Dimension of Polytopes

Let P be a polytope with vertex set $\mathcal{V}(P)$ and facets $\mathcal{F}(P)$. Given a subset $\mathcal{G}$ of $\mathcal{F}(P)$ a *realizer* for $(P, \mathcal{G})$ is a nonempty family $\mathcal{R}$ of linear orders on $\mathcal{V}(P)$ provided

(∗∗) For every facet $F \in \mathcal{G}$ and every vertex $x \in \mathcal{V}(P) \setminus \mathcal{V}(F)$, there is some $L \in \mathcal{R}$ so that $x > y$ in L for every $y \in \mathcal{V}(F)$.

The *dimension* of $(P, \mathcal{G})$, denoted $\dim(P, \mathcal{G})$, is then defined as the least positive integer t for which $(P, \mathcal{G})$ has a realizer of cardinality t. In the case $\mathcal{G} = \mathcal{F}(P)$ we simply write $\dim(P)$ and call this the *dimension of the polytope P*.

Theorem 0.1 *If P is a d-polytope with $d \geq 2$, i.e., a polytope whose affine hull is d-dimensional, then* $\dim(P) \geq d + 1$.

Proof. The proof is by induction on d. If $d = 2$ then the vertices and facets of P have the structure of the cycle C_n for $n = |\mathcal{V}(P)|$. In this case $\dim(P) \geq 3$.

Let P be a d-polytope embedded in $\mathbb{R}^d$ for some $d > 2$ with realizer $L_1, L_2, \ldots, L_t$. Let v be the highest vertex in L_t and consider a hyperplane H which separates v from all the other vertices of P.

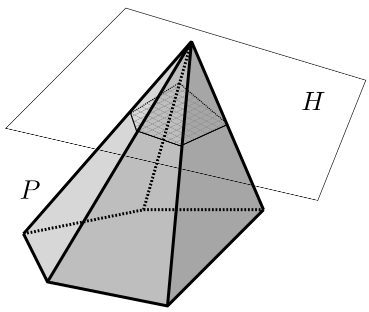

Figure. The *vertex figure* of the tip vertex of P.

The intersection $P \cap H$ is a $(d-1)$-polytope P/v, the so called *vertex figure* of P at v. The $(k-1)$-dimensional faces of P/v are in bijection with the k-dimensional faces of P that contain v. In particular an edge (u, v) of P corresponds to a vertex $u' = (u, v) \cap H$ of P/v and for every facet $\{u'_1, \ldots, u'_r\}$ of P/v there is a facet $\{v, u_1, \ldots, u_r, w_1, \ldots, w_s\}$ of P. Let $\mathcal{F}_v$ be the set of facets of P containing v. The correspondence $\mathcal{F}(P/v) \leftrightarrow \mathcal{F}_v$ shows that $\dim(P/v) \leq \dim(P, \mathcal{F}_v)$. Since P/v is $(d-1)$-dimensional $\dim(P/v) \geq d$ by induction. Now let $F \in \mathcal{F}_v$ and $w \notin \mathcal{V}(F)$, by the choice of v the order L_t cannot bring w over F. Therefore, $L_1, L_2, \ldots, L_{t-1}$ is a realizer for $(P, \mathcal{F}_v)$, i.e., $\dim(P, \mathcal{F}_v) \leq t - 1$. Combine the inequalities to deduce $t \geq d + 1$. □

It is known that for $d \geq 4$ a polytope in d-space can have a complete graph as skeleton (graph induced by 0-faces and 1-faces). Polytopes with this property are called *neighbourly*. We know that the dimension of complete graphs is unbounded. Monotonicity of dimension implies that for $d \geq 4$ there are d-polytopes with arbitrarily high dimension.