9. Übungsblatt zur Vorlesung: **Felsner/ Schröder**
Graphentheorie (DS II) 3. Januar 2022

Besprechungsdatum: 10. /13. Januar
`http://www.math.tu-berlin.de/~felsner/Lehre/dsII21.html`

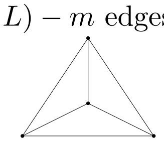
---

(1) Outerplanar graphs:
A graph G is *outerplanar*, if there exists a planar drawing of G, such that every vertex is at the outer face. Prove that the maximum number of edges of an outerplanar graph is exactly $2n-3$ if $n \geq 2$.
[Remark: Maximal outerplanar graphs are Laman graphs.]

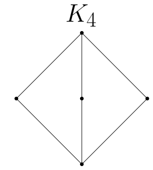
(2) Crossing Lemma:
For any sets P of n points and L of m lines in the plane, let $I(P,L)$ denote the set of incidences, that is

$$I(P,L) = \{(p,\ell) \in P \times L \mid p \in \ell\}$$

Prove that $|I(P,L)| \leq 4\max(n^{\frac{2}{3}}m^{\frac{2}{3}}, n) + m$.
[Hint: Make a sketch and find a graph with n vertices and $I(P,L) - m$ edges.]



K_4



$K_{2,3}$

(3) Kuratowski and Wagner for outerplanar graphs:
Prove that the following is equivalent:

(a) G is outerplanar.

(b) G has no subdivision of K_4 or $K_{2,3}$ as a subgraph.

(c) G does not have K_4 or $K_{2,3}$ as a minor.

(4) Adapt the random construction of Moon to obtain a drawing Γ of $K_{n,m}$ such that

$$\mathbb{E}(\mathrm{cr}(\Gamma)) = \frac{1}{4}\binom{n}{2}\binom{m}{2}$$

[Remark: It is conjectured that $\mathrm{cr}(K_{n,m}) = \lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$, only slightly less!]

(5) An application of k-planar graphs:
A graph G is k-planar, if there is a drawing of G in the plane, such that every edge is crossed at most k times. Prove the following variant of the crossing lemma for graphs G with n vertices and $m \geq 6n$ edges:

$$\mathrm{cr}(G) \geq \frac{1}{36}\frac{m^3}{n^2}$$

You may use the fact that every 1-planar graph has at most $4n-8$ edges and every 2-planar graph has at most $5n-10$ edges, see figures to the right for examples.