Multidimensional Sorting

Extended Abstract

In *Multidimensional Sorting*, Goodman and Pollack develop an analogue of ordinary sorting for finite point configurations in $\mathbb{R}^d$. Rather than recovering a linear order, geometric sorting aims to determine the *order type* of a labeled configuration

$$P = (P_1, \ldots, P_n),$$

that is, the orientation of every ordered $(d+1)$-tuple of points. The resulting structure captures the combinatorial geometry of the configuration independently of its particular coordinates and is equivalent to the oriented matroid structure realized by P.

For an ordered d-tuple

$$I = (i_1, \ldots, i_d),$$

let

$$\Lambda(I) = \{ j : (P_{i_1}, \ldots, P_{i_d}, P_j) \text{ is positively oriented} \}.$$

Thus, $\Lambda(I)$ specifies the points lying on the positive side of the oriented hyperplane determined by $P_{i_1}, \ldots, P_{i_d}$. Its associated counting function is

$$\lambda(I) = |\Lambda(I)|.$$

In the planar case, $\Lambda(i, j)$ is simply the set of points to the left of the directed line P_iP_j, while $\lambda(i, j)$ records only their number. Clearly, knowledge of Λ determines λ; the converse is the fundamental observation underlying geometric sorting.

The *Basic Theorem of Geometric Sorting* (Theorem 1.8) states that the λ-data determine the complete Λ-structure, and hence the order type of the configuration. Remarkably, the cardinalities of the positive half-spaces therefore contain the same order information as the corresponding sets themselves. The result holds in arbitrary dimension and does not require general position, allowing repeated points and affine dependencies such as collinear or coplanar subsets.

Goodman and Pollack then turn to the algorithmic computation of this representation. In the plane, the values $\lambda(i, j)$ for all ordered pairs can be obtained in

$$O(n^2 \log n)$$

time by sorting the remaining points according to their directions around each point. The higher-dimensional construction builds on the planar procedure and computes the complete λ-function for n points in $\mathbb{R}^d$ in

$$O(n^d \log n)$$

time. For $d = 1$, this specializes to the optimal $O(n \log n)$ bound for ordinary sorting, establishing geometric sorting as a genuine multidimensional generalization of the classical problem.

The paper subsequently considers the problem of comparing configurations whose points are arbitrarily labeled. Instead of considering all $n!$ possible relabelings, Goodman and Pollack introduce a distinguished family of *canonical orderings*. These are constructed from face flags of the convex hull and depend only on the geometric order structure of the configuration. The number of necessary canonical orderings is at most n in dimension two and $6n - 12$ in dimension three; for fixed higher dimension d, it is bounded by

$$c_d n^{\lfloor d/2 \rfloor}.$$

Thus, the factorial family of possible labelings is replaced by a polynomially bounded collection determined by the combinatorics of the convex hull.

Finally, these canonical orderings are used to solve the *comparison problem*: deciding whether two independently labeled configurations in $\mathbb{R}^d$ have the same order type. By generating the relevant canonical orderings and comparing their geometric sorting representations, order-type equivalence can be tested without enumerating arbitrary permutations. The paper thereby combines a compact encoding of order type, an efficient multidimensional sorting procedure, and a canonicalization method for configuration comparison, providing an algorithmic framework for the combinatorial classification of point sets.