sum_{r=n_2}^{\infty} \binom{l+r}{l} x^{l+r+1} \\ &= 1 + \frac{x}{1-2x} - \sum_{l=n_1}^{\infty} \sum_{k=l}^{\infty} \binom{l+(k-l)}{l} x^{k+1} - \sum_{r=n_2}^{\infty} \sum_{k=r}^{\infty} \binom{r+(k-r)}{r} x^{k+1} + \sum_l \sum_r (\dots) \\ &= \frac{1-x}{1-2x} - \sum_{l=n_1}^{\infty} \left(\frac{x}{1-x}\right)^{l+1} - \sum_{r=n_2}^{\infty} \left(\frac{x}{1-x}\right)^{r+1} + \sum_l \sum_r (\dots) \\ &= \frac{1-x}{1-2x} - \left(\frac{x}{1-x}\right)^{n_1+1} \frac{1}{1-\frac{x}{1-x}} - \left(\frac{x}{1-x}\right)^{n_2+1} \frac{1}{1-\frac{x}{1-x}} + \sum_l \sum_r (\dots) \\ &= \frac{1-x}{1-2x} \left(1 - \left(\frac{x}{1-x}\right)^{n_1+1} - \left(\frac{x}{1-x}\right)^{n_2+1}\right) + \sum_{l=n_1}^{\infty} \sum_{r=n_2}^{\infty} \binom{l+r}{l} x^{l+r+1}. \end{aligned}$$

We can now use the above equation in order to compute the following expression:

$$\begin{aligned}
& T_{n_1, n_2}(x) - \frac{x}{1-x} (T_{n_1-1, n_2}(x) + T_{n_1, n_2-1}(x)) + \left(\frac{x}{1-x}\right)^2 T_{n_1-1, n_2-1}(x) \\
&= \frac{1-2x}{1-x} + \frac{1}{(1-x)^2} \sum_{l=n_1}^{\infty} \sum_{r=n_2}^{\infty} \left[(1-x)^2 \binom{l+r}{l} - (1-x) \binom{l+r-1}{l} \right. \\
&\quad \left. - (1-x) \binom{l+r-1}{l-1} + \binom{l+r-2}{l-1} \right] \cdot x^{l+r+1} \\
&= \frac{1-2x}{1-x} + \frac{1}{(1-x)^2} \sum_{l=n_1}^{\infty} \sum_{r=n_2}^{\infty} \left[x^2 \binom{l+r}{l} - x \binom{l+r-1}{l} \right. \\
&\quad \left. - x \binom{l+r-1}{l-1} + \binom{l+r-2}{l-1} \right] \cdot x^{l+r+1} \\
&= \frac{1-2x}{1-x} + \frac{1}{(1-x)^2} \sum_{l \in \mathbb{Z}} \sum_{r \in \mathbb{Z}} \binom{l+r-2}{l-1} \left([l > n_1, r > n_2] - [l > n_1, r \geq n_2] \right. \\
&\quad \left. - [l \geq n_1, r > n_2] + [l \geq n_1, r \geq n_2] \right) \cdot x^{l+r+1} \\
&= \frac{1-2x}{1-x} + \frac{1}{(1-x)^2} \sum_{l \in \mathbb{Z}} \sum_{r \in \mathbb{Z}} \binom{l+r-2}{l-1} [l = n_1, r = n_2] \cdot x^{l+r+1} \\
&= \frac{1-2x}{1-x} + \frac{1}{(1-x)^2} \binom{n_1+n_2-2}{n_1-1} x^{n_1+n_2+1}.
\end{aligned}$$

In turn, we can further use the above to compute a very similar expression for $L_{n_1, n_2}(x)$, noting that the second term from before cancels nicely in the very last step:

$$\begin{aligned}
& L_{n_1, n_2}(x) - \frac{x}{1-x} (L_{n_1-1, n_2}(x) + L_{n_1, n_2-1}(x)) + \left(\frac{x}{1-x}\right)^2 L_{n_1-1, n_2-1}(x) \\
&= T_{n_1, n_2}(x) - \frac{x}{1-x} (T_{n_1-1, n_2}(x) + T_{n_1, n_2-1}(x)) + \left(\frac{x}{1-x}\right)^2 T_{n_1-1, n_2-1}(x) \\
&\quad - \left(\frac{x}{1-x}\right)^{n_1+n_2+1} \binom{n_1+n_2-2}{n_1-1} (1-x)^{n_1+n_2-1} = \frac{1-2x}{1-x}.
\end{aligned}$$

