Übungsblatt " Graphs, Order, and Geometry" SoSe 2024

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(1)

a. Let G be a comparability graph, show that G contains no odd cycle of length $\geqslant 5$ as induced subgraph.

b. Let G be a incomparability graph, show that G contains no odd cycle of length $\geqslant 5$ as induced subgraph.

(2) Characterize the down-set lattices of orders of width two.

(3) Let G be a graph with a girth $girth(G)$. Show that if $girth(G) > \chi(G)$, then G is a cover graph.

(4) For each m construct a partial order P_m with $\binom{m+1}{2}$ elements such that if $B_1, \ldots, B_k$ is a cover of P_m with the property that each B_i is a chain or an antichain, then $k \geqslant m$. (Later we will see that Greene-Kleitman Theory implies that every order with less than $\binom{m+1}{2}$ elements has such a cover with $k < m$.)

(5) Prove that the following conditions are equivalent:

a. G is a comparability graph of a poset of dimension at most 2;

b. G is a containment graph of intervals on a line;

c. G is a permutation graph.

(6) Let P be a poset and C be a chain in P. Prove that

$$\dim(P) \leqslant \dim(P - C) + 2.$$

(7) Let M be a subset of maximal elements of a poset P. Let $width(P \setminus M) \leqslant w$. Show that

$$\dim(P) \leqslant w + 1.$$

(8) Given a 2-dimensional poset P, find good time bounds for computing

a) The skeleton of P

b) The Ferrer's shape $\text{Fer}(P)$ of P.

c) A maximum k-chain in P.

(9) Find out about a result obtained by Logan–Shepp and Vershik–Kerov. What does it say about posets?

(10) Identify the faces of the order polytope $\mathcal{O}(P)$, i.e., describe a class of combinatorial objects which are in bijection to the faces. *Hint*: It may be helpful to think of $\hat{P}$, i.e., of P enriched with additional global $\mathbf{0}$ and $\mathbf{1}$ elements.

(11) What are the facets of the chain polytope $\mathcal{C}(P)$?

(12) For a graph $G = (V, E)$ we define

$$\mathcal{C}(G) = \{a \in [0,1]^V \mid \sum_{v \in C} a_v \leqslant 1 \quad \text{for every clique } C \text{ of } G\}.$$

If P is an order with comparability graph G, then $\mathcal{C}(P) = \mathcal{C}(G)$.

Show that in general $\mathcal{C}(G)$ may have corners which are not characteristic vectors of stable sets.