Topology
Winter term 2021/2022
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Exercise Session Sheet 6

Review of homework.

Exercise 1
Let $X := \mathbb{R}^n \setminus \{\mathbf{0}\}$ be the real euclidean space $\mathbb{R}^n$ with the origin $\mathbf{0}$ removed. We define an equivalence relation $\sim$ on X as follows: For $x, y \in X$ the relation $x \sim y$ holds precisely if x and y lie on a common 1-dimensional linear subspace $L_{x,y}$ of $\mathbb{R}^n$.

a) The quotient map $q\colon X \to X/{\sim}$ is open, i.e. it maps open sets to open sets.

b) Show that the quotient space $X/{\sim}$ is compact and Hausdorff.

Exercise 2
Give an example of a surjective map $f\colon X \to Y$ such that $X/{\sim_f}$ is not homeomorphic to Y.

Exercise 3
Let X be a set and J an arbitrary index set. For every $j \in J$, let Y_j be a topological space and let $f_j\colon Y_j \to X$ be a function. The statement dual to that of initial topologies yields **the** *final topology* $\mathcal{T}'$ on X, this time via functions $g\colon X \to Z$ for any topological space Z.

0) Make this precise with statements **A'** and **B'** and draw a commutative diagram.

a) Show that a final topology on X always exists.

b) Show that the final $\mathcal{T}'$ topology on X is always unique.

c) Determine a set-theoretic description for $\mathcal{T}'$, i.e. of the form $\mathcal{T}' = \{O \subset X\colon \ldots\}$.

Exercise 4
Formulate the statement of Exercise 3 for the coproduct $\coprod_{j\in J} Y_j$ in one sentence. Do you know an analogous statement in terms of, say, linear algebra, i.e. vector spaces and linear maps over $\mathbb{K}$?

Exercise 5
A polyhedron (or *polyhedral surface*) $S \subset \mathbb{R}^n$ as it was introduced in the first lecture can be pictured as a *cell complex* with V vertices, E edges and F faces. For each polygon of S create a copy of an appropriate polygon $p_k \subset \mathbb{R}^k$ and collect all these copies p_k in a set $\mathcal{S}$.
Describe an equivalence relation $\sim$ on the coproduct $\coprod_{p\in\mathcal{S}} p$, which records the cellular incidences on the surface S, i.e. which tracks the edges of S along the polygons. This sort of information is called *glueing data.*

Loops, homotopy, and mapping degree

Exercise 6
Let X be a topological space. Show that a loop in X is the same as a map $\mathbb{S}^1 \to X$.

Exercise 7
Show that if two maps $f, g\colon (X, p) \to (Y, q)$ are homotopic relative to $\{p\}$, then they induce the same maps on fundamental groups, i.e. $f_* = g_*$.

Let $\mathbb{S}^1 := \{z \in \mathbb{C}\colon |z| = 1\}$ and let $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ be a map. We chose $1 \in \mathbb{S}^1$ to be our base point and define a base point preserving variant $\overline{f}$ of f as follows:

$$\begin{aligned} \overline{f}\colon (\mathbb{S}^1, 1) &\longrightarrow (\mathbb{S}^1, 1) \\ z &\longmapsto \frac{f(z)}{f(1)} \ . \end{aligned}$$

Let $\mathbb{Z} \to \Phi\colon \pi_1(\mathbb{S}^1, 1)$ be the group **isomorphism** from the lecture. The *degree* $\deg(f)$ of f is the group homomorphism, which makes the following diagram commutative:

$$\begin{array}{ccc} \pi_1(\mathbb{S}^1, 1) & \xrightarrow{\quad \overline{f}_* \quad} & \pi_1(\mathbb{S}^1, 1) \\ \Phi \uparrow & & \uparrow \Phi \\ \mathbb{Z} & \xrightarrow{\quad \cdot \deg(f) \quad} & \mathbb{Z} \end{array}$$

Exercise 8
Let $f, g\colon \mathbb{S}^1 \to \mathbb{S}^1$ be homotopic maps. Show that $\deg(f) = \deg(g)$ holds.