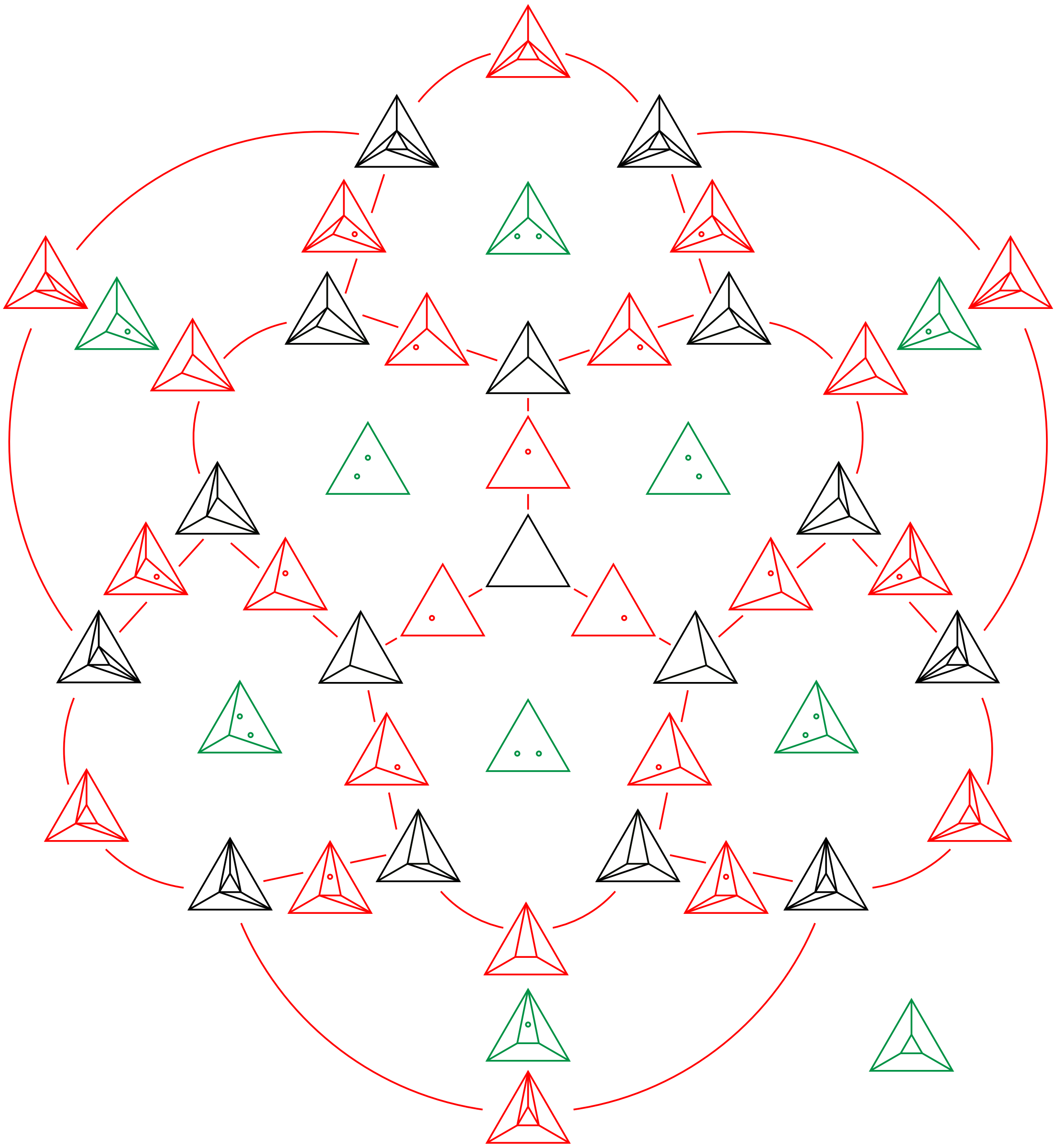

FIGURE 34. The refinement poset on regular subdivisions of the mother of all configuration.

**Definition 118.** The *volume vector* of a triangulation $T$ of a point set $\mathbf{P}$ is the vector of $\mathbb{R}^{\mathbf{P}}$ defined by

$$\Phi(T) := \left( \sum_{\mathbf{p} \in \triangle \in T} \mathrm{vol}(\triangle) \right)_{\mathbf{p} \in \mathbf{P}}$$

In other words, the coordinate corresponding to point $\mathbf{p} \in \mathbf{P}$ is the area of the star of $\mathbf{p}$ in $T$. The *secondary polytope* $\Sigma\mathsf{Poly}(\mathbf{P})$ of $\mathbf{P}$ is the convex hull of the volume vectors of all triangulations of $\mathbf{P}$,

$$\Sigma\mathsf{Poly}(\mathbf{P}) := \mathrm{conv}\left\{ \Phi(T) \mid T \text{ triangulation of } \mathbf{P} \right\}.$$

For example, the secondary polytope of the mother of all example is represented in Figure 35.

