Technische Universität Berlin Winter 2025/26
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Graph Theory (DS II) - Sheet 10

Exercise 10.1.

A graph G is outerplanar, if there exists a planar drawing of G, such that every vertex is incident to the outer face. Prove that the maximum number of edges of an outerplanar graph is exactly $2n-3$ if $n \geq 2$.
Show: Maximal outerplanar graphs are Laman graphs. Does the reverse also hold?

Exercise 10.2.

For any sets P of n points and L of m lines in the plane, let $I(P, L)$ denote the set of incidences, that is

$$I(P, L) = \{(p, \ell) \in P \times L \mid p \in \ell\}.$$

Prove that $|I(P, L)| \leq 4 \max(n^{\frac{2}{3}} m^{\frac{2}{3}}, n) + m$.
Hint: Make a sketch and find a graph with n vertices and $I(P, L) - m$ edges. Consider the crossing lemma.

Exercise 10.3.

Let the crossing edge number of a drawing Γ of a graph G be the number of edges in Γ that are crossed, no matter how many times they are crossed. Let $cre(G)$ denote the minimum of all crossing edge numbers of drawings of G.

(a) Prove that $cre(G) \leq 2cr(G)$.

(b) Find an example of a graph G with $cre(G) < 2cr(G)$

Exercise 10.4.

Adopt the construction of Moon to prove the existence of a drawing of $K_{n,m}$ with at most $\frac{1}{4}\binom{n}{2}\binom{m}{2}$ crossings.

Bonus Exercise

Consider a closed curve $\gamma\colon [0, 1] \to \mathbb{R}^2$ in the plane with a finite number of simple self crossings and without other self intersections. It cuts the plane into a finite number of regions. Consider the following graph. The vertices are the regions and two regions are connected by an edge if one can walk from one to the other by crossing the curve exactly once.
Find at least 3 interesting properties of these graphs. Can you characterize them?