A better proof of Theorem 8.9

In the book the *Laman Property* is defined on the basis of edge sets:

- $|E| = 2|V| - 3$ and $|E'| \leq 2|V[E']| - 3$ for all $\emptyset \neq E' \subset E$.

This definition is natural in the context of matroid theory but in our context it is unfortunate. The equivalent vertex based statement ist:

- $|E| = 2|V| - 3$ and $|E[X]| \leq 2|X| - 3$ for all $X \subset V$. (Laman Property)

For convenience we use the function $b(X) = 2|X| - 3 - |E[X]|$ which allows to restate the Laman Property as

- $b(V) = 0$ and $b(X) \geq 0$ for all $X \subset V$ with $|X| \geq 2$.

The hard part in the proof of Theorem 8.9 is the implication (3⇒1) which is restated in the following:

Proposition 0.1 *A graph with the Laman Property can be produced from a single edge by a sequence of* $\mathbf{H}_2$*- and* $\mathbf{H}_3$*-additions.*

Proof. Let G be a graph with the Laman Property. By induction it is enough to show that there is a graph G' with one vertex less, such that G' also has the Laman Property and G can be obtained from G' by an $\mathbf{H}_2$- or a $\mathbf{H}_3$-addition.

A simple counting argument shows that a graph with the Laman Property must have a vertex z of degree 2 or 3.

If $d(z) = 2$, then removing z and the two incident edges gives a graph G' with the Laman Property. Clearly, G is obtained from G' by $\mathbf{H}_2$-addition of z.

Suppose that $d(z) = 3$ and let $\{u, v, w\}$ be the neighbours of z. We begin with an easy observation:

($\star$) If $\{u, v, w\} \subset X$ and $z \notin X$, then $b(X) > 0$. Otherwise, if $b(X) = 0$, we would have $b(X + z) < 0$ which contradicts the Laman Property.

In particular the set $\{u, v, w\}$ can span at most two edges, i.e., $|E[u,v,w]| \leq 2$. We distinguish three cases.

Case 1: $|E[u,v,w]| = 2$.

Let (u, v) and (u, w) be edges. We claim that the graph G' obtained by deleting z and adding (v, w) has the Laman Property. Obviously G can be obtained from G' by $\mathbf{H}_3$-addition of z.

If G' is not Laman, then there is a set $X \subset V(G')$ such that $b_{G'}(X) < 0$. This implies $b_{G'}(X) \neq b(X)$, hence, $v, w \in X$, $z \notin X$ and $b(X) = 0$. From ($\star$) we obtain that $u \notin X$. It follows that $b(X+u+z) = 2(|X|+2)-3-|E[X]|-\#$ (edges in $E[X+u+z]$ incident to u or z) $\leq b(X) + 4 - 5 < 0$. The last inequality follows from the fact that all 5 edges spanned by $\{u, v, w, z\}$ are incident to u or z. The contradiction $b(X + u + z) < 0$ shows that G' has the Laman Property.

Case 2: $|E[u,v,w]| = 1$.

Let (u, v) and (u, w) be non-edges. Suppose that deleting z and adding (u, v) or (u, w) yields graphs which do not have the Laman Property. This implies that there are sets X_v and Y_w with $u, v \in X_v$, $z \notin X_v$ and $b(X_v) = 0$ and with $u, w \in Y_w$, $z \notin Y_w$ and $b(Y_w) = 0$. Again, ($\star$) implies $w \notin X_v$ and $v \notin Y_w$. Let $X = X_v - v$ and $Y = Y_w - w$.

In the following calculation we let $E[A|B]$ denote the set of edges with one end in A and one end in B.

$$\begin{aligned}
0 &= b(X+v)+b(Y+w)\\
&= 2(|X|+1)-3-|E[X]|-|E[X|v]|+2(|Y|+1)-3-|E[Y]|-|E[Y|w]|\\
&= (2|X\cap Y|-3)-|E[X\cap Y]|+(2(|X\cup Y|+2)-3)-|E[X\cup Y]|-|E[X|v]-|E[Y|w]|\\
&\geq b(X\cap Y)+(2(|X\cup Y|+2)-3)-|E[X\cup Y]|-|E[X\cup Y|v]-|E[X\cup Y|w]|-|E[v|w]|+|E[v|w]|\\
&= b(X\cap Y)+b(X\cup Y+v+w)+|E[v|w]|\\
&> 0.
\end{aligned}$$

For the last inequality note $b(X\cap Y)\geq -1$ with equality if $|X\cap Y|=1$ and $b(X\cup Y+v+w)\geq 1$ by $(\star)$ and $|E[v|w]|=1$ since $|E[u,v,w]|=1$. The contradiction proves that deleting z and adding either (u,v) or (u,w) yields a graph G' with the Laman Property. Obviously G can be obtained from G' by $\mathbf{H}_3$-addition of z.

Case 3: $|E[u,v,w]|=0$.

Suppose that deleting z and adding (u,v) or (u,w) or (v,w) yields three graphs which all fail to have the Laman Property. Then there are sets X and Y and Z with $u,v\in X$, $z\notin X$ and $b(X)=0$, with $u,w\in Y$, $z,u\notin Y$ and $b(Y)=0$ and with $v,w\in Z$, $z\notin Z$ and $b(Z)=0$. From $(\star)$ we know that $w\notin X$, $v\notin Y$ and $u\notin Z$. A calculation as in Case 2 only fails to produce the desired contradiction if $|X\cap Y|=|X\cap Z|=|Y\cap Z|=1$. In this case $(\star)$ implies $0<b(X\cup Y\cup Z)$ so that:

$$\begin{aligned}
0 &< b(X\cup Y\cup Z) \quad = \quad 2|X\cup Y\cup Z|-3-|E[X\cup Y\cup Z]|\\
&= 2(|X|+|Y|+|Z|-3)-3-|E[X]|-|E[Y]|-|E[Z]|-|E[X|Y]|-|E[Y|Z]|-|E[Z|X]|\\
&\leq b(X)+b(Y)+b(Z)\\
&= 0.
\end{aligned}$$

The contradiction proves that deleting z and adding either (u,v) or (u,w) or (v,w) yields a graph G' with the Laman Property. Obviously G can be obtained from G' by $\mathbf{H}_3$-addition of z. $\square$

Notes

- Besides being extremely unpleasant the old proof had the disadvantage that we did only prove the existence of a multigraph $G'=G_{i,j}$ with the property that G could be obtained from G' by an $\mathbf{H}_3$-addition.
- I'm thankful to András Frank for pointing out that the paper

  [93] A. FRANK AND L. SZEGŐ, *Constructive characterizations for packing and covering with trees*, Discr. Appl. Math., 131 (2003), pp. 347–371.

  containes a shorter proof of the equivalence of Laman Property and Henneberg Construction than the one given in the book. This new proof widely follows the approach taken there. The main result of [93] is a difficult generalization of this equivalence.