**Definition 119.** The *secondary cone* of a subdivision $S$ of a point set $\mathbf{P}$ is the polyhedral cone

$$C(S) := \left\{ \omega \in \mathbb{R}^{\mathbf{P}} \;\middle|\; S \text{ refines } S(\mathbf{P}, \omega) \right\}$$

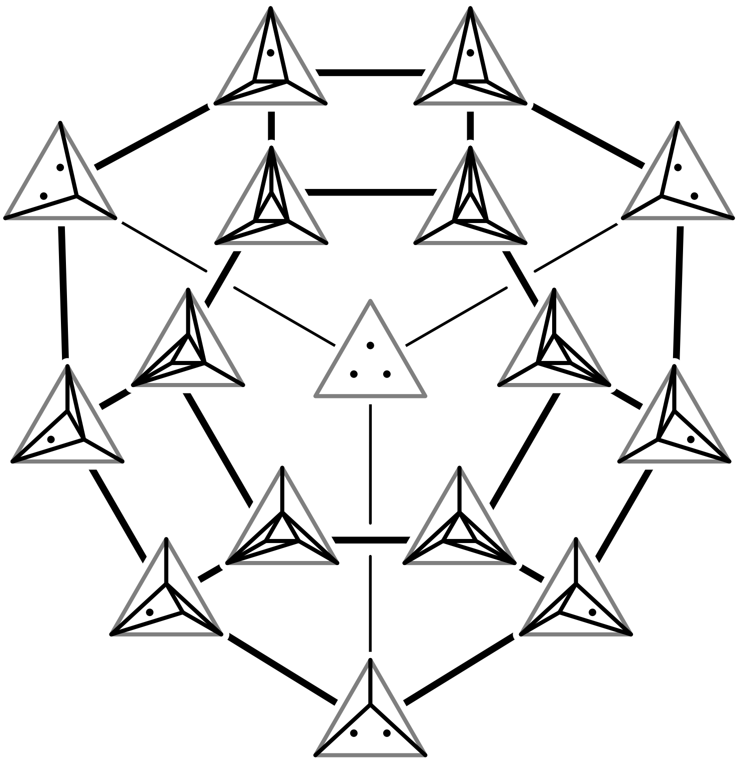

FIGURE 35. The secondary polytope of the mother of all example.

corresponding to all height functions whose lower convex hull projects to $S$. The *secondary fan* $\Sigma\mathsf{Fan}(\mathbf{P})$ of $\mathbf{P}$ is collection of the secondary cones of all subdivisions of $\mathbf{P}$,

$$\Sigma\mathsf{Fan}(\mathbf{P}) := \{C(S) \mid S \text{ subdivision of } \mathbf{P}\}.$$

It is a complete polyhedral fan.

**Example 120.** Consider the point configuration $\mathbf{P} = \{(0,0),(3,0),(0,3),(3,3),(1,1)\}$ of Example 114. The volume vectors of the four triangulations are given by:

$$(9, 9/2, 9/2, 9, 0) \qquad (9/2, 9, 9, 9/2, 0) \qquad (3, 15/2, 15/2, 9/2, 9/2) \qquad (3, 9/2, 9/2, 6, 9).$$

The projection of the secondary polytope $\Sigma\mathsf{Poly}(\mathbf{P})$ on the plane generated by the last two coordinate vectors is represented in Figure 36 (left). We look at what height functions produce the nine regular subdivisions of Figure 32. Without loss of generality (affine invariance), we restrict our attention to the height functions $\omega : \mathbf{P} \to \mathbb{R}$ with $\omega_1 = \omega_2 = \omega_3 = 0$. The nine regular subdivisions of Figure 32 then correspond to the following inequalities:

(i) $\omega_4 = \omega_5 = 0$,
(ii) $\omega_4 = 0$, $\omega_5 > 0$,
(iii) $\omega_4 > 0$, $\omega_5 = 0$,
(iv) $\omega_4 + 3\omega_5 = 0$, $\omega_5 < 0$,
(v) $\omega_4 < 0$, $\omega_4 - 3\omega_5 = 0$,
(vi) $\omega_4 < 0$, $\omega_4 - 3\omega_5 < 0$,
(vii) $\omega_4 > 0$, $\omega_5 > 0$,
(viii) $\omega_4 + 3\omega_5 > 0$, $\omega_5 < 0$,
(ix) $\omega_4 + 3\omega_5 < 0$, $\omega_4 - 3\omega_5 > 0$.