On the other hand, a direct computation shows that we also have the same equation for $R_{n_1, n_2}(x)$:

$$R_{n_1, n_2}(x) - \frac{x}{1-x} (R_{n_1-1, n_2}(x) + R_{n_1, n_2-1}(x)) + \left(\frac{x}{1-x}\right)^2 R_{n_1-1, n_2-1}(x) = \frac{1-2x}{1-x}.$$

Therefore, the proposition now easily follows by combining the previous two equations with the induction hypothesis. \square

Let now C_1 and C_2 be two arbitrary chains with n_1 and n_2 chain edges, respectively. By making use of Lemma 24 and the special case of Lemma 39 that we just proved in Proposition 41, we can

now compute

$$\begin{aligned}
\phi_{C_1 \vee C_2}(x) &= T_{C_1 \vee C_2}(x) + \left(\frac{x}{1-x} \right)^{n_1+n_2+1} T_{C_1 \vee C_2}(1-x) \\
&= \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2-1} t_{k_1}(C_1) \cdot t_{k_2}(C_2) \cdot x^{k_1+k_2} \left(T_{C_{\text{ccv}}(n_1-k_1) \vee C_{\text{ccv}}(n_2-k_2)}(x) \right. \\
&\quad \left. + \left(\frac{x}{1-x} \right)^{n_1+n_2+1-k_1-k_2} \cdot T_{C_{\text{ccv}}(n_1-k_1) \vee C_{\text{ccv}}(n_2-k_2)}(1-x) \right) \\
&= \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2-1} t_{k_1}(C_1) \cdot t_{k_2}(C_2) \cdot x^{k_1+k_2} \cdot \phi_{C_{\text{ccv}}(n_1-k_1) \vee C_{\text{ccv}}(n_2-k_2)}(x) \\
&= \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2-1} t_{k_1}(C_1) \cdot t_{k_2}(C_2) \cdot x^{k_1+k_2} \cdot \frac{1-x}{1-2x} \cdot \phi_{C_{\text{ccv}}(n_1-k_1)}(x) \cdot \phi_{C_{\text{ccv}}(n_2-k_2)}(x) \\
&= \frac{1-x}{1-2x} \cdot \left(\sum_{k_1=0}^{n_1-1} t_{k_1}(C_1) \cdot x^{k_1} \cdot \phi_{C_{\text{ccv}}(n_1-k_1)}(x) \right) \cdot \left(\sum_{k_2=0}^{n_2-1} t_{k_2}(C_2) \cdot x^{k_2} \cdot \phi_{C_{\text{ccv}}(n_2-k_2)}(x) \right) \\
&= \frac{1-x}{1-2x} \cdot \phi_{C_1}(x) \cdot \phi_{C_2}(x).
\end{aligned}$$

4.3 Proof of Theorem 30 (Poly Chains)

Recall once more Lemma 24. For the special case where $C_2 = C_{\text{ccv}}(n_2)$, we have that $t_0(C_2) = 1$ is the only non-zero triangulation number, and hence

$$T_{C_1 \vee C_{\text{ccv}}(n_2)}(x) = \sum_{k_1=0}^{n_1-1} t_{k_1}(C_1) \cdot x^{k_1} \cdot T_{C_{\text{ccv}}(n_1-k_1) \vee C_{\text{ccv}}(n_2)}(x).$$

After plugging this into the right-hand side of Lemma 24, we get the following variant of the statement, where the second sum is left unexpanded.

LEMMA 42. *Let C_1 be a chain with n_1 chain edges and let C_2 be any chain. Then,*

$$T_{C_1 \vee C_2}(x) = \sum_{k=0}^{n_1-1} t_k(C_1) \cdot x^k \cdot T_{C_{\text{ccv}}(n_1-k) \vee C_2}(x).$$

Let now C_0 be any fixed chain with m chain edges and let $N \geq 2$. Using commutativity as in Corollary 26, we can inductively apply Lemma 42 to the poly- C_0 chain and get

$$T_{C_{\text{poly}}(C_0, N)}(x) = \sum_{k_1=0}^{m-1} \cdots \sum_{k_N=0}^{m-1} t_{k_1}(\overline{C_0}) \cdots t_{k_N}(\overline{C_0}) \cdot x^{k_1+\cdots+k_N} \cdot T_{C_{\text{ccv}}(m-k_1) \vee \cdots \vee C_{\text{ccv}}(m-k_N)}(x).$$

