Topology
Winter term 2021/2022
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Exercise Session Sheet 9

Exercise 1 (Basics of affine linear algebra)

a) Recall what a convex combination resp. affine combination of points in $\mathbb{R}^n$ is.

b) Let $S \subset \mathbb{R}^n$ be any subset. Define the convex hull and the affine hull of S. Show that the convex hull of a compact set is compact (*trick question - guess a Theorem*).

c) A set of points $\{x_1, \ldots, x_m\} \in \mathbb{R}^m$ is called *affinely dependent* if one of the points x_i can be written as affine combination of the other points $\{x_1, \ldots, x_m\} \setminus \{x_i\}$. Otherwise, the set $\{x_1, \ldots, x_m\}$ is called *affinely independent*. Show that the set $\{x_1, \ldots, x_m\}$ is affinely dependent if and only if the set $\{(1, x_1), \ldots, (1, x_m)\} \subset \mathbb{R}^{n+1}$ is linearly dependent. (What is this statement good for?)

d) A simplex σ is the convex hull of affinely independent points $\{x_1, \ldots, x_m\}$. Show that any map $f\colon \{x_1, \ldots, x_m\} \to \mathbb{R}^k$ extends uniquely to an affine map defined on the affine hull of σ. How does this statement relate to the lecture?

Exercise 2
Let K be a (geometric) simplicial complex in $\mathbb{R}^n$.

a) The realization $|K|$ is closed and bounded subset of $\mathbb{E}^n$ and hence compact.

b) Each point in $|K|$ is contained in the relative interior of exactly one simplex.

Exercise 3
Let $\mathcal{S}$ be an abstract simplicial complex and let $\mathcal{T} \subset \mathcal{S}$ be a subcomplex. For now, let the realization of such complexes be given by the disjoint union of the simplices glued together along shared faces. Make this precise and construct a closed embedding $\iota\colon |\mathcal{T}| \to |\mathcal{S}|$.

Exercise 4
What is a *path between two vertices* in an abstract simplicial complex $\mathcal{S}$? Show that this induces an equivalence relation $\sim$. Let C be an equivalence class and $\sigma \in \mathcal{S}$ any simplex. Show that $\sigma \cap C = \emptyset$ or $\sigma \subset C$. Describe the connected components of $|\mathcal{S}|$.