The corresponding secondary fan $\Sigma\mathsf{Fan}(\mathbf{P})$ is represented in Figure 36 (middle). Finally, the refinement poset of regular subdivisions of $\mathbf{P}$ is represented in Figure 36 (right).

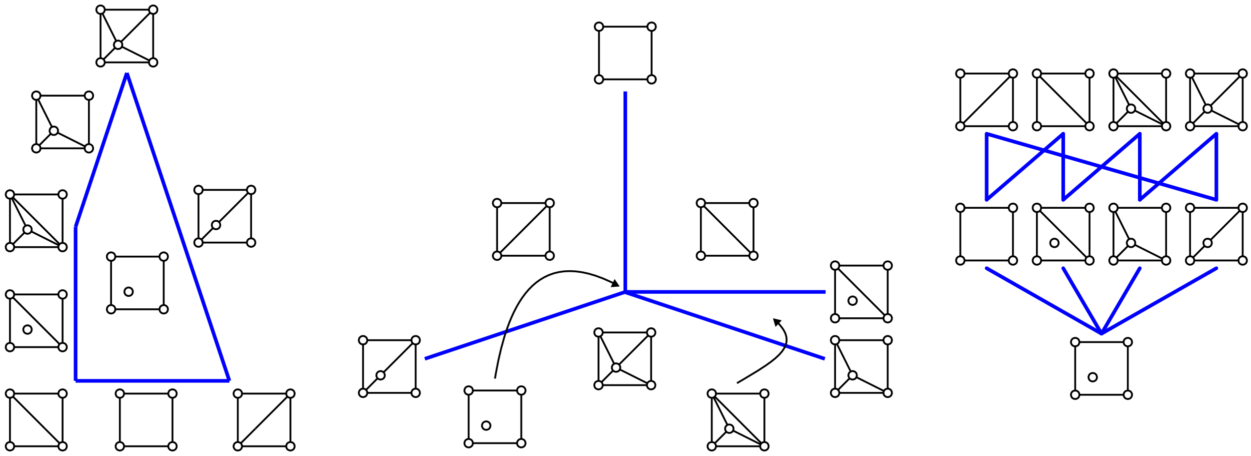

FIGURE 36. The secondary polytope (left), the secondary fan (middle), and the poset of regular subdivisions (right) of the set $\{(0,0),(3,0),(0,3),(3,3),(1,1)\}$.

**Theorem 121** (Gelfand, Kapranov, and Zelevinsky [GKZ94]). *Let $\mathbf{P}$ be a planar point set in general position.*

*(i) The dimension of the secondary polytope $\Sigma\mathsf{Poly}(\mathbf{P})$ is $|\mathbf{P}|-3$.*
*(ii) The secondary fan $\Sigma\mathsf{Fan}(\mathbf{P})$ is the inner normal fan of the secondary polytope $\Sigma\mathsf{Poly}(\mathbf{P})$.*
*(iii) The face lattice of the secondary polytope $\Sigma\mathsf{Poly}(\mathbf{P})$ is isomorphic to the refinement poset of regular subdivisions of $\mathbf{P}$.*

*Proof.* We start with (i). The lower bound on $\dim(\Sigma\mathsf{Poly}(\mathbf{P}))$ is obtained by induction on $|\mathbf{P}|$. It is clear when $|\mathbf{P}|=3$ since the secondary polytope is reduced to a single point. For $|\mathbf{P}|\geq 4$, consider an arbitrary point $\mathbf{p}\in\mathbf{P}$. If $\mathbf{p}$ lies in the convex hull of $\mathbf{P}\smallsetminus\mathbf{p}$, a triangulation $T$ of $\mathbf{P}$ is a triangulation of $\mathbf{P}\smallsetminus\mathbf{p}$ iff $\Phi(T)_{\mathbf{p}}=0$. Therefore,

$$\Sigma\mathsf{Poly}(\mathbf{P}\smallsetminus\mathbf{p})=\Sigma\mathsf{Poly}(\mathbf{P})\cap\left\{\mathbf{x}\in\mathbb{R}^{\mathbf{P}}\;\middle|\;x_{\mathbf{p}}=0\right\}.$$

Similarly, if $\mathbf{p}$ is on the convex hull of $\mathbf{P}$, we obtain that

$$\Sigma\mathsf{Poly}(\mathbf{P}\smallsetminus\mathbf{p})=\Sigma\mathsf{Poly}(\mathbf{P})\cap\left\{\mathbf{x}\in\mathbb{R}^{\mathbf{P}}\;\middle|\;x_{\mathbf{p}}=\mathrm{vol}(\mathrm{conv}(\mathbf{P}))-\mathrm{vol}(\mathrm{conv}(\mathbf{P}\smallsetminus\mathbf{p}))\right\}.$$

