6. Übungsblatt zur Vorlesung: **Felsner/ Schröder**
Graphentheorie (DS II) 29. November 2021

Besprechungsdatum: 6. /9. Dezember
`http://www.math.tu-berlin.de/~felsner/Lehre/dsII21.html`

(1) Say there are n parking lots in a row, numbered $1, \ldots, n$ and n cars in a waiting line to get a parking spot. Each car (driver) C_i has a favourite spot a_i, that they try to get, as soon as it is their turn. If a_i is already taken, C_i parks in the next free parking lot, if there is any left. We call $(a_1, \ldots, a_n)$ *parking function* of length n if all cars will be parked when this strategy is applied.
Show: If $(a_1, \ldots, a_n)$ is a parking function, then so is $(a_{\pi(1)}, \ldots, a_{\pi(n)})$ for all permutations $\pi \in S_n$.

(2) Let T be a tree. Let G be the graph that contains a vertex for every vertex-edge incidence, that is for every pair $(v, e) \in V(T) \times E(T)$, such that $v \in e$, and an edge whenever two such incidences share the vertex or the edge, more formally

$$E(G) = \{\{(v, e), (v', e')\} \subseteq V(T) \times E(T) \mid v = v' \vee e = e'\}$$

Prove that its square $G^2 = (V(G), \{\{u, w\} \mid \text{dist}_G(u, w) \leq 2\})$ has a Hamilton cycle.

(3) Graph parameters, Turán graphs

(a) Find a graph G, whose parameters $\alpha(G), \omega(G), \chi(G), \theta(G)$ are pairwise different.

(b) Find $k, n \in \mathbb{N}$ such that $T_k(n)$ is the smallest Turán graph (with respect to the number of edges), that contains the Petersen graph as a subgraph.

(c) Find $k, n \in \mathbb{N}$ such that $T_k(n)$ is the smallest regular Turán graph (with respect to the number of edges), that contains the Petersen graph as a subgraph.

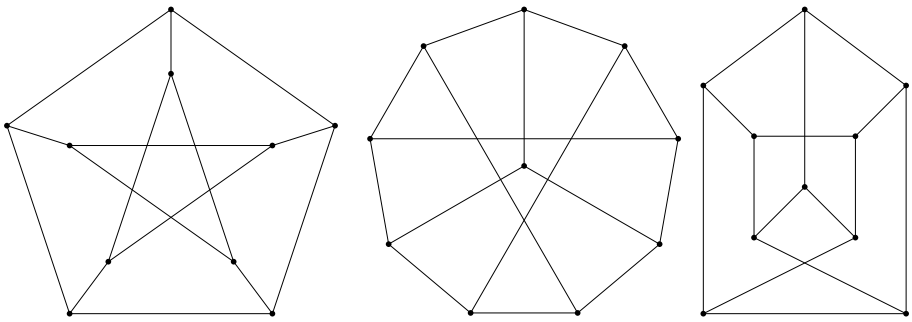

Figure 1: These are some of the more beautiful drawings of the Petersen graph.

(4) A word $a_1 a_2 \cdots a_n$ is *repetition-free*, if no letter appears twice, that is $a_i \neq a_j \; \forall i \neq j$. Let $S_n(m)$ be the *shift graph* on the repetition-free words of length n using the alphabet $\Omega = \{0, \ldots, m-1\}$: Between two words v and w, there is a directed edge (v, w) if and only if there are some $v_0, ..., v_n \in \Omega : v = v_0, \ldots, v_{n-1}$ and $w = v_1, \ldots, v_n$.

(a) For which parameters n, m is $S_n(m)$ Eulerian?

(b) For which parameters n, m is $S_n(m)$ Hamiltonian?