Multidimensional Sorting

Extended Abstract

Sorting is one of the most familiar operations in mathematics and computer science. Given a list of real numbers, sorting places them in increasing order and thereby records all their pairwise order relations. This is more informative than merely finding the minimum and maximum: a sorted list captures the complete order pattern of the data. In their paper *Multidimensional Sorting*, Goodman and Pollack ask what the corresponding idea should be for a finite set of points in the plane or in higher-dimensional space. Since points in the plane do not have a single natural left-to-right order, the usual notion of sorting must be replaced by a geometric one.

The key idea is that the "order" of a point configuration is encoded by orientation. In the plane, three points may form a counterclockwise turn, a clockwise turn, or be collinear. If this information is known for every ordered triple of points, then one knows the *order type* of the configuration. This order type describes the combinatorial shape of the point set: it records how the points are arranged relative to one another, without depending on exact distances or angles. In d-dimensional space, the analogous information is obtained from the orientations of all $(d+1)$-tuples of points.

Goodman and Pollack introduce a procedure called *geometric sorting*. Instead of listing all orientations directly, they consider a more compact counting description. In the plane, take every directed line determined by two points P_i and P_j, and count how many of the remaining points lie to the left of this directed line. These numbers define a function

$$\lambda(i, j).$$

At first sight, this counting information seems weaker than knowing exactly which points lie on which side. The central result of the paper shows that this impression is misleading: the complete order type of the configuration can be reconstructed from these counts. Thus, knowing how many points lie on the positive side of each directed line, or more generally each oriented hyperplane in higher dimensions, is enough to recover the full geometric order structure.

This result is the *Basic Theorem of Geometric Sorting*. It is particularly striking because it works not only for points in general position, but also in degenerate cases, such as collinear triples in the plane or coplanar quadruples in three-dimensional space. The theorem therefore gives a robust analogue of ordinary sorting: just as sorting numbers produces a compact representation of their order relations, geometric sorting produces a compact representation of the order relations among points.

The paper also provides an efficient algorithm for carrying out this sorting process. For n points in $\mathbb{R}^d$, Goodman and Pollack compute the corresponding geometric sorting data in time

$$O(n^d \log n).$$

When $d = 1$, this becomes the classical $O(n \log n)$ time bound for sorting numbers, so the algorithm genuinely extends ordinary sorting to higher dimensions.

A second major problem considered in the paper is the comparison of configurations. Two point sets may be numbered differently and still have the same order type. Naively, one might have to test all $n!$ possible renumberings. Goodman and Pollack show that this is unnecessary: by using certain distinguished orderings of the points, called *canonical orderings*, the number of relevant renumberings can be reduced dramatically. This makes it possible to decide efficiently whether two configurations have the same geometric order structure.

Geometric sorting is therefore both a conceptual and an algorithmic generalization of ordinary sorting. It replaces the linear order of numbers by the order type of a point configuration, gives a compact way to encode this information, and provides efficient methods for computing and comparing it. The authors also point to applications in pattern recognition, stereochemistry, and cluster analysis, where one often wants to compare shapes or configurations by their relative structure rather than by their precise metric measurements.