It immediately follows by induction that $\dim(\Sigma\mathsf{Poly}(\mathbf{P}))\geq|\mathbf{P}|-3$. To prove the reverse inequality, we exhibit three independent linear relations satisfied by the volume vectors of the triangulations of $\mathbf{P}$. First, since a triangulation $T$ of $\mathbf{P}$ decomposes the convex hull of $\mathbf{P}$ into triangles, we obtain:

$$\mathrm{vol}(\mathrm{conv}(\mathbf{P}))=\sum_{\triangle\in T}\mathrm{vol}(\triangle)=\sum_{\triangle\in T}\sum_{\mathbf{p}\in\triangle}\frac{\mathrm{vol}(\triangle)}{3}=\frac{1}{3}\sum_{\mathbf{p}\in\mathbf{P}}\sum_{\mathbf{p}\in\triangle\in T}\mathrm{vol}(\triangle)=\frac{1}{3}\sum_{\mathbf{p}\in\mathbf{P}}\Phi(T)_{\mathbf{p}}.$$

The other two linear relations are obtained from the center of mass $\mathrm{cm}(\mathrm{conv}(\mathbf{P}))$ of the convex hull of $\mathbf{P}$:

$$\mathrm{vol}(\mathrm{conv}(\mathbf{P}))\cdot\mathrm{cm}(\mathrm{conv}(\mathbf{P}))=\sum_{\triangle\in T}\mathrm{vol}(\triangle)\cdot\mathrm{cm}(\triangle)=\sum_{\triangle\in T}\mathrm{vol}(\triangle)\cdot\left(\frac{1}{3}\sum_{\mathbf{p}\in\triangle}\mathbf{p}\right)=\frac{1}{3}\sum_{\mathbf{p}\in\mathbf{P}}\Phi(T)_{\mathbf{p}}\cdot\mathbf{p},$$

since the center of mass of a triangle $\mathbf{pqr}$ coincides with its vertex barycenter $(\mathbf{p}+\mathbf{q}+\mathbf{r})/3$. Note that this equality between two points in the plane gives two independent relations.

We now prove (ii). Consider a lifting function $\omega:\mathbf{P}\to\mathbb{R}$ and a triangulation $T$ of $\mathbf{P}$. Let $f_{T,\omega}:\mathbb{R}^2\to\mathbb{R}$ denote the piecewise linear map such that $f_{T,\omega}(\mathbf{p})=\omega(\mathbf{p})$ for $\mathbf{p}\in\mathbf{P}$, and which is affine on each triangle of $T$. Then the volume below the surface defined by $f_{T,\omega}$ is

$$\int_{\mathrm{conv}(\mathbf{P})}f_{T,\omega}(\mathbf{x})\,d\mathbf{x}=\sum_{\triangle\in T}\int_{\triangle}f_{T,\omega}(\mathbf{x})\,d\mathbf{x}=\sum_{\triangle\in T}\frac{\mathrm{vol}(\triangle)}{3}\sum_{\mathbf{p}\in\triangle}\omega(\mathbf{p})=\frac{1}{3}\sum_{\mathbf{p}\in\mathbf{P}}\omega(\mathbf{p})\cdot\sum_{\mathbf{p}\in\triangle\in T}\mathrm{vol}(\triangle)=\frac{\langle\Phi(T)|\omega\rangle}{3}.$$

It follows that for any lifting function $\omega : \mathbf{P} \to \mathbb{R}$ and any triangulation $T$ of $\mathbf{P}$ distinct from the regular triangulation $S(\mathbf{P}, \omega)$ induced by $\omega$, we have

$$\langle \Phi(S(\mathbf{P}, \omega)) | \omega \rangle < \langle \Phi(T) | \omega \rangle.$$

Said differently, for any regular triangulation $T$ of $\mathbf{P}$, the normal cone of $\Phi(T)$ in $\Sigma\mathsf{Poly}(\mathbf{P})$ is the secondary cone $C(T)$ of $T$. This achieves the proof of Point (ii).

Finally, Point (iii) is immediate from (ii) and the definition of the secondary fan. $\square$

**Exercice 122.** Compute the volume vectors of all triangulations of the mother of all configuration. What happens to the volume vectors of the two non-regular triangulations? What happens if we slightly rotate the three outer vertices clockwise? (Hint: show that one of the two triangulations of Figure 33 becomes regular while the other remains non-regular). Deduce that some triangulations are non-regular even under small perturbations of their vertex sets.