Topology
Winter term 2021/2022
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Exercise Session Sheet 7

Definition. A map $\iota\colon A \to X$ is called *embedding* if ι is injective and the topology on A is the initial topology with respect to the map ι.

Exercise 1
Show that $\iota\colon A \to X$ is an embedding if and only if ι is a homeomorphism onto its image $\iota(A) \subset X$ equipped with the subspace topology.

Definition. Let X, Y be topological spaces, $A \subset X$ a subspace and $f\colon A \to Y$ a map. Let $\iota_X\colon X \to X+Y$ and $\iota_Y\colon Y \to X+Y$ be the canonical inclusions. Further, we define $\sim_f$ to be the equivalence relation on $X+Y$ *generated* by all relations of the form $\iota_X(a) \sim_f \iota_Y(f(a))$ for $a \in A$. We get the following maps

$$\chi\colon X \xrightarrow{\iota_X} X+Y \xrightarrow{q} X \cup_f Y,$$
$$\iota\colon Y \xrightarrow{\iota_Y} X+Y \xrightarrow{q} X \cup_f Y,$$

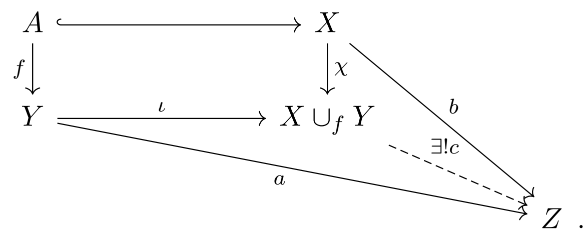
where $q\colon X+Y \to X \cup_f Y$ denotes the quotient map.

Exercise 2 Let the setup be given as in the above definition.

a) Show that $\iota\colon Y \to X \cup_f Y$ is an embedding.

b) Is $\chi\colon X \to X \cup_f Y$ an embedding? Prove your conjecture.

c) State and prove the universal mapping property for the space $X \cup_f Y$ and the maps χ, ι and f, which is indicated in the following commutative diagram:



In this situation we say that X is *glued* to Y along the map f.

Exercise 3
Show that $\mathbb{RP}^n = \mathbb{S}^n/{\sim_a}$ is homeomorphic to the space $\mathbb{D}^n/{\sim}$ where $\mathbb{D}^n$ is the closed unit ball $\mathbb{D}^n := \{x \in \mathbb{R}^n \colon \|x\| \leq 1\}$ and $\sim$ is the equivalence relation on $\mathbb{D}^n$ generated by setting $x \sim y$ if $x = \pm y$ holds on the boundary of $\mathbb{D}^n$, i.e. if $x, y \in \mathbb{S}^{n-1}$ are antipodal.

Exercise 4
Show that $\mathbb{RP}^2$ is obtained by glueing a 2-disk $\mathbb{D}^2$ to the sphere $\mathbb{S}^1 \subset \mathbb{C}$ along the map $u_2 \colon \mathbb{S}^1 \to \mathbb{S}^1, z \mapsto z^2$.

Exercise 5
Let G be a topological group and let $H < G$ be a subgroup.

a) Show that the factor map $q \colon G \to G/H$ is open. As usual, G/H denotes the set of left cosets $G/H := \{gH : g \in G\}$.

b) The space G/H is discrete precisely of H is an open subgroup.

c) Let G be a compact group and H be a closed subgroup of G. Show that H is open if and only if H has finite index in G

Exercise 6
Let G be a group and $S \subset G$ a subset. Describe the smallest normal subgroup of G containing S.