Topology
Winter term 2021/2022
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Exercise Session Sheet 4

Exercise 1
Show that every infinite subset of a compact topological space X has a limit point in X.

Compactness in metric spaces.

Definition. For this section, let (X, d) be a metric space. For any subset $A \subset X$, the *diameter* $\operatorname{diam}(A) \in [0, \infty]$ is the supremum of all distances $d(x, y)$ for $x, y \in A$. The subset A is called *bounded* if $\operatorname{diam}(A) < \infty$ and it's called *totally bounded* if for every $\varepsilon > 0$, A can be covered by finitely many open ε-balls.

Exercise 2
In a compact metric space, every sequence has a convergent subsequence, i.e. every compact space is *sequentially compact.*

Exercise 3
Give an example of a metric space X and some closed and bounded subset $A \subset X$ which is not compact.

Exercise 4 (Lebesgue-Lemma)
Let X be a sequentially compact metric space and let $\mathcal{T}$ be an open cover of X. Show that there exists a $\delta > 0$ such that any subset of X of diameter smaller than δ is contained in some member of $\mathcal{T}$.

Exercise 5
Show that a metric space X is totally bounded precisely if every sequence has a subsequence which is a Cauchy sequence.
Note: This implies that sequentially compact spaces are totally bounded!

Exercise 6 (**Bolzano-Weierstrass** for metric spaces)
A subset $A \subset (X, d)$ is compact if and only if A is sequentially compact.

Exercise 7 (**Bolzano-Weierstrass** for metric spaces, reformulation)
Let X be a metric space. Show that a subset $A \subset X$ is compact if and only if A is complete and totally bounded.

Exercise 8
Let X be a topological space with base $\mathcal{B}$. Show that X is compact if and only if every open cover of X by members of $\mathcal{B}$ has a finite subcover.

Product topology

Exercise 9 Let X and Y be arbitrary topological spaces. Show that the product $X \times Y$ has property $\mathfrak{p}$ precisely if both its factors X and Y have property $\mathfrak{p}$ for

a) $\mathfrak{p} =$ Hausdorff,

b) $\mathfrak{p} =$ connected,

c) $\mathfrak{p} =$ path-connected,

d) $\mathfrak{p} =$ metrizable,

e) $\mathfrak{p} =$ compact .

connectedness and path-connectedness

Exercise 10
Show that path-connecteness defines an equivalence relation.

Definition. A topological space X is called *locally path-connected* if for every point $x \in X$ and every neighborhood U of x there exists a path-connected neighborhood $V \subset U$ of x.
Examples: Topological manifolds.

Exercise 11
Let X be a locally path-connected topological space. Show that the path-connected components are open and closed and coincide with the connected components.

Exercise 12
Give an example of a complete metric space X an open connected subset $C \subset X$, which is not path-connected.