For any particular summand in the above expression, let a_i be the number of indices $j = 1, \dots, N$ that satisfy $k_j = m - i$. Then, this particular summand can also be written as

$$t_0(\overline{C_0})^{a_m} \cdot t_1(\overline{C_0})^{a_{m-1}} \cdots t_{m-1}(\overline{C_0})^{a_1} \cdot x^{0a_m+1a_{m-1}+\cdots+(m-1)a_1} \cdot T_{C_{\text{gdc}}(a_1, \dots, a_m)}(x).$$

Since the number of summands with these particular values a_i is equal to $\binom{N}{a_1, \dots, a_m}$, we hence get

$$T_{C_{\text{poly}}(C_0, N)}(x) = \sum_{\substack{0 \leq a_1, \dots, a_m \leq N \\ a_1 + \cdots + a_m = N}} \binom{N}{a_1, \dots, a_m} \cdot \left(\prod_{k=1}^m t_{m-k}(\overline{C_0})^{a_k} \cdot x^{(m-k)a_k} \right) \cdot T_{C_{\text{gdc}}(a_1, \dots, a_m)}(x).$$

Finally, note that the corresponding leading coefficients on both sides are equal to

$$U(C_{\text{poly}}(C_0, N)) = \sum_{\substack{0 \leq a_1, \dots, a_m \leq N \\ a_1 + \dots + a_m = N}} \binom{N}{a_1, \dots, a_m} \cdot \left(\prod_{k=1}^m t_{m-k}(\overline{C_0})^{a_k} \right) \cdot U(C_{\text{gdc}}(a_1, \dots, a_m)).$$

We will now use this equation to prove both an upper and a lower bound for $U(C_{\text{poly}}(C_0, N))$.

Upper bound. By Theorem 40 and the multinomial theorem, we get

$$\begin{aligned} U(C_{\text{poly}}(C_0, N)) &\leq \sum_{\substack{0 \leq a_1, \dots, a_m \leq N \\ a_1 + \dots + a_m = N}} \binom{N}{a_1, \dots, a_m} \cdot \prod_{k=1}^m \left(t_{m-k}(\overline{C_0}) \cdot 2^k(k+1) \right)^{a_k} \\ &= \left(\sum_{k=1}^m t_{m-k}(\overline{C_0}) \cdot 2^k(k+1) \right)^N. \end{aligned}$$

The claimed upper bound now follows by recalling that the poly- C_0 chain has $n = Nm$ chain edges.

Lower bound. The entropy bound for multinomial coefficients states that

$$\binom{N}{a_1, \dots, a_m} \geq \frac{1}{N^m} e^{H(\frac{a_1}{N}, \dots, \frac{a_m}{N})}, \quad H(\alpha_1, \dots, \alpha_m) = - \sum_{k=1}^m \alpha_k \ln(\alpha_k).$$

By Theorem 36 and the above entropy bound, we get for some constant c_m that

$$\begin{aligned} U(C_{\text{poly}}(C_0, N)) &\geq \frac{1}{(N+1)^{c_m}} \sum_{\substack{0 \leq a_1, \dots, a_m \leq N \\ a_1 + \dots + a_m = N}} \binom{N}{a_1, \dots, a_m} \cdot \prod_{k=1}^m \left(t_{m-k}(\overline{C_0}) \cdot 2^k(k+1) \right)^{a_k} \\ &\geq \frac{1}{(N+1)^{c_m+m}} \sum_{\substack{0 \leq a_1, \dots, a_m \leq N \\ a_1 + \dots + a_m = N}} e^{H(\frac{a_1}{N}, \dots, \frac{a_m}{N})} \cdot \prod_{k=1}^m \left(t_{m-k}(\overline{C_0}) \cdot 2^k(k+1) \right)^{a_k}. \end{aligned}$$

We can lower bound the right-hand side by picking any one summand. We pick the largest one by using Lemma 37 (we might have to round the a_k slightly to get integer values, but the effect of this vanishes as N goes to infinity). As desired, the resulting lower bound is therefore

$$U(C_{\text{poly}}(C_0, N)) = \Omega \left(\frac{1}{(N+1)^{c_m+m}} \left(\sum_{k=1}^m t_{m-k}(\overline{C_0}) \cdot 2^k(k+1) \right)^N \right) \quad (\text{as } N \rightarrow \infty).$$

