Topology
Winter term 2021/2022
Prof. Dr. Michael Joswig, Holger Eble

Exercise Session Sheet 5

Review of homework

Exercise 1
Show that $\mathbb{Q}^d$ is not locally compact.

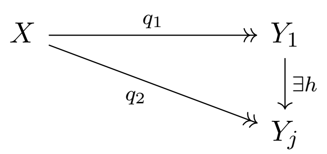
Exercise 2
Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be a compact Hausdorff spaces. Let $X \subset X_1, X_2$ be a non-compact subspace of both X_1 and X_2 such that $X_1 \setminus X = \{\infty_1\}$ and $X_2 \setminus X = \{\infty_2\}$ holds. Show that $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ are homeomorphic.

Exercise 3
Give two non-homeomorphic spaces with homeomorphic one-point compactifications.

Identification spaces

Exercise 4
Show that a continous map from a compact space onto a Hausdorff space is an identification map.

Exercise 5
Let X, Y_0 and Y_1 be topological spaces and let $q_1 \colon X \to Y_1$ and $q_2 \colon X \to Y_2$ be identification maps such that for all $x, x' \in X$ we have $q_1(x) = q_1(x')$ if and only if we have $q_2(x) = q_2(x')$. Then there is a unique map $h \colon Y_1 \to Y_2$ with $h \circ q_1 = q_2$. **In particular, h is a homeomorphism.**



Exercise 6
Show that $[0,1]/\{0,1\}$ is homeomorphic to $\mathbb{S}^1$.

Exercise 7
Mimic Exercise 6 (in some way) and show that $\mathbb{D}^{n+1}/\mathbb{S}^n$ is homeomorphic to $\mathbb{S}^{n+1}$.

Projective space

Definition. Let $\mathbb{RP}^n$ be defined as $\mathbb{S}^n/\!\sim$ where $\sim$ identifies antipodal points.

Exercise 8
Show that $\mathbb{RP}^n$ is homeomorphic to the space $\mathbb{D}^n/\!\sim$ where $\sim$ is the equivalence relation generated by setting $x \sim y$ precisely if $x = y$ holds or $x, y \in \mathbb{S}^{n-1}$ are antipodal.