Flipping Non-Crossing Spanning Trees

Two extended abstracts

The following abstracts are based on a seminar presentation by Lucas Burggraf, covering a selection of results of a 2024 paper titled *Flipping Non-Crossing Spanning Trees* by Håvard Bakke Bjerkevik, Linda Kleist, Torsten Ueckerdt and Birgit Vogtenhuber.

Abstract for first year math students

Imagine you are designing a fiber-optic communication network to connect n fixed cities on a flat map. To save on cable, you want to use the absolute minimum number of straight-line cables needed so that every city can still communicate with every other city. Mathematically, this minimal structure is a *spanning tree*. To avoid interference, you have one crucial constraint: no two cables may cross each other.

Now suppose an underground line breaks, or a new routing policy requires you to change the network layout. You cannot tear down the whole network overnight. Instead, you must transition gradually: remove one cable and add a new one to restore connectivity, making sure no cables cross at any point.

This brings us to a central question in discrete and combinatorial geometry: Given two valid network designs on the same set of cities, what is the minimum number of single-cable swaps needed to transform the first design into the second? This minimal number of moves is called the *flip distance*.

To more precisely discuss this topic, we need to first introduce some key concepts. The set P is a set of n points in the flat 2-dimensional plane, no three points of which are on the same straight line. This condition is often also referred to as points in general position. We also define a *non-crossing spanning tree* T as a graph whose vertices (also called nodes) are the points in P with exactly $n-1$ straight-line edges such that there is a path between any two vertices and none of the edges cross each other. For simplicity we will consider a tree to be the same as its corresponding edge set.

For any given tree T, and an edge $e \in T$ we can consider removing the edge and adding a new edge $e' \notin T$ to the tree. If this swap yields a new tree, i.e. a connected graph on n vertices with $n-1$ edges and no two edges crossing, then we consider this to be a valid flip. For the newly obtained tree $T' = (T \setminus \{e\}) \cup \{e'\}$ we now know it is related to T by a single flip and thus we can say that the *flip-distance* of these two trees is equal to 1. We can write this as $\operatorname{dist}(T, T') = 1$. This notation can be used generally to denote the minimal number of flips needed to get from any tree T_1 to T_2 as $\operatorname{dist}(T_1, T_2)$.

If we consider every possible non-crossing spanning tree on a given point set P, we can define this to be a graph as well. In this case the vertices each represent a certain non-crossing spanning tree and we can add an edge between two such vertices if the corresponding trees have flip-distance one. The obtained structure is called a *flip graph* and we denote it by $\mathcal{F}(P)$.

When studying flip graphs, one of the main questions concerns the worst-case distance two trees can have. This is called the diameter of the flip graph and we denote it by $\mathrm{diam}(\mathcal{F}(P))$. For a point set P in general position the diameter has been shown to be between $\lfloor 3/2 \cdot n \rfloor - 5$ and $2n - 4$. The lower of the two bounds was improved to $14/9 \cdot n - O(1)$ in the paper mentioned in the preamble. We have a better estimate of the diameter if we consider the set of points to be in convex position, to simplify we can imagine the points distributed along a circular path. For this case we can show that the diameter is somewhere in between $14/9 \cdot n - O(1)$ and $5/3 \cdot n - 3$, which is a significant improvement on the previous bounds in this case.

The most important tool, that was used to improve the bounds in the case of points in convex position was, to introduce another layer of abstraction in the form of a so-called conflict graph. This is a graph that tries to represent conflicts that different possible flips in the underlying non-crossing spanning tree could create. That is, for example if swapping one edge with another would create a crossing we can have this information stored in the conflict graph. Finding an efficient way to resolve these conflicts is an essential step in reducing the number of flips needed in the flip graph.

Abstract for experts in the field of Discrete and Combinatorial Geometry

For a set P of n points in general position in the plane, the flip graph $\mathcal{F}(P)$ has a vertex for each non-crossing spanning tree on P and an edge between any two trees that differ by a single edge exchange. The paper mentioned in the preamble presents significantly improved lower and upper bounds on the diameter of $\mathcal{F}(P)$ for points in convex position. These results also improve on the lower bound for points in general position. The lower bound was improved to $^{14}/_{9} \cdot n - O(1)$. The upper bound on $\operatorname{diam}(\mathcal{F}(P))$ for points in convex position was shown to be at most $^{5}/_{3} \cdot n - 3$.

Central to the proofs is a reformulation of the problem of finding minimum distance paths in the flip graph. The key insight is to find large acyclic subsets in an associated directed *conflict graph* $H(T, T')$, for given trees T and T'.

Let P be a set of n points in the plane in general position. A non-crossing spanning tree on P is a spanning tree whose edges are straight line segments that do not cross except at common endpoints. An edge flip transforms a tree T into another non-crossing spanning tree T' by deleting one edge $e \in T$ and inserting an edge $e' \notin T$. The flip graph $\mathcal{F}(P)$ has as its vertex set all non-crossing spanning trees on P, with edges joining pairs related by a single flip.

When P is in convex position, all n-element sets yield isomorphic flip graphs, which we denote by $\mathcal{F}_n$. Determining the diameter $\operatorname{diam}(\mathcal{F}_n)$ has been an active area of research. A previously established lower bound of $\lfloor ^{3}/_{2} \cdot n \rfloor - 5$ stood unimproved for 25 years. Similarly following long-standing $2n - O(1)$ bounds, a recent breakthrough reduced the upper bound to $\approx 1,95n$. Both of these bounds were improved, as mentioned above, using the conflict graph method. Now the gap between the lower and upper bounds has narrowed to $\sim {}^{1}/_{9} \cdot n$.

To establish both the lower and upper bounds, a directed auxiliary structure termed the conflict graph $H(T, T')$ is introduced. For a given pair of trees T and T' we first associate edges of one tree with edges of the other in a one-to-one correspondence, that will prescribe which kind of flip we will preform. Then we define the vertices in $H(T, T')$ to correspond to certain edge pairs, that might cause issues if flipped without caution. Directed arcs encode required dependencies: an arc $(e_1, e_1') \to (e_2, e_2')$ indicates that the flip e_1 to e_1' has to happen before the flip e_2 to e_2'. If the order of these flips were to be reversed a cycle or a crossing would be created.

Finding a minimal flip sequence from T to T' is asymptotically equivalent to identifying a maximum-cardinality acyclic subgraph in $H(T, T')$. Up to lower-order additive terms, this reduces the geometric problem of spatial non-crossing constraints to a graph-theoretic optimization over $H(T, T')$, offering a systematic toolkit for analyzing flip graph diameters. These methods could possibly also be used to resolve other reconfiguration problems.