4.4 Proof of Theorem 30 (Twin Chains)

Let C_0 again be any fixed chain with m chain edges and let $N \geq 2$. Observe then with Lemma 22 that we have in fact $T_{C_{\text{poly}}(C_0, N)}(x) = T_{C_0}(x)^N$ for the flipped poly- C_0 chains. Therefore, expanding the right-hand side with the multinomial theorem yields the coefficients

$$Y_{k,N} := t_k(\overline{C_{\text{poly}}(C_0, N)}) = \sum_{\substack{0 \leq a_1, \dots, a_m \leq N \\ a_1 + \dots + a_m = N \\ (m-1)a_1 + \dots + a_{m-1} = k}} \binom{N}{a_1, \dots, a_m} \prod_{\ell=1}^m t_{m-\ell}(C_0)^{a_\ell}.$$

On the other hand, applying Lemma 42 twice to the twin- C_0 chains yields

$$T_{C_{\text{twin}}(C_0, N)} = \sum_{k_1=0}^{mN-1} \sum_{k_2=0}^{mN-1} Y_{k_1, N} \cdot Y_{k_2, N} \cdot x^{k_1+k_2} \cdot T_{C_{\text{ccv}}(mN-k_1) \vee E \vee C_{\text{ccv}}(mN-k_2)}(x),$$

with leading coefficients

$$\begin{aligned} U(C_{\text{twin}}(C_0, N)) &= \sum_{k_1=0}^{mN-1} \sum_{k_2=0}^{mN-1} \gamma_{k_1, N} \cdot \gamma_{k_2, N} \cdot U(C_{\text{ccv}}(mN - k_1) \vee E \vee C_{\text{ccv}}(mN - k_2)) \\ &= \sum_{k_1=0}^{mN-1} \sum_{k_2=0}^{mN-1} \gamma_{k_1, N} \cdot \gamma_{k_2, N} \cdot \binom{2mN - k_1 - k_2}{mN - k_1}, \end{aligned}$$

where the last step is the same as in the analysis of the classic double chain; see [17]. We will again use the above equation to prove both an upper and a lower bound for $U(C_{\text{twin}}(C_0, N))$.

Upper bound. After bounding the binomial coefficient by $2^{2mN-k_1-k_2}$, we can factorize to

$$U(C_{\text{twin}}(C_0, N)) \leq \left(\sum_{k=0}^{mN-1} \gamma_{k, N} \cdot 2^{mN-k} \right)^2.$$

Plugging in the definition of $\gamma_{k, N}$ and noting that $mN - k = 1a_1 + 2a_2 + \dots + ma_m$, we can again simplify with the multinomial theorem, similar to what we did for poly- C_0 chains, and get

$$U(C_{\text{twin}}(C_0, N)) \leq \left(\sum_{k=1}^m t_{m-k}(C_0) \cdot 2^k \right)^{2N}.$$

Lower bound. By only keeping terms with $k_1 = k_2$, we get

$$\begin{aligned} U(C_{\text{twin}}(C_0, N)) &\geq \sum_{k=0}^{mN-1} \gamma_{k, N}^2 \cdot \binom{2mN - 2k}{mN - k} \\ &\geq \sum_{k=0}^{mN-1} \gamma_{k, N}^2 \cdot \frac{2^{2mN-2k}}{2^{mN}} = \frac{4^{mN}}{2^{mN}} \sum_{k=0}^{mN-1} (2^{-k} \cdot \gamma_{k, N})^2. \end{aligned}$$

By dropping all the summands except for one specific k to be determined later, and by plugging in the definition of $\gamma_{k, N}$ and again noting that $mN - k = 1a_1 + 2a_2 + \dots + ma_m$, we get

$$\begin{aligned} U(C_{\text{twin}}(C_0, N)) &\geq \frac{4^{mN}}{2^{mN}} (2^{-k} \cdot \gamma_{k, N})^2 = \frac{1}{2^{mN}} (2^{mN-k} \cdot \gamma_{k, N})^2 \\ &= \frac{1}{2^{mN}} \left(\sum_{\substack{0 \leq a_1, \dots, a_m \leq N \\ a_1 + \dots + a_m = N \\ (m-1)a_1 + \dots + a_{m-1} = k}} \binom{N}{a_1, \dots, a_m} \prod_{\ell=1}^m (t_{m-\ell}(C_0) \cdot 2^\ell)^{a_\ell} \right)^2. \end{aligned}$$

After applying the entropy bound, we pick the largest summand for the best possible k via Lemma 37 and get

$$U(C_{\text{twin}}(C_0, N)) = \Omega \left(\frac{1}{2^{mN^{m+1}}} \left(\sum_{\ell=1}^m t_{m-\ell}(C_0) \cdot 2^\ell \right)^{2N} \right) \quad (\text{as } N \rightarrow \infty).$$

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